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D-Smarandache Curves According to The Sabban Frame of The Spherical Indicatrix Curve

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Abstract. In this study, we first gave the Darboux vector according to the alternative frame. Then we formed a Sabban frame of spherical indicatrix curve of D-alternative vector defined by a differentiable curve. Then the geodesic curvature of this vector is calculated according to this frame. Finally we defined Smarandache curves generated by the Sabban frame and give some characterizations of them.

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1. Introduction

In differential geometry, special curves have an important role. One of these curves is a Smarandache curve. Smarandache curves are first defined by M. Turgut and S. Yılmaz in 2008 [6]. Special Smarandache curves also have been studied by some authors [1,2]. Let $\alpha = \alpha(s)$ be a regular unit speed curve in E^3. The Frenet frame and alternative frame of this curve are and $\{N, C, W\}$, respectively. Here, N is normal vector, W is unit Darboux vector and $C = W \wedge N$ [4]. In this paper, we created the Smarandache curves according to the alternative frame of the unit speed curve. Then we introduced alternative frame and its properties. Finally we calculated geodesic curvature of these curves according to alternative frame.

2. Preliminaries

Let $\alpha = \alpha(s)$ be a regular curve with unit speed. Then the Frenet apparatus of the curve (α) [3]

$$
\begin{aligned}
T(s) &= \alpha'(s), \quad N(s) = \frac{\alpha''(s)}{\| \alpha''(s) \|}, \quad B(s) = T(s) \wedge N(s), \\
\kappa(s) &= \| \alpha''(s) \|, \qquad \tau(s) = \frac{\langle \alpha'(s) \wedge \alpha''(s), \alpha'''(s) \rangle}{\| \alpha' \wedge \alpha'' \|^2}, \\
T' &= \kappa N, \quad N' = -\kappa T + \tau B, \quad B' = -\tau N.
\end{aligned}
$$

In Euclidean 3-space any regular curve $\alpha(s)$ depending on the Frenet vectors moves around the axis of Darboux vector and the Darboux vector and defining a unit vector field are given as [4]

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$$W=\frac{\tau}{\sqrt{\kappa^2+\tau^2}}T+\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}B,\quad C=W\wedge N=-\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}T+\frac{\tau}{\sqrt{\kappa^2+\tau^2}}B.$$

So build another orthonormal moving frame along the curve $\alpha(s)$. This frame defined as alternative frame and is represented by $\{N, C, W\}$. The derivative formulae of the alternative frame is given by [4]

$$N'=\beta C,\quad C'=-\beta N+\gamma W,\quad W'=-\gamma C,\quad \beta=\sqrt{\kappa^2+\tau^2},\quad \gamma=\frac{\kappa^2}{\kappa^2+\tau^2}(\frac{\tau}{\kappa})'.$$

The relationship between the Frenet frame and alternative frame is

$$\begin{bmatrix}N\\C\\W\end{bmatrix}=\begin{bmatrix}0&1&0\\-\bar{\kappa}&0&\bar{\tau}\\\bar{\tau}&0&\bar{\kappa}\end{bmatrix}\begin{bmatrix}T\\N\\B\end{bmatrix}\quad or\quad \begin{bmatrix}T\\N\\B\end{bmatrix}=\begin{bmatrix}0&-\bar{\kappa}&\bar{\tau}\\1&0&0\\0&\bar{\tau}&\bar{\kappa}\end{bmatrix}\begin{bmatrix}N\\C\\W\end{bmatrix}\ ,\ \bar{\kappa}=\frac{\kappa}{\beta}\ ,\ \bar{\tau}=\frac{\tau}{\beta}.$$

Principal normal vector N is common both frames. Let $\gamma : I \rightarrow S^2$ be a unit speed spherical curve and s arc-length parameter of γ.Let us denote $t(s)=\gamma'(s)$ and $d(s)=\gamma(s)\wedge t(s)$. This frame is called the Sabban frame of γ on S^2. Then we have the following spherical Frenet formulae of γ

$$\gamma'(s)=t(s),\quad t'(s)=-\gamma(s)+\kappa_g(s)d(s),\quad d'(s)=-\kappa_g(s)t(s), \tag{2.1}$$

$$\kappa_g(s)=\langle t'(s), d(s)\rangle \tag{2.2}$$

where $\kappa_g(s)$ is the geodesic curvature of the curve of γ on S^2 [5].

3. Smarandache Curves of Alternative Frame

Theorem 3.1. *Let $\alpha(s)$ be unit speed curve and alternative frame $\{N, C, W\}$. The alternative Darboux vector $\bar{D}$ of the curve α is given by*

$$\bar{D}=\gamma N+\beta W. \tag{3.1}$$

Proof. The alternative Darboux vector of the curve $\bar{D}$ can be written as follow (*Figure* 1)

$$\bar{D}=aN+bC+cW.$$

Taking the cross product of (3.1) and N, we get

$$\begin{aligned}N'=\bar{D}\wedge N\quad&\Rightarrow\quad \beta C=(aN+bC+cW)\wedge N\\&\Rightarrow\quad \beta C=-bW+cC\\&\Rightarrow\quad b=0, c=\beta.\end{aligned}$$

Taking the cross product of (3.1) and C, we get

$$\begin{aligned}C'=\bar{D}\wedge C\quad&\Rightarrow\quad -\beta N+\gamma W=(aN+bC+cW)\wedge C\\&\Rightarrow\quad -\beta N+\gamma W=aW-cN\\&\Rightarrow\quad a=\gamma, c=\beta.\end{aligned}$$

Taking the cross product of (3.1) and W, we get

$$\begin{aligned}W'=\bar{D}\wedge W\quad&\Rightarrow\quad -\gamma C=(aN+bC+cW)\wedge W\\&\Rightarrow\quad -\gamma C=-aC-bN\\&\Rightarrow\quad a=\gamma, b=0.\end{aligned}$$

Thus, alternative Darboux vector $\bar{D}$ is obtained as $\bar{D}=\gamma N+\beta W$ and the unit Darboux vector D is given by

$$D = pN + qW, \quad p = \frac{\gamma}{\sqrt{\gamma^2+\beta^2}}, \quad q = \frac{\beta}{\sqrt{\gamma^2+\beta^2}}. \tag{3.2}$$

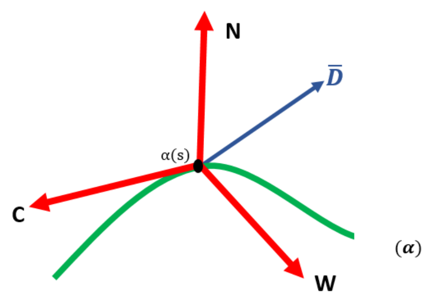

FIGURE 1. Alternative Darboux vector

□

Let $D = D(s)$ and $\alpha_D(s) = D(s)$ be a unit speed regular spherical curves on S^2, $\{D, T_D, (D \wedge T_D)\}$ and $\{D_{\alpha_D}, T_{D_{\alpha_D}}, (D \wedge T_D)_{\alpha_D}\}$ be the Sabban frame of these curves, respectively. If we take the derivative of the equation $\alpha_D(s) = D(s)$, then T_D vector is

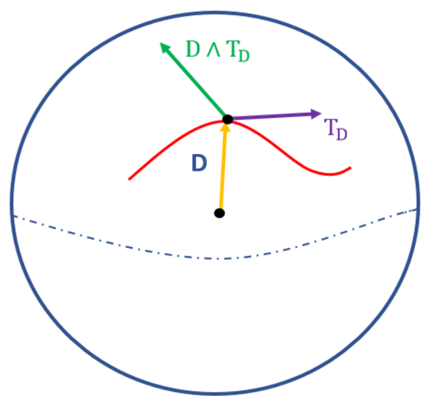

FIGURE 2. $\{D, T_D, D \wedge T_D\}$ Sabban Frame

$$\begin{aligned} T_D.\frac{d_{s^*}}{d_s} &= p'N + q'W, \\ T_D &= \frac{p'}{\sqrt{(p')^2+(q')^2}}N + \frac{q'}{\sqrt{(p')^2+(q')^2}}W, \quad \frac{d_{s^*}}{d_s} = \sqrt{(p')^2+(q')^2} \quad . \end{aligned}$$

Considering the $D(s)$ and T_D vectors we can write,

$$D \wedge T_D = \underbrace{\frac{p'q - pq'}{\sqrt{(p')^2+(q')^2}}}_{=1} C.$$

If we take the derivative of this equation then $(D \wedge T_D)'$ vector is

$$
\begin{aligned}
(D \wedge T_D)' . \frac{d_{s^*}}{d_s} &= -\beta N + \gamma W, \\
(D \wedge T_D)' &= \frac{-\beta}{\sqrt{(p')^2 + (q')^2}} N + \frac{\gamma}{\sqrt{(p')^2 + (q')^2}} W, \\
(D \wedge T_D)' &= \underbrace{-\Big(\frac{p'\beta - q'\gamma}{\sqrt{(p')^2 + (q')^2}} \Big)}_{\kappa_g^D} \underbrace{\Big(\frac{-p'}{\sqrt{(p')^2 + (q')^2}} N + \frac{q'}{\sqrt{(p')^2 + (q')^2}} W \Big)}_{T_D}.
\end{aligned}
\tag{3.3}
$$

Accordingly, the $\{D, T_D, (D \wedge T_D)\}$ Sabban frame is obtained from the D vector.If we take the derivative of the equation (3.1), then T_D' vector is

$$
\begin{aligned}
T_D' . \frac{d_{s^*}}{d_s} &= \Big(\frac{p'}{\sqrt{(p')^2 + (q')^2}} \Big)' N + \Big(\frac{p'}{\sqrt{(p')^2 + (q')^2}} \Big) N' + \Big(\frac{q'}{\sqrt{(p')^2 + (q')^2}} \Big)' W + \Big(\frac{q'}{\sqrt{(p')^2 + (q')^2}} \Big) W', \\
T_D' . \frac{d_{s^*}}{d_s} &= \Big(\frac{p'}{\sqrt{(p')^2 + (q')^2}} \Big)' N + \Big(\frac{p'\beta - q'\gamma}{\sqrt{(p')^2 + (q')^2}} \Big) C + \Big(\frac{q'}{\sqrt{(p')^2 + (q')^2}} \Big)' W, \\
T_D' &= \frac{\Big(\frac{p'}{q'} \Big)' (q')^2}{((p')^2 + (q')^2)^2} \Big(q' N - p' W + \frac{p'\beta - q'\gamma}{(p')^2 + (q')^2} . C \Big), \\
T_D' &= -\frac{\Big(\frac{p'}{q'} \Big)' (q')^2 \beta^2 \Big(\frac{\gamma}{\beta} \Big)'}{\underbrace{\Big((p')^2 + (q')^2 \Big)^2 + (\beta^2 + \gamma^2)}_{=1}} \underbrace{\frac{\gamma N + \beta W}{\sqrt{\beta^2 + \gamma^2}}}_{D} + \underbrace{\frac{p'\beta - q'\gamma}{(p')^2 + (q')^2}}_{\kappa_g^D} C.
\end{aligned}
\tag{3.4}
$$

From the equation (2.2), (3.3) and (3.4) the geodesic curvature of $\alpha_D(s) = D(s)$ is

$$
\kappa_g^D(s) = \frac{p'\beta - q'\gamma}{(p')^2 + (q')^2}.
$$

If we take the derivative of this equation (3.2) then p' and q' are

$$
p = \frac{\gamma}{\gamma^2 + \beta^2} \Rightarrow p' = \frac{\beta(\gamma'\beta - \beta'\gamma)}{(\gamma^2 + \beta^2)^{\frac{3}{2}}}, \quad q = \frac{\beta}{\gamma^2 + \beta^2} \Rightarrow q' = -\frac{\gamma(\gamma'\beta - \beta'\gamma)}{(\gamma^2 + \beta^2)^{\frac{3}{2}}}
\tag{3.5}
$$

and

$$
(p')^2 + (q')^2 = \frac{(\gamma'\beta - \beta'\gamma)^2}{\Big(\gamma^2 + \beta^2 \Big)^2}, \quad p'\beta - q'\gamma = \frac{(\gamma'\beta - \beta'\gamma)}{\sqrt{\gamma^2 + \beta^2}}.
\tag{3.6}
$$

Using the equation (2.2), (3.5) and (3.6) we can write κ_g^D geodesic curvature is

$$
\kappa_g^D = \frac{(\gamma^2 + \beta^2)^{\frac{3}{2}}}{\gamma'\beta - \beta'\gamma}.
\tag{3.7}
$$

Then from the equation (2.1), (3.3) and (3.4) we have the following spherical Sabban formulae of $\alpha_D(s)$,

$$
D' = T_D, \quad T_D' = -D + \kappa_g^D (D \wedge T_D), \quad (D \wedge T_D)' = -\kappa_g^D T_D.
$$

Definition 3.2. Let (D) be a spherical curve of $\alpha(s)$, D and T_D be Sabban vectors of (D). Then DT_D-Smarandache curve can be identified as

$$\alpha_{DT_D} = \frac{1}{\sqrt{2}}(D + T_D) \quad or \quad \alpha_{DT_D} = \frac{(\beta+\gamma)N + (\beta-\gamma)W}{\sqrt{2}\sqrt{\gamma^2+\beta^2}}. \tag{3.8}$$

Theorem 3.3. *The geodesic curvature according to DT_D-Smarandache curve is*

$$\kappa_g^{DT_D} = \frac{\Big(\gamma'\beta-\beta'\gamma\Big)^4\Big((\lambda_1+\lambda_2)(\gamma^2+\beta^2)^{\frac{3}{2}} + 2\lambda_3(\gamma'\beta-\beta'\gamma)\Big)}{\Big(2(\gamma'\beta-\beta'\gamma)^2 + (\gamma^2+\beta^2)^3\Big)^{\frac{5}{2}}}$$

where

$$\begin{aligned}
\lambda_1 &= \Big(\frac{(\gamma^2+\beta^2)^{\frac{3}{2}}}{\gamma'\beta-\beta'\gamma}\Big)\Big(\frac{(\gamma^2+\beta^2)^{\frac{3}{2}}}{\gamma'\beta-\beta'\gamma}\Big)' - \Big(\frac{(\gamma^2+\beta^2)^{\frac{3}{2}}}{\gamma'\beta-\beta'\gamma}\Big)^2 - 2,\\
\lambda_2 &= -\Big(\frac{(\gamma^2+\beta^2)^{\frac{3}{2}}}{\gamma'\beta-\beta'\gamma}\Big)^4 - 3\Big(\frac{(\gamma^2+\beta^2)^{\frac{3}{2}}}{\gamma'\beta-\beta'\gamma}\Big)^2 - \Big(\frac{(\gamma^2+\beta^2)^{\frac{3}{2}}}{\gamma'\beta-\beta'\gamma}\Big)\Big(\frac{(\gamma^2+\beta^2)^{\frac{3}{2}}}{\gamma'\beta-\beta'\gamma}\Big)' - 2,\\
\lambda_3 &= \Big(\frac{(\gamma^2+\beta^2)^{\frac{3}{2}}}{\gamma'\beta-\beta'\gamma}\Big)^3 + 2\Big(\frac{(\gamma^2+\beta^2)^{\frac{3}{2}}}{\gamma'\beta-\beta'\gamma}\Big) + 2\Big(\frac{(\gamma^2+\beta^2)^{\frac{3}{2}}}{\gamma'\beta-\beta'\gamma}\Big)'.
\end{aligned}$$

Proof. If we take the derivative of the equation (3.8) then T_{DT_D} vector is

$$\begin{aligned}
T_{DT_D}\frac{ds^*}{ds} &= \frac{1}{\sqrt{2}}\Big(-D + T_D + \kappa_g^D(D\wedge T_D)\Big),\\
T_{DT_D} &= \frac{1}{\sqrt{2+(\kappa_g^D)^2}}\Big(-D + T_D + \kappa_g^D(D\wedge T_D)\Big), \quad \frac{ds^*}{ds} = \frac{\sqrt{2+(\kappa_g^D)^2}}{\sqrt{2}}.
\end{aligned} \tag{3.9}$$

Considering the equations (3.8) and (3.9), we have

$$\begin{aligned}
\alpha_{DT_D}\wedge T_{DT_D} &= \frac{1}{\sqrt{2}\sqrt{2+(\kappa_g^D)^2}}(D+T_D)\wedge\Big(-D + T_D + \kappa_g^D(D\wedge T_D)\Big),\\
\alpha_{DT_D}\wedge T_{DT_D} &= \frac{1}{\sqrt{2}\sqrt{2+(\kappa_g^D)^2}}\Big(\kappa_g^D D - \kappa_g^D T_D + 2(D\wedge T_D)\Big).
\end{aligned} \tag{3.10}$$

If we take the derivative of the equation (3.9), then T'_{DT_D} vector is

$$
\begin{aligned}
T'_{DT_D}\frac{ds^*}{ds} &= \Big(\frac{1}{\sqrt{2+(\kappa_g^D)^2}}\Big)'\Big(-D+T_D+\kappa_g^D(D\wedge T_D)\Big)+\Big(\frac{1}{\sqrt{2+(\kappa_g^D)^2}}\Big)\Big(-D+T_D+\kappa_g^D(D\wedge T_D)\Big)',\\
T'_{DT_D}\frac{ds^*}{ds} &= -\frac{\kappa_g^D\Big(\kappa_g^D\Big)'\Big(-D+T_D+\kappa_g^D(D\wedge T_D)\Big)}{\Big(2+(\kappa_g^D)^2\Big)^{\frac{3}{2}}}+\frac{-D-(1+(\kappa_g^D)^2)T_D+(\kappa_g^D+(\kappa_g^D)'(D\wedge T_D))}{\sqrt{2+(\kappa_g^D)^2}},\\
T'_{DT_D} &= \frac{\sqrt{2}\Big(\kappa_g^D(\kappa_g^D)'-(\kappa_g^D)^2-2\Big)}{\Big(2+(\kappa_g^D)^2\Big)^2}D-\frac{\sqrt{2}\Big((\kappa_g^D)^4+3(\kappa_g^D)^2+\kappa_g^D(\kappa_g^D)'+2\Big)}{\Big(2+(\kappa_g^D)^2\Big)^2}T_D \qquad (3.11)\\
&\quad+\frac{\sqrt{2}\Big((\kappa_g^D)^3+2\kappa_g^D+2(\kappa_g^D)'\Big)}{\Big(2+(\kappa_g^D)^2\Big)^2}(D\wedge T_D),\\
T'_{DT_D} &= \frac{\sqrt{2}\Big(\lambda_1 D+\lambda_2 T_D+\lambda_3(D\wedge T_D)\Big)}{\Big(2+(\kappa_g^D)^2\Big)^2}.
\end{aligned}
$$

Where the coefficients are

$$\lambda_1=\kappa_g^D(\kappa_g^D)'-(\kappa_g^D)^2-2,\quad \lambda_2=-(\kappa_g^D)^4-3(\kappa_g^D)^2-\kappa_g^D(\kappa_g^D)'-2,\quad \lambda_3=(\kappa_g^D)^3+2\kappa_g^D+2(\kappa_g^D)'. \qquad (3.12)$$

Using the equation (2.2), (3.10) and (3.11) we can write $\kappa_g^{DT_D}$ geodesic curvature is

$$\kappa_g^{DT_D}=\frac{1}{\Big(2+(\kappa_g^D)^2\Big)^{\frac{5}{2}}}(\lambda_1\kappa_g^D-\lambda_2\kappa_g^D+2\lambda_3). \qquad (3.13)$$

Considering the equations (3.7), (3.12) and (3.13) we can write $\kappa_g^{DT_D}$ geodesic curvature is

$$\kappa_g^{DT_D}=\frac{\Big(\gamma'\beta-\beta'\gamma\Big)^4}{\Big(2(\gamma'\beta-\beta'\gamma)^2+(\gamma^2+\beta^2)^3\Big)^{\frac{5}{2}}}\Big((\lambda_1+\lambda_2)(\gamma^2+\beta^2)^{\frac{3}{2}}+2\lambda_3(\gamma'\beta-\beta'\gamma)\Big). \qquad \square$$

Definition 3.4. Let (D) be a spherical curve of $\alpha(s)$, D and $D\wedge T_D$ be Sabban vectors of (D). Then $D(D\wedge T_D)$-Smarandache curve can be identified as

$$\alpha_{D(D\wedge T_D)}=\frac{1}{\sqrt{2}}(D+T_D)\ \ or\ \ \alpha_{D(D\wedge T_D)}=\frac{\gamma N+\sqrt{\gamma^2+\beta^2}C+\beta W}{\sqrt{2}\sqrt{\gamma^2+\beta^2}}. \qquad (3.14)$$

Theorem 3.5. *The geodesic curvature according to $D(D\wedge T_D)$-Smarandache curve is*

$$\kappa_g^{D(D\wedge T_D)}=\frac{\gamma'\beta-\beta'\gamma+(\gamma^2+\beta^2)^{\frac{3}{2}}}{\gamma'\beta-\beta'\gamma-(\gamma^2+\beta^2)^{\frac{3}{2}}}.$$

Proof. If we take the derivative of the equation (3.14) then $T_{D(D\wedge T_D)}$ vector is

$$\begin{aligned} T_{D(D\wedge T_D)}\frac{ds^*}{ds} &= \frac{1}{\sqrt{2}}(T_D - \kappa_g^D T_D), \\ T_{D(D\wedge T_D)} &= T_D, \quad \frac{ds^*}{ds} = \frac{1-\kappa_g^D}{\sqrt{2}}. \end{aligned} \tag{3.15}$$

Considering the equations (3.14) and (3.15), we have

$$\alpha_{D(D\wedge T_D)} \wedge T_{D(D\wedge T_D)} = \frac{1}{\sqrt{2}}\big(-D + (D\wedge T_D)\big). \tag{3.16}$$

If we take the derivative of the equation (3.15), then $T'_{D(D\wedge T_D)}$ vector is

$$T'_{D(D\wedge T_D)} = \frac{\sqrt{2}}{1-\kappa_g^D}\big(-D + \kappa_g^D (D\wedge T_D)\big). \tag{3.17}$$

Using the equation (2.2), (3.7), (3.16) and (3.17), we can write $\kappa_g^{D(D\wedge T_D)}$ geodesic curvature is

$$\kappa_g^{D(D\wedge T_D)} = \frac{1+\kappa_g^D}{1-\kappa_g^D} \ \ or \ \ \kappa_g^{D(D\wedge T_D)} = \frac{\gamma'\beta-\beta'\gamma+(\gamma^2+\beta^2)^{\frac{3}{2}}}{\gamma'\beta-\beta'\gamma-(\gamma^2+\beta^2)^{\frac{3}{2}}}.$$ □

Definition 3.6. Let (D) be a spherical curve of $\alpha(s)$, T_D and $D\wedge T_D$ be Sabban vectors of (D). Then $T_D(D\wedge T_D)$-Smarandache curve can be identified as

$$\alpha_{T_D(D\wedge T_D)} = \frac{T_D + (D\wedge T_D)}{\sqrt{2}} \ \ or \ \ \alpha_{T_D(D\wedge T_D)} = \frac{\beta N + \sqrt{\gamma^2+\beta^2}C - \gamma W}{\sqrt{2}\sqrt{\gamma^2+\beta^2}}. \tag{3.18}$$

Theorem 3.7. *The geodesic curvature according to $T_D(D\wedge T_D)$-Smarandache curve is*

$$\kappa_g^{T_D(D\wedge T_D)} = \frac{(\gamma'\beta-\beta'\gamma)^4\Big(2\lambda_1(\gamma^2+\beta^2)^{\frac{3}{2}} + (-\lambda_2+\lambda_3)(\gamma'\beta-\beta'\gamma)\Big)}{\Big((\gamma'\beta-\beta'\gamma)^2 + 2(\gamma^2+\beta^2)^3\Big)^{\frac{5}{2}}}$$

where

$$\begin{aligned} \lambda_1 &= 2\sqrt{2}\Big(\frac{(\gamma^2+\beta^2)^{\frac{3}{2}}}{\gamma'\beta-\beta'\gamma}\Big)\Big(\frac{(\gamma^2+\beta^2)^{\frac{3}{2}}}{\gamma'\beta-\beta'\gamma}\Big)' + \sqrt{2}\Big(\frac{(\gamma^2+\beta^2)^{\frac{3}{2}}}{\gamma'\beta-\beta'\gamma}\Big) + 2\sqrt{2}\Big(\frac{(\gamma^2+\beta^2)^{\frac{3}{2}}}{\gamma'\beta-\beta'\gamma}\Big)^3, \\ \lambda_2 &= \sqrt{2} + \sqrt{2}\Big(\frac{(\gamma^2+\beta^2)^{\frac{3}{2}}}{\gamma'\beta-\beta'\gamma}\Big)' + 3\sqrt{2}\Big(\frac{(\gamma^2+\beta^2)^{\frac{3}{2}}}{\gamma'\beta-\beta'\gamma}\Big)^2 + 2\sqrt{2}\Big(\frac{(\gamma^2+\beta^2)^{\frac{3}{2}}}{\gamma'\beta-\beta'\gamma}\Big)^4, \\ \lambda_3 &= \sqrt{2}\Big(\frac{(\gamma^2+\beta^2)^{\frac{3}{2}}}{\gamma'\beta-\beta'\gamma}\Big)' - \sqrt{2}\Big(\frac{(\gamma^2+\beta^2)^{\frac{3}{2}}}{\gamma'\beta-\beta'\gamma}\Big)^2 + 2\sqrt{2}\Big(\frac{(\gamma^2+\beta^2)^{\frac{3}{2}}}{\gamma'\beta-\beta'\gamma}\Big)^4. \end{aligned}$$

Proof. If we take the derivative of the equation (3.18) then $T_{T_D(D\wedge T_D)}$ vector is

$$\begin{aligned} T_{T_D(D\wedge T_D)}\frac{ds^*}{ds} &= \frac{1}{\sqrt{2}}\Big(-D - \kappa_g^D T_D + \kappa_g^D (D\wedge T_D)\Big), \\ T_{T_D(D\wedge T_D)} &= \frac{1}{\sqrt{1+2(\kappa_g^D)^2}}\Big(-D - \kappa_g^D T_D + \kappa_g^D (D\wedge T_D)\Big), \quad \frac{ds^*}{ds} = \frac{1+(\kappa_g^D)^2}{\sqrt{2}}. \end{aligned} \tag{3.19}$$

Considering the equations (3.18) and (3.19), we have

$$\begin{aligned}\alpha_{T_D(D\wedge T_D)}\wedge T_{T_D(D\wedge T_D)} &= \frac{1}{\sqrt{2+4(\kappa_g^D)^2}}\Big((T_D+(D\wedge T_D))\wedge(-D-\kappa_g^D T_D+\kappa_g^D(D\wedge T_D)\Big),\\ \alpha_{T_D(D\wedge T_D)}\wedge T_{T_D(D\wedge T_D)} &= \frac{1}{\sqrt{2+4(\kappa_g^D)^2}}\Big(2\kappa_g^D D-T_D+(D\wedge T_D)\Big).\end{aligned} \tag{3.20}$$

If we take the derivative of the equation (3.19), then $T'_{T_D(D\wedge T_D)}$ vector is

$$T'_{T_D(D\wedge T_D)}\frac{ds^*}{ds} = \Big(\frac{1}{\sqrt{1+2(\kappa_g^D)^2}}\Big)'\Big(-D-\kappa_g^D T_D+\kappa_g^D(D\wedge T_D)\Big)+\Big(\frac{1}{\sqrt{1+2(\kappa_g^D)^2}}\Big)\Big(-D-\kappa_g^D T_D+\kappa_g^D(D\wedge T_D)\Big)',$$

$$\begin{aligned}T'_{T_D(D\wedge T_D)}\frac{ds^*}{ds} &= -\frac{2\kappa_g^D(\kappa_g^D)'}{\Big(1+2(\kappa_g^D)^2\Big)^{\frac{3}{2}}}\Big(-D-\kappa_g^D T_D+\kappa_g^D(D\wedge T_D)\Big)+\frac{1}{\sqrt{1+2(\kappa_g^D)^2}}\Big(-\kappa_g^D-(1+(\kappa_g^D)'+(\kappa_g^D)^2)\Big)T_D\\ &+\Big((-(\kappa_g^D)^2+(\kappa_g^D)')(D\wedge T_D)\Big),\end{aligned}$$

$$\begin{aligned}T'_{T_D(D\wedge T_D)} &= \frac{\sqrt{2}\Big(2\kappa_g^D(\kappa_g^D)'+\kappa_g^D+2(\kappa_g^D)^3\Big)}{\Big(1+2(\kappa_g^D)^2\Big)^2}D-\frac{\sqrt{2}\Big(1+(\kappa_g^D)'+3(\kappa_g^D)^2+2(\kappa_g^D)^4\Big)}{\Big(1+2(\kappa_g^D)^2\Big)^2}T_D\\ &+\frac{\sqrt{2}\Big((\kappa_g^D)'-(\kappa_g^D)^2+2(\kappa_g^D)^4\Big)}{\Big(1+2(\kappa_g^D)^2\Big)^2}(D\wedge T_D),\end{aligned} \tag{3.21}$$

$$T'_{T_D(D\wedge T_D)} = \frac{\sqrt{2}\Big(\lambda_1 D-\lambda_2 T_D+\lambda_3(D\wedge T_D)\Big)}{\Big(1+2(\kappa_g^D)^2\Big)^2}$$

where the coefficients are

$$\lambda_1=\Big(2\kappa_g^D(\kappa_g^D)'+\kappa_g^D+2(\kappa_g^D)^3\Big),\quad \lambda_2=-1-(\kappa_g^D)'-3(\kappa_g^D)^2-2(\kappa_g^D)^4,\quad \lambda_3=(\kappa_g^D)'-(\kappa_g^D)^2+2(\kappa_g^D)^4. \tag{3.22}$$

Using the equation (2.2), (3.20) and (3.21), we can write $\kappa_g^{T_D(D\wedge T_D)}$ geodesic curvature is

$$\kappa_g^{T_D(D\wedge T_D)}=\frac{1}{\Big(1+2(\kappa_g^D)^2\Big)^{\frac{5}{2}}}(2\lambda_1\kappa_g^D-\lambda_2+\lambda_3). \tag{3.23}$$

Considering the equations (3.7), (3.22) and (3.23) we can write $\kappa_g^{T_D(D\wedge T_D)}$ geodesic curvature is

$$\kappa_g^{T_D(D\wedge T_D)}=\frac{(\gamma'\beta-\beta'\gamma)^4\Big(2\lambda_1(\gamma^2+\beta^2)^{\frac{3}{2}}+(-\lambda_2+\lambda_3)(\gamma'\beta-\beta'\gamma)\Big)}{\Big((\gamma'\beta-\beta'\gamma)^2+2(\gamma^2+\beta^2)^3\Big)^{\frac{5}{2}}}.$$

□

Definition 3.8. Let (D) be a spherical curve of $\alpha(s)$, D, T_D and $D \wedge T_D$ be Sabban vectors of (D). Then $DT_D(D \wedge T_D)$-Smarandache curve can be identified as

$$\alpha_{DT_D(D\wedge T_D)} = \frac{1}{\sqrt{3}}\Big(D + T_D + (D \wedge T_D)\Big) \ \ or \ \ \alpha_{DT_D(D\wedge T_D)} = \frac{(\beta+\gamma)N + \sqrt{\beta_2+\gamma^2}C + (\beta-\gamma)W}{\sqrt{3}\sqrt{\gamma^2+\beta^2}}. \tag{3.24}$$

Theorem 3.9. *The geodesic curvature according to $DT_D(D \wedge T_D)$-Smarandache curve is*

$$\begin{aligned}
\kappa_g^{DT_D(D\wedge T_D)} &= \frac{(\gamma'\beta-\beta'\gamma)^4\Big(-\gamma'\beta+\beta'\gamma+2(\gamma^2+\beta^2)^{\frac{3}{2}}\Big)}{4\sqrt{2}\Big((\gamma'\beta-\beta'\gamma)^2-(\gamma^2+\beta^2)^{\frac{3}{2}}(\gamma'\beta-\beta'\gamma)+(\gamma^2+\beta^2)^3\Big)^{\frac{5}{2}}}\lambda_1 \\
&-\frac{(\gamma'\beta-\beta'\gamma)^4\Big(\gamma'\beta-\beta'\gamma+(\gamma^2+\beta^2)^{\frac{3}{2}}\Big)}{4\sqrt{2}\Big((\gamma'\beta-\beta'\gamma)^2-(\gamma^2+\beta^2)^{\frac{3}{2}}(\gamma'\beta-\beta'\gamma)+(\gamma^2+\beta^2)^3\Big)^{\frac{5}{2}}}\lambda_2 \\
&+\frac{(\gamma'\beta-\beta'\gamma)^4\Big(\gamma'\beta-\beta'\gamma-(\gamma^2+\beta^2)^{\frac{3}{2}}\Big)}{4\sqrt{2}\Big((\gamma'\beta-\beta'\gamma)^2-(\gamma^2+\beta^2)^{\frac{3}{2}}(\gamma'\beta-\beta'\gamma)+(\gamma^2+\beta^2)^3\Big)^{\frac{5}{2}}}\lambda_3
\end{aligned}$$

where

$$\begin{aligned}
\lambda_1 &= \Big(\frac{(\gamma^2+\beta^2)^{\frac{3}{2}}}{\gamma'\beta-\beta'\gamma}\Big)'\Big(\frac{2(\gamma^2+\beta^2)^{\frac{3}{2}}-\gamma'\beta+\beta'\gamma}{\gamma'\beta-\beta'\gamma}\Big)+4\frac{(\gamma^2+\beta^2)^{\frac{3}{2}}}{\gamma'\beta-\beta'\gamma}-4\Big(\frac{(\gamma^2+\beta^2)^{\frac{3}{2}}}{\gamma'\beta-\beta'\gamma}\Big)^2+2\Big(\frac{(\gamma^2+\beta^2)^{\frac{3}{2}}}{\gamma'\beta-\beta'\gamma}\Big)^3-2, \\
\lambda_2 &= \Big(\frac{(\gamma^2+\beta^2)^{\frac{3}{2}}}{\gamma'\beta-\beta'\gamma}\Big)'\Big(1-3\frac{(\gamma^2+\beta^2)^{\frac{3}{2}}}{\gamma'\beta-\beta'\gamma}+2\Big(\frac{(\gamma^2+\beta^2)^{\frac{3}{2}}}{\gamma'\beta-\beta'\gamma}\Big)^2\Big) \\
&-2\Big(1+\Big(\frac{(\gamma^2+\beta^2)^{\frac{3}{2}}}{\gamma'\beta-\beta'\gamma}\Big)'+\Big(\frac{(\gamma^2+\beta^2)^{\frac{3}{2}}}{\gamma'\beta-\beta'\gamma}\Big)^2\Big)\Big(1-\Big(\frac{(\gamma^2+\beta^2)^{\frac{3}{2}}}{\gamma'\beta-\beta'\gamma}\Big)'+\Big(\frac{(\gamma^2+\beta^2)^{\frac{3}{2}}}{\gamma'\beta-\beta'\gamma}\Big)^2,\Big) \\
\lambda_3 &= \Big(\frac{(\gamma^2+\beta^2)^{\frac{3}{2}}}{\gamma'\beta-\beta'\gamma}\Big)'\Big(\frac{(\gamma^2+\beta^2)^{\frac{3}{2}}}{\gamma'\beta-\beta'\gamma}-2\Big(\frac{(\gamma^2+\beta^2)^{\frac{3}{2}}}{\gamma'\beta-\beta'\gamma}\Big)^2\Big) \\
&+2\Big(1+\Big(\frac{(\gamma^2+\beta^2)^{\frac{3}{2}}}{\gamma'\beta-\beta'\gamma}\Big)'+\Big(\frac{(\gamma^2+\beta^2)^{\frac{3}{2}}}{\gamma'\beta-\beta'\gamma}\Big)^2\Big)\Big(1-\Big(\frac{(\gamma^2+\beta^2)^{\frac{3}{2}}}{\gamma'\beta-\beta'\gamma}\Big)'+\Big(\frac{(\gamma^2+\beta^2)^{\frac{3}{2}}}{\gamma'\beta-\beta'\gamma}\Big)^2\Big).
\end{aligned}$$

Proof. If we take the derivative of the equation (3.24) then $T_{DT_D(D\wedge T_D)}$ vector is

$$\begin{aligned}
T_{DT_D(D\wedge T_D)}\frac{ds^*}{ds} &= \frac{1}{\sqrt{3}}\Big(-D+(1-\kappa_g^D)T_D+\kappa_g^D(D\wedge T_D)\Big), \\
T_{DT_D(D\wedge T_D)} &= \frac{-D+(1-\kappa_g^D)T_D+\kappa_g^D(D\wedge T_D)}{\sqrt{2}\sqrt{1-\kappa_g^D+(\kappa_g^D)^2}}, \quad \frac{ds^*}{ds} = \frac{\sqrt{2}}{\sqrt{3}}\sqrt{1-\kappa_g^D+(\kappa_g^D)^2}.
\end{aligned} \tag{3.25}$$

Considering the equations (3.24) and (3.25), we have

$$\alpha_{DT_D(D\wedge T_D)} \wedge T_{DT_D(D\wedge T_D)} = \frac{\Big(D + T_D + D\wedge T_D\Big)\wedge\Big(-D + (1-\kappa_g^D)T_D + \kappa_g^D(D\wedge T_D)\Big)}{\sqrt{6}\sqrt{1-\kappa_g^D + (\kappa_g^D)^2}},$$
$$\alpha_{DT_D(D\wedge T_D)} \wedge T_{DT_D(D\wedge T_D)} = \frac{(2\kappa_g^D - 1)D - (1+2\kappa_g^D)T_D + (2-\kappa_g^D)(D\wedge T_D)}{\sqrt{6}\sqrt{1-\kappa_g^D + (\kappa_g^D)^2}}. \qquad (3.26)$$

If we take the derivative of the equation (3.25), then $T'_{DT_D(D\wedge T_D)}$ vector is

$$T'_{DT_D(D\wedge T_D)}\frac{ds^*}{ds} = \Big(\frac{1}{\sqrt{2}\sqrt{1-(\kappa_g^D)+(\kappa_g^D)^2}}\Big)'\Big(-D + (1-\kappa_g^D)T_D + \kappa_g^D(D\wedge T_D)\Big)$$
$$+\Big(\frac{1}{\sqrt{2}\sqrt{1-(\kappa_g^D)+(\kappa_g^D)^2}}\Big)\Big(-D + (1-\kappa_g^D)T_D + \kappa_g^D(D\wedge T_D)\Big)',$$

$$T'_{DT_D(D\wedge T_D)}\frac{ds^*}{ds} = \frac{(\kappa_g^D)'\Big(1-2(\kappa_g^D)\Big)\Big(-D + (1-\kappa_g^D)T_D + \kappa_g^D(D\wedge T_D)\Big)}{2\sqrt{2}\Big(1-(\kappa_g^D)+(\kappa_g^D)^2\Big)^{\frac{3}{2}}}$$
$$+\frac{(\kappa_g^D - 1)D - \Big(1 + (\kappa_g^D)' + (\kappa_g^D)^2\Big)T_D + \Big(\kappa_g^D - (\kappa_g^D)^2 + (\kappa_g^D)'\Big)(D\wedge T_D)}{\sqrt{2}\sqrt{1-(\kappa_g^D)+(\kappa_g^D)^2}},$$

$$T'_{DT_D(D\wedge T_D)} = \frac{\sqrt{3}}{4}.\frac{(\kappa_g^D)'\Big(1-2(\kappa_g^D)\Big) + 2(\kappa_g^D - 1)\Big(1-(\kappa_g^D)+(\kappa_g^D)^2\Big)}{\Big(1-(\kappa_g^D)+(\kappa_g^D)^2\Big)^2}D$$
$$+\frac{\sqrt{3}}{4}.\frac{(\kappa_g^D)'\Big(1-3(\kappa_g^D)+2(\kappa_g^D)^2\Big) - 2\Big(1+(\kappa_g^D)'+(\kappa_g^D)^2\Big)\Big(1-(\kappa_g^D)+(\kappa_g^D)^2\Big)}{\Big(1-(\kappa_g^D)+(\kappa_g^D)^2\Big)^2}T_D \qquad (3.27)$$
$$+\frac{\sqrt{3}}{4}.\frac{(\kappa_g^D)'\Big((\kappa_g^D) - 2(\kappa_g^D)^2\Big) + 2\Big(1+(\kappa_g^D)'+(\kappa_g^D)^2\Big)\Big(1-(\kappa_g^D)+(\kappa_g^D)^2\Big)}{\Big(1-(\kappa_g^D)+(\kappa_g^D)^2\Big)^2}\Big(D\wedge T_D\Big),$$

$$T'_{DT_D(D\wedge T_D)} = \frac{\sqrt{3}}{4\Big(1-(\kappa_g^D)+(\kappa_g^D)^2\Big)^2}\Big(\lambda_1 D - \lambda_2 T_D + \lambda_3(D\wedge T_D)\Big).$$

where the coefficients are

$$\begin{aligned}
\lambda_1 &= (\kappa_g^D)'(1-2\kappa_g^D)+2(\kappa_g^D-1)\Big(1-\kappa_g^D+(\kappa_g^D)^2\Big),\\
\lambda_2 &= (\kappa_g^D)'\Big(1-3\kappa_g^D+2(\kappa_g^D)^2\Big)-2\Big(1+(\kappa_g^D)'+(\kappa_g^D)^2\Big)\Big(1-\kappa_g^D+(\kappa_g^D)^2\Big),\\
\lambda_3 &= (\kappa_g^D)'\Big(\kappa_g^D-2(\kappa_g^D)^2\Big)+2\Big(1+(\kappa_g^D)'+(\kappa_g^D)^2\Big)\Big(1-\kappa_g^D+(\kappa_g^D)^2\Big).
\end{aligned} \tag{3.28}$$

Using the equation (2.2), (3.26), (3.28) and (3.27),we can write $\kappa_g^{DT_D(D\wedge T_D)}$ geodesic curvature is

$$\kappa_g^{T_D(D\wedge T_D)} = \frac{1}{\Big(4\sqrt{2}1-(\kappa_g^D)+(\kappa_g^D)^2\Big)^{\frac{5}{2}}}\Big((-1+2\kappa_g^D)\lambda_1-(1+\kappa_g^D)\lambda_2+(1-\kappa_g^D)\lambda_3\Big). \tag{3.29}$$

Considering the equations (3.7), (3.16) and (3.29) the proof is completed. □

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