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Research Article

Subclasses of starlike functions related to Blaschke products

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Abstract

In this paper we examine subclasses of the class of starlike functions defined by the set of zeros of Schwarz functions. Distortion and the growth theorems are shown. Bounds of the classical coefficient functionals are also computed.

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1. Introduction

Let $\mathcal{H}$ be the class of all analytic functions in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{A}$ its subclass of all standardly normalized functions f by $f(0) := 0$ and $f'(0) := 1$. Subclasses of $\mathcal{H}$, particularly subclasses of univalent functions in $\mathcal{A}$, are the basic subject to study in the geometric function theory. Many of subfamilies of $\mathcal{A}$ have an analytical description expressed in term of Carathéodory class of functions, i.e., the family $\mathcal{P}$ of functions $p \in \mathcal{H}$ normalized by $p(0) := 1$ having a positive real part. Based on famous Riesz Theorem on the factorization of functions in the Hardy classes, so in particular, of Schwarz functions, i.e., of analytic self mappings of $\mathbb{D}$ with a fixed point at the origin, forming the class denoted here by $\mathcal{B}_0$, we can distinguish subclasses of $\mathcal{B}_0$ related to the Blaschke product. Since there is a one-to-one relationship between the class $\mathcal{P}$ and the class $\mathcal{B}_0$, the factorization of the class $\mathcal{B}_0$ can be transferred to the class $\mathcal{P}$, so in the next step to the subclasses of $\mathcal{A}$ which are related to class $\mathcal{P}$. Such is the class $\mathcal{S}^*$ introduced by Alexander [1], whose elements are all starlike functions, i.e., $f \in \mathcal{A}$ belongs to $\mathcal{S}^*$ if it univalently maps $\mathbb{D}$ onto a domain $f(\mathbb{D})$ starlike with respect to the origin. It means that $[0, w] \subset f(\mathbb{D})$ for each $w \in f(\mathbb{D})$. Therefore the distribution of zero sets of Schwarz functions plays a fundamental role for related subfamilies in $\mathcal{S}^*$.

In this paper we study the basic properties of subclasses in $\mathcal{S}^*$ defined by the mentioned method. We prove growth and distortion theorems (Theorems 3.3 and 3.4). In the last section we show that the estimates of some coefficient functionals over such defined subclasses of $\mathcal{S}^*$ can be expressed in term of a given set of of zeros and that the new results

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are more detailed than the classical. In this matter, our computing is restricted to the case when the Blaschke product is reduced to one factor. In other words, Schwarz functions defining an appropriate subclasses of $\mathcal{S}^*$ are considered with one distinguished zero different from the origin.

Given $m \in \mathbb{N}$, let $\mathcal{A}_m$ of be the subset of $\mathcal{H}$ of all f of the form

$$f(z) = z + \sum_{k=m}^{\infty} a_{k+1} z^{k+1}, \quad z \in \mathbb{D}. \tag{1.1}$$

Let $\mathcal{S}$ be the class of all univalent functions in $\mathcal{A}$. Given $m \in \mathbb{N}$, let

$$\mathcal{S}^*_m := \mathcal{S}^* \cap \mathcal{A}_m.$$

Given $r \in (0,1)$, let $\mathbb{T}_r := \{z \in \mathbb{C} : |z| = r\}$ and let $\mathbb{T} := \mathbb{T}_1$. Let $\mathcal{B}$ be the class of all $\omega \in \mathcal{H}$ such that $|\omega(z)| \le 1$ for $z \in \mathbb{D}$, and $\mathcal{B}^0$ be its subclass of non-vanishing functions in $\mathbb{D}$.

Let $\mathbb{D}^0 := \mathbb{D} \setminus \{0\}$. Given $k \in \mathbb{N}$, let $\Lambda_k := (\mathbb{D}^0)^k$. Let $\Lambda_0 := \{0\}^1$ and $\Lambda_\infty = (\mathbb{D}^0)^\infty$. Let

$$\Lambda := \bigcup_{k \in \mathbb{N} \cup \{0,\infty\}} \Lambda_k.$$

Given $\alpha \in \mathbb{D}$, let

$$\varphi_\alpha(z) := \frac{z - \alpha}{1 - \overline{\alpha} z}, \quad z \in \mathbb{D},$$

denote the Blaschke factor. A sequence of points $\mathit{\Lambda} = (\alpha_k) \in \Lambda_\infty$ is said to satisfy the Blaschke condition if

$$\sum_{k=1}^{\infty} (1 - |\alpha_k|) < \infty,$$

which guaranties convergence of the product

$$B_{\mathit{\Lambda}}(z) := \prod_{k \in \mathbb{N}} \frac{-|\alpha_k|}{\alpha_k} \varphi_{\alpha_k}(z), \quad z \in \mathbb{D}.$$

A function $B(z) := z^m B_{\mathit{\Lambda}}(z),\ z \in \mathbb{D}$, with $m \in \mathbb{N} \cup \{0\}$, is called the Blaschke product. When $\mathit{\Lambda}(\mathbb{N}) = \emptyset$, set $B_{\mathit{\Lambda}}(z) := 1,\ z \in \mathbb{D}$, and then

$$B_{\mathit{\Lambda}}(z) = z^m,\ z \in \mathbb{D}.$$

2. Definition of the class $\mathcal{S}^*(m, \mathit{\Lambda})$

For $f \in \mathcal{H}$ let $Z(f)$ denote the set of all zeros of f in $\mathbb{D}^0$ counting with their multiplicities. Clearly, $Z(f) \in \Lambda$. It is known, that the sequence $Z(\omega)$ of each bounded analytic function ω, so in particular, of each Schwarz function, satisfies the Blaschke condition. By Riesz Theorem (e.g., [5, p. 283], [2, p. 20]) each $\omega \in \mathcal{B}_0$ has a unique canonical factorization

$$\omega(z) = z^m B_{Z(\omega)} \varphi(z), \quad z \in \mathbb{D},$$

where $m \in \mathbb{N}$ and $\varphi \in \mathcal{B}^0$. Thus $B(z) = z^m B_{Z(\omega)}(z)$ for $z \in \mathbb{D}$, is the Blaschke product with the same zeros as the function ω. Vice versa, each function

$$\omega(z) := z^m B_{\mathit{\Lambda}}(z) \varphi(z), \quad z \in \mathbb{D}, \tag{2.1}$$

with $m \in \mathbb{N}$, $\mathit{\Lambda} \in \Lambda$ satisfying the Blaschke condition and $\varphi \in \mathcal{B}^0$, is a Schwarz function. This is a starting point for further considerations.

Definition 2.1. Let $m \in \mathbb{N}$ and let $\mathit{\Lambda} \in \Lambda$ satisfy the Blaschke condition. By $\mathcal{B}(m, \mathit{\Lambda})$ we denote the class of functions of the form (2.1), where $\varphi \in \mathcal{B}$.

Let $\mathcal{B}^0(m, \mathit{\Lambda})$ be the class of functions of the form (2.1), where $\varphi \in \mathcal{B}^0$.

When $B_{\mathit{\Lambda}} \equiv 1$, i.e., when $\mathit{\Lambda}(\mathbb{N}) = \emptyset$, we will write $\mathcal{B}(m)$ and $\mathcal{B}^0(m)$ instead of $\mathcal{B}(m, \mathit{\Lambda})$ and $\mathcal{B}^0(m, \mathit{\Lambda})$, respectively.

Definition 2.2. Let $m \in \mathbb{N}$ and let $\Lambda \in \Lambda$ satisfy the Blaschke condition. By $\mathcal{P}(m, \Lambda)$ we denote the class of functions of the form

$$p(z) = \frac{1+\omega(z)}{1-\omega(z)}, \quad z \in \mathbb{D}, \tag{2.2}$$

where $\omega \in \mathcal{B}(m, \Lambda)$, i.e., of the form

$$p(z) = \frac{1+z^m B_\Lambda(z)\varphi(z)}{1-z^m B_\Lambda(z)\varphi(z)}, \quad z \in \mathbb{D}, \ \varphi \in \mathcal{B}. \tag{2.3}$$

Let $\mathcal{P}^0(m, \Lambda)$ be the class of functions of the form (2.2), where $\omega \in \mathcal{B}^0(m, \Lambda)$.

When $B_\Lambda \equiv 1$, we will write $\mathcal{P}(m)$ and $\mathcal{P}^0(m)$ instead of $\mathcal{P}(m, \Lambda)$ and $\mathcal{P}^0(m, \Lambda)$, respectively.

The classes $\mathcal{P}(m, \Lambda)$ were introduced in [8], where their basic properties have been proved also.

By using the classes $\mathcal{P}(m, \Lambda)$ we now define the corresponding classes of starlike functions. Let us recall that $f \in \mathcal{S}^*$ if and only if

$$zf'(z) = f(z)p(z), \quad z \in \mathbb{D}, \tag{2.4}$$

for some $p \in \mathcal{P}$ (see [10], [3, p. 41]).

Definition 2.3. Let $m \in \mathbb{N}$ and let $\Lambda \in \Lambda$ satisfy the Blaschke condition. By $\mathcal{S}^*(m, \Lambda)$ we denote the class of functions $f \in \mathcal{A}$ satisfying (2.4), where $p \in \mathcal{P}(m, \Lambda)$, i.e., such that

$$zf'(z) = f(z)\frac{1+z^m B_\Lambda(z)\varphi(z)}{1-z^m B_\Lambda(z)\varphi(z)}, \quad z \in \mathbb{D}, \ \varphi \in \mathcal{B}. \tag{2.5}$$

Let $\mathcal{S}^{*,0}(m, \Lambda)$ be the class of the functions satisfying (2.4), where $p \in \mathcal{P}^0(m, \Lambda)$.

When $\Lambda(\mathbb{N}) = \emptyset$, for short we will write $\mathcal{S}^*(m)$.

It is clear that $\mathcal{S}^*(m, \Lambda) \subset \mathcal{S}^*_m$.

Theorem 2.4. *Let $m \in \mathbb{N}$ and let $\Lambda \in \Lambda$ satisfy the Blaschke condition. A function f is in $\mathcal{S}^*(m, \Lambda)$ if and only if*

$$f(z) = z \exp\left(2\int_0^z \frac{\zeta^{m-1} B_\Lambda(\zeta)\varphi(\zeta)}{1-\zeta^m B_\Lambda(\zeta)\varphi(\zeta)} d\zeta\right), \quad z \in \mathbb{D}, \tag{2.6}$$

where $\varphi \in \mathcal{B}$.

Proof. By (2.5) we have

$$\frac{f'(z)}{f(z)} - \frac{1}{z} = \frac{p(z)-1}{z} = \frac{2z^{m-1}B_\Lambda(z)\varphi(z)}{1-z^m B_\Lambda(z)\varphi(z)}, \quad z \in \mathbb{D}.$$

Hence

$$\log \frac{f(z)}{z} = 2\int_0^z \frac{\zeta^{m-1} B_\Lambda(\zeta)\varphi(\zeta)}{1-\zeta^m B_\Lambda(\zeta)\varphi(\zeta)} d\zeta, \quad z \in \mathbb{D}, \ \log 1 := 0,$$

which yields (2.6).

Conversely, since every f of the form (2.6) satisfies the condition (2.5), so it belongs to $\mathcal{S}^*(m, \Lambda)$. □

3. Growth and distortion theorems

Given $r \in (0, 1)$ and $f \in \mathcal{H}$, let

$$M_r(f) := \max_{z \in \mathbb{T}_r} |f(z)|.$$

Particularly, let $M_r(\Lambda) := M_r(B_\Lambda)$.

Lemma 3.1. *Let $m \in \mathbb{N}$ and let $\Lambda \in \Lambda$ satisfy the Blaschke condition. If $p \in \mathcal{P}(m,\Lambda)$, then for $z \in \mathbb{T}_r$, $r \in (0,1)$,*

$$|p(z)| \leq \frac{1 + r^m M_r(\Lambda)}{1 - r^m M_r(\Lambda)}, \tag{3.1}$$

and

$$\operatorname{Re} p(z) \geq \frac{1 - r^m M_r}{1 + r^m M_r}. \tag{3.2}$$

Proof. Since $p \in \mathcal{P}(m,\Lambda)$ is the form (2.3) with $|\varphi(z)| \leq 1,\ z \in \mathbb{D}$, from (2.3) we have

$$\begin{aligned}|p(z)| &= \left|\frac{1 + z^m B_\Lambda(z)\varphi(z)}{1 - z^m B_\Lambda(z)\varphi(z)}\right| \\ &\leq \frac{1 + |z^m B_\Lambda(z)\varphi(z)|}{1 - |z^m B_\Lambda(z)\varphi(z)|} \leq \frac{1 + r^m M_r(\Lambda)}{1 - r^m M_r(\Lambda)}, \quad z \in \mathbb{T}_r,\end{aligned}$$

which shows the inequality (3.1). Moreover

$$\begin{aligned}\operatorname{Re} p(z) =& \operatorname{Re} \frac{1 + z^m B_\Lambda(z)\varphi(z)}{1 - z^m B_\Lambda(z)\varphi(z)} \\ =& \frac{1 - |z^m B_\Lambda(z)\varphi(z)|^2}{|1 - z^m B_\Lambda(z)\varphi(z)|^2} \geq \frac{1 - |z^m B_\Lambda(z)\varphi(z)|}{1 + |z^m B_\Lambda(z)\varphi(z)|} \\ \geq& \frac{1 - r^m M_r(\Lambda)}{1 + r^m M_r(\Lambda)}, \quad z \in \mathbb{T}_r,\end{aligned}$$

which confirms the inequality (3.2) □

Directly from the above lemma we have the following theorem.

Theorem 3.2. *Let $m \in \mathbb{N}$ and let $\Lambda \in \Lambda$ satisfy the Blaschke condition. If $f \in \mathcal{S}^*(m,\Lambda)$, then for $z \in \mathbb{T}_r$, $r \in (0,1)$,*

$$\left|\frac{zf'(z)}{f(z)}\right| \leq \frac{1 + r^m M_r(\Lambda)}{1 - r^m M_r(\Lambda)},$$

and

$$\operatorname{Re} \frac{zf'(z)}{f(z)} \geq \frac{1 - r^m M_r(\Lambda)}{1 + r^m M_r(\Lambda)}. \tag{3.3}$$

The growth theorem for the class $\mathcal{S}^*(m,\Lambda)$ is as follows.

Theorem 3.3. *Let $m \in \mathbb{N}$ and let $\Lambda \in \Lambda$ satisfy the Blaschke condition. If $f \in \mathcal{S}^*(m,\Lambda)$, then for $z \in \mathbb{T}_r$, $r \in (0,1)$,*

$$|f(z)| \leq r \exp\left(2\int_0^r \frac{t^{m-1} M_t(\Lambda)}{1 - t^m M_t(\Lambda)} dt\right), \tag{3.4}$$

and

$$|f(z)| \geq r \exp\left(-2\int_0^r \frac{t^{m-1} M_t(\Lambda)}{1 + t^m M_t(\Lambda)} dt\right). \tag{3.5}$$

Proof. From (2.6) with $z := re^{i\theta}, \theta \in \mathbb{R}$, we have

$$\begin{aligned}|f(z)| =&|z| \left|\exp\left(2\int_0^z \frac{\zeta^{m-1}B_\Lambda(\zeta)\varphi(\zeta)}{1-\zeta^m B_\Lambda(\zeta)\varphi(t)}d\zeta\right)\right|\\ \leq& r\exp\left(2\left|\int_0^r \frac{t^{m-1}e^{im\theta}B_\Lambda(te^{i\theta})\varphi(te^{i\theta})}{1-t^m e^{im\theta}B_\Lambda(te^{i\theta})\varphi(te^{i\theta})}dt\right|\right)\\ \leq& r\exp\left(2\int_0^r \frac{t^{m-1}|B_\Lambda(te^{i\theta})||\varphi(te^{i\theta})|}{|1-t^m e^{im\theta}B_\Lambda(te^{i\theta})\varphi(te^{i\theta})|}dt\right)\\ \leq& r\exp\left(2\int_0^r \frac{2t^{m-1}M_t(\Lambda)}{1-t^m M_t(\Lambda)}dt\right),\end{aligned}$$

which shows the inequality (3.4).

Let $\zeta := te^{i\theta},\ t \in (0, r]$. Using (3.3) we have

$$\begin{aligned}&t\frac{\partial}{\partial t}\log\left|\frac{f(te^{i\theta})}{te^{i\theta}}\right| = t\frac{\partial}{\partial t}\operatorname{Re}\left\{\log\left(\frac{f(te^{i\theta})}{te^{i\theta}}\right)\right\}\\ &= \operatorname{Re}\left\{\frac{\zeta f'(\zeta)}{f(\zeta)} - 1\right\} \geq \frac{1-t^m M_t(\Lambda)}{1+t^m M_t(\Lambda)} - 1\\ &= \frac{-2t^m M_t}{1+t^m M_t}.\end{aligned}$$

Hence

$$\log\left|\frac{f(\zeta)}{\zeta}\right| \geq -2\int_0^r \frac{t^{m-1}M_t(\Lambda)}{1+t^m M_t(\Lambda)}dt,$$

which yields

$$|f(\zeta)| \geq |\zeta|\exp\left(-2\int_0^r \frac{t^{m-1}M_t(\Lambda)}{1+t^m M_t(\Lambda)}dt\right).$$

Particularly, it holds for $\zeta := z$ which shows the inequality (3.5). □

The distortion theorem for the class $\mathcal{S}^*(m, \Lambda)$ is the following.

Theorem 3.4. *Let $m \in \mathbb{N}$ and let $\Lambda \in \Lambda$ satisfy the Blaschke condition. If $f \in \mathcal{S}^*(m, \Lambda)$, then for $z \in \mathbb{T}_r$, $r \in (0, 1)$,*

$$|f'(z)| \leq \frac{1+r^m M_r(\Lambda)}{1-r^m M_r(\Lambda)}\exp\left(2\int_0^r \frac{t^{m-1}M_t(\Lambda)}{1-t^m M_t(\Lambda)}dt\right) \tag{3.6}$$

and

$$|f'(z)| \geq \frac{1-r^m M_r(\Lambda)}{1+r^m M_r(\Lambda)}\exp\left(-2\int_0^r \frac{t^{m-1}M_t(\Lambda)}{1+t^m M_t(\Lambda)}dt\right) \tag{3.7}$$

Proof. Since

$$f'(z) = \frac{f(z)}{z}p(z), \quad z \in \mathbb{D},$$

both inequalities below (3.6) and (3.7) follow from (3.1) with (3.4), and from (3.2) with (3.5), respectively. □

Below we present some statements which are particular cases of Theorems 3.3 and 3.4.

Theorem 3.5. *Let $m \in \mathbb{N}$. If $f \in \mathcal{S}^*(m)$, then for $z \in \mathbb{T}_r$, $r \in (0, 1)$,*

$$|f(z)| \leq \frac{r}{(1-r^m)^{\frac{2}{m}}}, \tag{3.8}$$

and

$$|f(z)| \geq \frac{r}{(1+r^m)^{\frac{2}{m}}}. \tag{3.9}$$

Both inequalities are sharp with the extremal functions

$$f(z) = \frac{z}{(1-z^m)^{\frac{2}{m}}}, \quad z \in \mathbb{D}, \tag{3.10}$$

and

$$f(z) = \frac{z}{(1+z^m)^{\frac{2}{m}}}, \quad z \in \mathbb{D}, \tag{3.11}$$

for (3.8) *and* (3.9), *respectively/*

Proof. From (3.4) and (3.5) with $M_t(\Lambda) = 1$ for $t \in [0, r]$, for $z \in \mathbb{T}_r$, $r \in (0, 1)$, we respectively have

$$\begin{aligned} |f(z)| &\leq r \exp\left(2\int_0^r \frac{t^{m-1}}{1-t^m} dt\right) \\ &= r \exp\left(-\frac{2}{m}\log(1-r^m)\right) = \frac{r}{(1-r^m)^{\frac{2}{m}}} \end{aligned}$$

i.e., the inequality (3.8), and

$$\begin{aligned} |f(z)| &\geq r \exp\left(-2\int_0^r \frac{t^{m-1}M_t}{1+t^m M_t} dt\right) \\ &= r \exp\left(-\frac{2}{m}\log(1+r^m)\right) = \frac{r}{(1+r^m)^{\frac{2}{m}}}, \end{aligned}$$

i.e., the inequality (3.9).

Equalities in (3.8) and (3.9) hold, respectively, for the functions (3.10) and (3.11) at $z := r$. □

Theorem 3.6. *Let* $m \in \mathbb{N}$. *If* $f \in \mathcal{S}^*(m)$, *then for* $z \in \mathbb{T}_r$, $r \in (0, 1)$,

$$|f'(z)| \leq \frac{1+r^m}{(1-r^m)^{\frac{2}{m}+1}} \tag{3.12}$$

and

$$|f'(z)| \geq \frac{1-r^m}{(1+r^m)^{\frac{2}{m}+1}}. \tag{3.13}$$

Both inequalities are sharp with the extremal functions (3.10) *and* (3.11), *respectively for* (3.12) *and* (3.13).

Proof. From (3.6) and (3.7) with $M_t(\Lambda) = 1$ for $t \in [0, r]$, for $z \in \mathbb{T}_r$, $r \in (0, 1)$, we respectively have

$$\begin{aligned} |f'(z)| &\leq \frac{1+r^m}{1-r^m} \exp\left(2\int_0^r \frac{t^{m-1}}{1-t^m} dt\right) \\ &= \frac{1+r^m}{1-r^m} \exp\left(-\frac{2}{m}\log(1-r^m)\right) = \frac{1+r^m}{(1-r^m)^{\frac{2+m}{m}}} \end{aligned}$$

i.e., the inequality (3.12), and

$$\begin{aligned} |f'(z)| &\geq \frac{1-r^m}{1+r^m} \exp\left(-2\int_0^r \frac{t^{m-1}}{1+t^m} dt\right) \\ &= \frac{1-r^m}{1+r^m} \exp\left(\frac{-2}{m}\log(1+r^m)\right) = \frac{1-r^m}{(1+r^m)^{\frac{2+m}{m}}}. \end{aligned}$$

i.e., the inequality (3.13).

Sharpness of (3.12) and (3.13) is clear. □

When $\Lambda = (\alpha)$, where $\alpha \in \mathbb{D}^0$, then

$$M_t(\Lambda) = \frac{t+|\alpha|}{1+|\alpha|t}, \quad t \in (0,1).$$

Then Theorems 3.3 and 3.4 reduce respectively to

Theorem 3.7. *Let $m \in \mathbb{N}$ and $\alpha \in \mathbb{D}^0$. If $f \in \mathcal{S}^*(m;(\alpha))$, then for $z \in \mathbb{T}_r$, $r \in (0,1)$,*

$$|f(z)| \le r \exp\left(2\int_0^r \frac{t^{m-1}(t+|\alpha|)}{-t^m(t+|\alpha|)+|\alpha|t+1}dt\right) \tag{3.14}$$

and

$$|f(z)| \ge r \exp\left(-2\int_0^r \frac{t^{m-1}(t+|\alpha|)}{t^m(t+|\alpha|)+|\alpha|t+1}dt\right). \tag{3.15}$$

Theorem 3.8. *Let $m \in \mathbb{N}$ and $\alpha \in \mathbb{D}^0$. If $f \in \mathcal{S}^*(m;(\alpha))$, then for $z \in \mathbb{T}_r$, $r \in (0,1)$,*

$$|f'(z)| \le \frac{r^m(r+|\alpha|)+|\alpha|r+1}{-r^m(r+|\alpha|)+|\alpha|r+1} \exp\left(2\int_0^r \frac{t^{m-1}(t+|\alpha|)}{-t^m(t+|\alpha|)+|\alpha|t+1}dt\right) \tag{3.16}$$

and

$$|f'(z)| \ge \frac{-r^m(r+|\alpha|)+|\alpha|r+1}{r^m(r+|\alpha|)+|\alpha|r+1} \exp\left(-2\int_0^r \frac{t^{m-1}(t+|\alpha|)}{t^m(t+|\alpha|)+|\alpha|t+1}dt\right). \tag{3.17}$$

In particular, when $m = 1$ we have the following results.

Theorem 3.9. *Let $\alpha \in \mathbb{D}^0$. If $f \in \mathcal{S}^*(1,(\alpha))$, then for $z \in \mathbb{T}_r$, $r \in (0,1)$,*

$$|f(z)| \le \frac{r}{(1-r)^{1+|\alpha|}(1+r)^{1-|\alpha|}} \tag{3.18}$$

and

$$|f(z)| \ge \frac{r}{1+2|\alpha|r+r^2}. \tag{3.19}$$

Proof. From (3.14) and (3.15) for $z \in \mathbb{T}_r$, $r \in (0,1)$, we respectively have

$$\begin{aligned}
|f(z)| &\le \frac{1}{r}\exp\left(-2\int_0^r \frac{|\alpha|t+1}{t\,(t^2-1)}dt\right)\\
&= \frac{1}{r}\exp\left(2\log r - (1+|\alpha|)\log(1-r) + (|\alpha|-1)\log(1+r)\right)\\
&= \frac{r\,(1+r)^{|\alpha|-1}}{(1-r)^{|\alpha|+1}}
\end{aligned}$$

i.e., the inequality (3.18), and

$$\begin{aligned}
|f(z)| &\ge \frac{1}{r}\exp\left(2\int_0^r \frac{|\alpha|t+1}{t\,(t^2+2|\alpha|t+1)}dt\right)\\
&= \frac{1}{r}\exp\left(2\log r - \log(r^2+2|\alpha|r+1)\right)\\
&= \frac{r}{(1-r)^{1+|\alpha|}(1+r)^{1-|\alpha|}}.
\end{aligned}$$

i.e., the inequality (3.19). □

Theorem 3.10. *Let $\alpha \in \mathbb{D}^0$. If $f \in \mathcal{S}^*(1, (\alpha))$, then for $z \in \mathbb{T}_r$, $r \in (0,1)$,*

$$|f'(z)| \leq \frac{1 + 2|\alpha|r + r^2}{(1-r)^{2+|\alpha|}(1+r)^{2-|\alpha|}} \tag{3.20}$$

and

$$|f'(z)| \geq \frac{1 - r^2}{(r^2 + 2|\alpha|r + 1)^2}. \tag{3.21}$$

Proof. From (3.16) and (3.17) for $z \in \mathbb{T}_r$, $r \in (0,1)$, we respectively have

$$\begin{aligned}
|f'(z)| &\leq \frac{r^2 + 2|\alpha|r + 1}{r^2(1 - r^2} \exp\left(-2\int_0^r \frac{|\alpha|t + 1}{t\,(t^2 - 1)} dt\right) \\
&= \frac{r^2 + 2|\alpha|r + 1}{r^2(1 - r^2} \exp\left(2\ln r - (1 + |\alpha|)\ln(1 - r) + (|\alpha| - 1)\ln(1 + r)\right) \\
&= \frac{1 + 2|\alpha|r + r^2}{(1-r)^{2+|\alpha|}(1+r)^{2-|\alpha|}}
\end{aligned}$$

i.e., the inequality (3.20), and

$$\begin{aligned}
|f'(z)| &\geq \frac{1 - r^2}{r^2(r^2 + 2|\alpha|r + 1)} \exp\left(2\int_0^r \frac{|\alpha|t + 1}{t\,(t^{m+1} + |\alpha|t^m + |\alpha|t + 1)} dt\right) \\
&= \frac{1 - r^2}{r^2(r^2 + 2|\alpha|r + 1)} \exp\left(2\ln r - \ln(r^2 + 2|\alpha|r + 1)\right) \\
&= \frac{1 - r^2}{(1 + 2|\alpha|r + r^2)^2}.
\end{aligned}$$

i.e., the inequality (3.21). □

4. Coefficients functionals

In this section we discuss some basic coefficients problems for the class $\mathcal{S}^*(m, (\alpha))$, where $m \in \mathbb{N}$ and $\alpha \in \mathbb{D}^0$. Let $f \in \mathcal{S}^*(m, (\alpha))$. Then

$$zf'(z)\left(1 - z^m\varphi(z)\varphi_\alpha(z)\right) = f(z)\left(1 + z^m\varphi(z)\varphi_\alpha(z)\right), \quad z \in \mathbb{D}, \tag{4.1}$$

for some $\varphi \in \mathcal{B}$, i.e., equivalently

$$zf'(z)\left(1 - \overline{\alpha}z - z^m\varphi(z)(z - \alpha)\right) = f(z)\left(1 - \overline{\alpha}z + z^m\varphi(z)(z - \alpha)\right), \quad z \in \mathbb{D}.$$

Substituting into the above equation the series (1.1) and the series

$$\varphi(z) = \sum_{n=0}^{\infty} b_n z^n, \quad z \in \mathbb{D}, \tag{4.2}$$

by comparing the corresponding coefficients we get

$$ma_{m+1} = -2\alpha b_0. \tag{4.3}$$

and when $m > 1$,

$$(m+1)a_{m+2} = -2\alpha b_1 + 2(1 - |\alpha|^2)b_0. \tag{4.4}$$

Moreover, when $m = 1$, then for $n \geq 2$,

$$\begin{aligned}
&(n-1)a_n + \left[n\alpha b_0 - (n-2)\overline{\alpha}\right] a_{n-1} + (n-1)(\alpha b_1 - b_0)a_{n-2} \\
&+ (n-2)(\alpha b_2 - b_1)a_{n-3} + \cdots + 4(\alpha b_{n-4} - b_{n-5})a_3 + 3(\alpha b_{n-3} - b_{n-4})a_2 \\
&= -2\alpha b_{n-2} + 2b_{n-3}.
\end{aligned}$$

Thus particularly,

$$a_3 = -\alpha b_1 + 3\alpha^2 b_0^2 + \left(1 - |\alpha|^2\right) b_0, \tag{4.5}$$

and

$$\begin{aligned} a_4 = & -\frac{2}{3}\alpha b_2 + \frac{2}{3}(1-|\alpha|^2)b_1 + \frac{2}{3}\overline{\alpha}(1-|\alpha|^2)b_0 \\ & + \frac{10}{3}\alpha^2 b_0 b_1 - 4\alpha^3 b_0^3 - \frac{10}{3}\alpha(1-|\alpha|^2)b_0^2. \end{aligned} \tag{4.6}$$

Since $|b_0| = |\varphi(0)| \le 1$ (e.g., [4, Vol. I, pp. 84-85]), from (4.3) we get

Theorem 4.1. *Let $m \in \mathbb{N}$ and $\alpha \in \mathbb{D}^0$. If $f \in \mathcal{S}^*(m,(\alpha))$ is of the form* (1.1)*, then*

$$|a_{m+1}| \le \frac{2}{m}|\alpha|. \tag{4.7}$$

The result is sharp. The equality in (4.7) *holds for the function f given by* (4.1) *with $\varphi \equiv -\mathrm{e}^{-\mathrm{i}\theta}$, where $\theta := \operatorname{Arg}\alpha \in [0, 2\pi)$.*

Given $m \in \mathbb{N}$ and $\lambda \in \mathbb{R}$, consider the functional

$$\Phi_{m,\lambda}(f) := |a_{m+2} - \lambda a_{m+1}^2|$$

over the class $\mathcal{S}^*$ of functions f of the form (1.1). Particularly, the functional $\Phi_\lambda := \Phi_{1,\lambda}$ plays a fundamental role in many extremal coefficients problem. Keogh and Merkes [6, Theorem 1] proved that for the whole class $\mathcal{S}^*$ the following result holds:

$$\left|a_3 - \lambda a_2^2\right| \le \begin{cases} |3-4\lambda|, & \lambda \in (-\infty, 1/2] \cup [1, +\infty), \\ 1, & \lambda \in [1/2, 1]. \end{cases} \tag{4.8}$$

We compute first the upper bound of Φ_λ in the class $\mathcal{S}^*(1,(\alpha))$. It should be expected that the result is more detailed then the estimates in (4.8) and so is.

Theorem 4.2. *Let $\alpha \in \mathbb{D}^0$ and $f \in \mathcal{S}^*(1,(\alpha))$ be of the form* (1.1)*. If $|\alpha| \in (0, \sqrt{2}-1]$, then*

$$\left|a_3 - \lambda a_2^2\right| \le |\alpha|^2 (|3-4\lambda| - 1) + 1, \quad \lambda \in \mathbb{R}. \tag{4.9}$$

If $|\alpha| \in (\sqrt{2}-1, 1)$, then

$$\left|a_3 - \lambda a_2^2\right| \tag{4.10}$$

$$\le \begin{cases} |\alpha|^2(|3-4\lambda|-1)+1, & \lambda \le \dfrac{5|\alpha|^2 - 2|\alpha| + 1}{8|\alpha|^2} \vee \lambda \ge \dfrac{7|\alpha|^2 + 2|\alpha| - 1}{8|\alpha|^2}, \\ \dfrac{(|\alpha|^2+1)^2 - 4|3-4\lambda||\alpha|^3}{4|\alpha|(1-|\alpha||3-4\lambda|)}, & \lambda \in \left(\dfrac{5|\alpha|^2 - 2|\alpha| + 1}{8|\alpha|^2}, \dfrac{7|\alpha|^2 + 2|\alpha| - 1}{8|\alpha|^2}\right). \end{cases}$$

Particulary, for $|\alpha| \in (0,1)$,

$$|a_3| \le 2|\alpha|^2 + 1, \tag{4.11}$$

$$\left|a_3 - \frac{1}{2}a_2^2\right| \le 1, \tag{4.12}$$

$$\left|a_3 - \frac{3}{4}a_2^2\right| \le \begin{cases} 1 - |\alpha|^2, & |\alpha| \in (0, \sqrt{2}-1], \\ \dfrac{(|\alpha|^2+1)^2}{4|\alpha|} & |\alpha| \in (\sqrt{2}-1, 1), \end{cases} \tag{4.13}$$

and

$$\left|a_3 - a_2^2\right| \le 1. \tag{4.14}$$

The result is sharp. Let $\alpha := |\alpha|\mathrm{e}^{\mathrm{i}\theta}$, $\theta \in [0, 2\pi)$. Equality in (4.9) *and in the first inequality in* (4.10) *holds for the function f given by* (4.1) *with $\varphi \equiv \pm\mathrm{e}^{-2\mathrm{i}\theta}$. Equality in the second inequality in* (4.10) *holds for the function f given by* (4.1) *with*

$$\varphi(z) := \pm\mathrm{e}^{-2\mathrm{i}\theta}\frac{\mathrm{e}^{-\mathrm{i}\theta}z - x_0}{1 - \mathrm{e}^{-\mathrm{i}\theta}x_0 z}, \quad z \in \mathbb{D}, \tag{4.15}$$

where

$$x_0 := \frac{1-|\alpha|^2}{2|\alpha|(1-|\alpha||3-4\lambda|)}. \tag{4.16}$$

Proof. Since

$$|b_1| \le 1 - |b_0|^2, \tag{4.17}$$

(see e.g., [4, Vol. II, p. 78]), from (4.3) for $m = 1$ and (4.5) we have

$$\begin{aligned} \left|a_3 - \lambda a_2^2\right| &= \left|-\alpha b_1 + (3-4\lambda)\alpha^2 b_0^2 + \left(1-|\alpha|^2\right) b_0\right| \\ &\le |\alpha| \left(1 - |b_0|^2\right) + \left(1-|\alpha|^2\right)|b_0| + |\alpha|^2 |3-4\lambda| \, |b_0|^2 \\ &= |\alpha| \left(|\alpha| \, |3-4\lambda| - 1\right) |b_0|^2 + \left(1-|\alpha|^2\right)|b_0| + |\alpha| = \gamma(|b_0|), \end{aligned} \tag{4.18}$$

where

$$\gamma(x) := |\alpha| \left(|\alpha| \, |3-4\lambda| - 1\right) x^2 + \left(1-|\alpha|^2\right) x + |\alpha| , \quad x \in [0,1].$$

(a) For $|\alpha| \, |3-4\lambda| - 1 \ge 0$, i.e., for

$$\lambda \in \left(-\infty, \frac{3\,|\alpha|-1}{4\,|\alpha|}\right] \cup \left[\frac{3\,|\alpha|+1}{4\,|\alpha|}, +\infty\right),$$

we have $\gamma'(x) \ge 0,\ x \in [0,1]$, and hence

$$\gamma(x) \le \gamma(1) = |\alpha|^2 \left(|3-4\lambda| - 1\right) + 1, \quad x \in [0,1]. \tag{4.19}$$

(b) Let now $|\alpha| \, |3-4\lambda| - 1 < 0$, i.e., let

$$\lambda \in \left(\frac{3\,|\alpha|-1}{4\,|\alpha|}, \frac{3\,|\alpha|+1}{4\,|\alpha|}\right). \tag{4.20}$$

Note that $\gamma'(x) = 0$ only for $x := x_0$, where x_0 is given by (4.16). Thus $x_0 \ge 1$ if and only if

$$|3-4\lambda| \ge \frac{|\alpha|^2 + 2|\alpha| - 1}{2|\alpha|^2},$$

which in view of (4.20) holds: when $|\alpha| \in (0, \sqrt{2}-1]$ for λ as in (4.20), and when $|\alpha| \in (\sqrt{2}-1, 1)$ for

$$\lambda \in \left(\frac{3\,|\alpha|-1}{4\,|\alpha|}, \frac{5\,|\alpha|^2 - 2\,|\alpha| + 1}{8\,|\alpha|^2}\right] \cup \left[\frac{7\,|\alpha|^2 + 2\,|\alpha| - 1}{8\,|\alpha|^2}, \frac{3\,|\alpha|+1}{4\,|\alpha|}\right). \tag{4.21}$$

Hence and by the case (a) it follows that $\gamma'(x) \ge 0$ for $x \in [0,1]$, so the inequality (4.19) holds: when $|\alpha| \in (0, \sqrt{2}-1]$ for all $\lambda \in \mathbb{R}$, and when $|\alpha| \in (\sqrt{2}-1, 1)$ for λ as in (4.21). This and (4.18) prove that the inequality (4.9) and the first inequality in (4.10) are true.

The second inequality in (4.10) is a consequence of the inequality

$$\gamma(x) \le \gamma(x_0) = \frac{|\alpha|^4 - 4\,|3-4\lambda| \, |\alpha|^3 + 2\,|\alpha|^2 + 1}{4\,|\alpha| - 4\,|\alpha|^2 \, |3-4\lambda|}$$

which holds for

$$\lambda \in \left(\frac{5\,|\alpha|^2 - 2\,|\alpha| + 1}{8\,|\alpha|^2}, \frac{7\,|\alpha|^2 + 2\,|\alpha| - 1}{8\,|\alpha|^2}\right).$$

The inequalities (4.11)-(4.14) are particular cases of the inequalities (4.9) and (4.10) for $\lambda = 0$, $\lambda = 1/2$, $\lambda = 3/4$ and $\lambda = 1$, respectively.

It remains to discuss the sharpness. Let $\alpha := |\alpha| e^{i\theta},\ \theta \in [0, 2\pi)$. It is easy to verify that the equality in (4.9) and in the first inequality in (4.10) holds for the function f given by (4.1) either with $\varphi \equiv e^{-2i\theta}$ or with $\varphi \equiv -e^{-2i\theta}$. Let $\lambda \le 3/4$. For

$$\varphi(z) = -e^{-2i\theta} \frac{e^{-i\theta} z - x_0}{1 - e^{-i\theta} x_0 z} = e^{-2i\theta} x_0 - (1 - x_0^2) e^{-3i\theta} z + \ldots, \quad z \in \mathbb{D},$$

we have

$$\begin{aligned}\left|a_3-\lambda a_2^2\right| &= \left|-\alpha b_1+\left(1-|\alpha|^2\right)b_0+\alpha^2(3-4\lambda)b_0^2\right|\\ &= \left||\alpha|(1-x_0^2)\mathrm{e}^{-2\mathrm{i}\theta}+\left(1-|\alpha|^2\right)x_0\mathrm{e}^{-2\mathrm{i}\theta}+(3-4\lambda)|\alpha|^2x_0^2\mathrm{e}^{-2\mathrm{i}\theta}\right|\\ &= |\alpha|(1-x_0^2)+\left(1-|\alpha|^2\right)x_0+(3-4\lambda)|\alpha|^2x_0^2,\end{aligned}$$

which yields equality in the second inequality in (4.10). The case $\lambda > 3/4$ follows in a similar way. □

Remark 4.3. One can be checked that the upper bounds in (4.9) and (4.10) do not exceed of the upper bounds in (4.8). Setting $|\alpha| = 1$ the inequalities (4.9) and (4.10) reduce to the inequality (4.8).

We consider now the case $m > 1$.

Theorem 4.4. *Let $\alpha \in \mathbb{D}^0$, $m > 1$ and $f \in \mathcal{S}^*(m,(\alpha))$ be of the form* (1.1)*. If $|\alpha| \in (0, \sqrt{2}-1]$, then*

$$\left|a_{m+2}-\lambda a_{m+1}^2\right| \le \left(\frac{4|\lambda|}{m^2}-\frac{2}{m+1}\right)|\alpha|^2+\frac{2}{m+1}, \quad \lambda\in\mathbb{R}. \tag{4.22}$$

If $|\alpha| \in (\sqrt{2}-1, 1)$, then

$$\left|a_{m+2}-\lambda a_{m+1}^2\right| \tag{4.23}$$

$$\le \begin{cases} \left(\dfrac{4|\lambda|}{m^2}-\dfrac{2}{m+1}\right)|\alpha|^2+\dfrac{2}{m+1}, & |\lambda| \ge \dfrac{m^2(|\alpha|^2+2|\alpha|-1)}{4|\alpha|^2(m+1)},\\ \dfrac{m^2(|\alpha|^2+1)^2-8(m+1)|\lambda||\alpha|^3}{2(m+1)|\alpha|(m^2-2(m+1)|\alpha||\lambda|)}, & |\lambda| < \dfrac{m^2(|\alpha|^2+2|\alpha|-1)}{4|\alpha|^2(m+1)}. \end{cases}$$

Particularly,

$$|a_{m+2}| \le \begin{cases} \dfrac{2}{m+1}(1-|\alpha|^2), & |\alpha| \in (0,\sqrt{2}-1],\\ \dfrac{(|\alpha|^2+1)^2}{2(m+1)|\alpha|}, & |\alpha| \in (\sqrt{2}-1,1), \end{cases} \tag{4.24}$$

and for $|\alpha| \in (0,1)$,

$$\left|a_{m+2}-\frac{m^2}{2(m+1)}a_{m+1}^2\right| \le \frac{2}{m+1}. \tag{4.25}$$

The result is sharp. Equality in (4.22) *and in the first inequality in* (4.23) *holds for the function f given by* (4.1) *with $\varphi \equiv \pm\mathrm{e}^{-2\mathrm{i}\theta}$, where $\theta := \operatorname{Arg}\alpha \in [0,2\pi)$. Equality in the second inequality in* (4.23) *holds for the function f given by* (4.1) *with φ given by* (4.15) *where*

$$x_0 = \frac{(1-|\alpha|^2)m^2}{2|\alpha|\left(m^2-2(m+1)|\alpha||\lambda|\right)}. \tag{4.26}$$

Proof. From (4.3), (4.4) and (4.17) we have

$$\begin{aligned}\left|a_{m+2}-\lambda a_{m+1}^2\right| &= \left|-\frac{2}{m+1}\alpha b_1-\frac{4\lambda}{m^2}\alpha^2b_0^2+\frac{2}{m+1}\left(1-|\alpha|^2\right)b_0\right|\\ &\le \frac{2}{m+1}|\alpha|\left(1-|b_0|^2\right)+\frac{2}{m+1}\left(1-|\alpha|^2\right)|b_0|+\frac{4|\lambda|}{m^2}|\alpha|^2|b_0|^2\\ &= 2|\alpha|\left(\frac{2|\lambda|}{m^2}|\alpha|-\frac{1}{m+1}\right)|b_0|^2+\frac{2}{m+1}\left(1-|\alpha|^2\right)|b_0|+\frac{2}{m+1}|\alpha|\\ &=: \gamma(|b_0|),\end{aligned} \tag{4.27}$$

where for $x \in [0,1]$,

$$\gamma(x) := 2\,|\alpha| \left(\frac{2|\lambda|}{m^2}\,|\alpha| - \frac{1}{m+1}\right) x^2 + \frac{2}{m+1}\left(1-|\alpha|^2\right) x + \frac{2}{m+1}\,|\alpha|\,.$$

(a) For $2|\lambda||\alpha|/m^2 - 1/(m+1) \geq 0$, i.e., for

$$\lambda \in \left(-\infty, \frac{-m^2}{2\,|\alpha|\,(m+1)}\right] \cup \left[\frac{m^2}{2\,|\alpha|\,(m+1)}, +\infty\right),$$

we have $\gamma'(x) \geq 0$ for $x \in [0,1]$, and consequently

$$\gamma(x) \leq \gamma(1) = \frac{4|\lambda|}{m^2}\,|\alpha|^2 + \frac{2}{m+1}\left(1-|\alpha|^2\right), \quad x \in [0,1]. \tag{4.28}$$

(b) Let now $2|\lambda||\alpha|/m^2 - 1/(m+1) < 0$, i.e., let

$$\lambda \in \left(\frac{-m^2}{2\,|\alpha|\,(m+1)}, \frac{m^2}{2\,|\alpha|\,(m+1)}\right). \tag{4.29}$$

Note that $\gamma'(x) = 0$ only for $x := x_0$, where x_0 is given by (4.26). Thus $x_0 \geq 1$ if and only if

$$|\lambda| > \frac{m^2\left(|\alpha|^2 + 2|\alpha| - 1\right)}{4(m+1)|\alpha|^2},$$

which in view of (4.29) holds: when $|\alpha| \in (0, \sqrt{2}-1]$ for λ as in (4.29), and when $|\alpha| \in (\sqrt{2}-1, 1)$ for

$$\frac{m^2(|\alpha|^2 + 2\,|\alpha| - 1)}{4\,|\alpha|^2\,(m+1)} \leq |\lambda| < \frac{m^2}{2\,|\alpha|\,(m+1)}. \tag{4.30}$$

Hence and by the case (a) it follows that $\gamma'(x) \geq 0$ for $x \in [0,1]$, so the inequality (4.28), holds when $|\alpha| \in (0, \sqrt{2}-1]$ for all $\lambda \in \mathbb{R}$, and when $|\alpha| \in (\sqrt{2}-1, 1)$ for λ as in (4.30). This and (4.27) prove the inequality (4.22) and the first inequality in (4.23).

The second inequality in (4.23) is a consequence of the inequality

$$\gamma(x) \leq \gamma(x_0) = \frac{m^2(|\alpha|^2+1)^2 - 8\,(m+1)\,|\lambda|\,|\alpha|^3}{2\,(m+1)\,|\alpha|\,(m^2 - 2\,(m+1)\,|\alpha|\,|\lambda|)}$$

which holds for

$$|\lambda| < \frac{m^2(|\alpha|^2 + 2\,|\alpha| - 1)}{4\,|\alpha|^2\,(m+1)}.$$

The inequalities (4.24) and (4.25) are particular cases of the inequalities (4.22) and (4.23) for $\lambda = 0$, and $\lambda = m^2/2(m+1)$, respectively.

The sharpness follows analogously as in Theorem 4.2. □

The sharp bounds for the second and third coefficient in the class $\mathcal{S}^*(1,(\alpha))$ for all $\alpha \in \mathbb{D}^0$ were given in (4.7) and (4.11), respectively. Now we will deal with the fourth coefficient.

Theorem 4.5. *Let $\alpha \in \mathbb{D}^0$ and $f \in \mathcal{S}^*(1,(\alpha))$ be of the form* (1.1)*. If $|\alpha| \in (0, \alpha_0]$, where $\alpha_0 \approx 0.27248$ is the unique zero of the polynomial $(0,1) \ni t \mapsto 7t^3 - 8t^2 + 9t - 2$, then*

$$\begin{aligned} |a_4| \leq & \frac{2}{81|\alpha|^4(6|\alpha|-5)^2}\Big[-520|\alpha|^9 + 807|\alpha|^8 + 2208|\alpha|^7 \\ & - 3079|\alpha|^6 - 297|\alpha|^5 + 1014|\alpha|^4 + 5|\alpha|^3 - 90|\alpha|^2 + 24|\alpha| - 2 \\ & + \sqrt{\psi(|\alpha|)}\Big(86|\alpha|^6 - 230|\alpha|^5 + 36|\alpha|^4 + 66|\alpha|^3 \\ & + 28|\alpha|^2 - 16|\alpha| + 2\Big)\Big], \end{aligned} \tag{4.31}$$

where

$$\psi(|\alpha|) := 43|\alpha|^6 - 115|\alpha|^5 + 18|\alpha|^4 + 33|\alpha|^3 + 14|\alpha|^2 - 8|\alpha| + 1. \tag{4.32}$$

If $|\alpha| \in (\alpha_0, 1)$, *then*

$$|a_4| \le 4|\alpha|. \tag{4.33}$$

Proof. Let $t := |\alpha| \in (0,1)$. Since (see e.g., [4, Vol. II, p. 78]) for $k \in \mathbb{N}$,

$$|b_k| \le 1 - |b_0|^2, \tag{4.34}$$

from (4.6) we have

$$\begin{aligned}|a_4| \le &\frac{2}{3}\Big[t|b_2| + (1-t^2)|b_1| + t(1-t^2)|b_0| + 5t^2|b_0||b_1| \\ &+6t^3|b_0|^3 + 5t(1-t^2)|b_0|^2\Big] \\ \le &\frac{2}{3}\Big[t(1-|b_0|^2) + (1-t^2)(1-|b_0|^2) + t(1-t^2)|b_0| \\ &+5t^2|b_0|(1-|b_0|^2) + 6t^3|b_0|^3 + 5t(1-t^2)|b_0|^2\Big] \\ \le &\frac{2}{3}\Big[t^2(6t-5)|b_0|^3 + (-5t^3+t^2+4t-1)|b_0|^2 \\ &+(-t^3+5t^2+t)|b_0| - t^2 + t + 1\Big] =: \frac{2}{3}\gamma_t(|b_0|),\end{aligned} \tag{4.35}$$

where

$$\begin{aligned}\gamma_t(x) =&t^2(6t-5)x^3 + (-5t^3+t^2+4t-1)x^2 \\ &+ (-t^3+5t^2+t)x - t^2 + t + 1, \quad x \in [0,1].\end{aligned}$$

For $t = 5/6$ we have

$$\gamma_{5/6}(x) = \frac{1}{216}\left(29x^2 + 805x + 246\right) \le 5, \quad x \in [0,1]. \tag{4.36}$$

Let $t \in I := (0,1) \setminus \{5/6\}$. Note that

$$\gamma_t(0) = -t^2 + t + 1 > 0, \quad \gamma_t(1) = 6t > 0, \quad t \in (0,1),$$

and for $x \in (0,1)$,

$$\gamma_t'(x) = 3t^2(6t-5)x^2 + 2(-5t^3+t^2+4t-1)x + t(-t^2+5t+1).$$

We have

$$\Delta = 4\psi(t) := 4\left(43t^6 - 115t^5 + 18t^4 + 33t^3 + 14t^2 - 8t + 1\right), \quad t \in I.$$

Since $\Delta = 0$ only for $t = t_0 \approx 0.833709$, so $\gamma_{t_0}'(x) = 3t_0^2(6t_0-5)(x-t_0)^2 \ge 0$ for $x \in [0,1]$. Thus the function γ_{t_0} is increasing and therefore

$$\gamma_{t_0}(x) \le \gamma_{t_0}(1) = 6t_0, \quad x \in [0,1]. \tag{4.37}$$

For $t \in (t_0, 1)$ we have $\Delta < 0$ and since $t_0 > 5/6$, it follows that $\gamma_t' > 0$. Thus γ_t is an increasing functions and therefore for $t \in (t_0, 1)$,

$$\gamma_t(x) \le \gamma_t(1) = 6t, \quad x \in [0,1]. \tag{4.38}$$

Consider the case $\Delta = 4\psi(t) > 0$, i.e., the case $t \in (0, t_0) \setminus \{5/6\}$. Then γ_t' extended to the real axis has two zeros

$$x_{1,2} = \frac{5t^3 - t^2 - 4t + 1 \mp \sqrt{\psi(t)}}{3t^2(6t-5)}.$$

Note first that

$$5t^3 - t^2 - 4t + 1 > 0, \quad t \in (0, t_1), \tag{4.39}$$

and

$$5t^3 - t^2 - 4t + 1 < 0, \quad t \in (t_1, t_0), \tag{4.40}$$

where $t_1 \approx 0.25440$ is the unique zero of the polynomial $(0, t_0) \ni t \mapsto 5t^3 - t^2 - 4t + 1$.

From (4.39) and (4.40) it follows that $x_1 < 0$ for $t \in (5/6, t_0)$, and $x_1 > 0$ for $t \in [t_1, 5/6)$. For $t \in (0, t_1)$ the condition $x_1 > 0$ is equivalent to the inequality

$$(5t^3 - t^2 - 4t + 1)^2 < 43t^6 - 115t^5 + 18t^4 + 33t^3 + 14t^2 - 8t + 1,$$

which is equivalent to the true inequality

$$3t^3(6t^3 - 35t^2 + 19t + 5) > 0. \tag{4.41}$$

Thus $x_1 > 0$ for $t \in (0, 5/6)$. Observe now that for $t \in (0, 5/6)$ the condition $x_1 < 1$ is equivalent to

$$-13t^3 + 14t^2 - 4t + 1 > \sqrt{\psi(t)}, \tag{4.42}$$

which is obviously false for $t \in [t_2, 5/6)$, where $t_2 \approx 0.81524$ is the unique real zero of the polynomial $t \mapsto -13t^3 + 14t^2 - 4t + 1$. For $t \in (0, t_2)$ the inequality (4.42) is equivalent to

$$3t^2(6t - 5)(7t^3 - 8t^2 + 9t - 2) > 0$$

which holds for $t \in (0, t_3)$, where $t_3 \approx 0.27248$ is the unique real zero of the polynomial $t \mapsto 7t^3 - 8t^2 + 9t - 2$. Summarizing, $x_1 \in (0, 1)$ if and only if $t \in (0, t_3)$.

Now we show that $x_2 < 0$ for $t \in (0, t_0) \setminus \{5/6\}$. Indeed, for $t \in (5/6, t_0)$ the condition $x_2 < 0$ in view of (4.39) is equivalent to

$$5t^3 - t^2 - 4t + 1 < -\sqrt{\psi(t)},$$

which is equivalent to the true inequality

$$3t^3(6t^3 - 35t^2 + 19t + 5) < 0.$$

For $t \in (0, t_1]$ the condition $x_2 < 0$ is true by (4.39). For $t \in (t_1, 5/6)$ the condition $x_2 < 0$ is equivalent to the true inequality (4.41).

Since for $t \in (t_3, 5/6)$,

$$\gamma_t'(0) = t(-t^2 + 5t + 1) > 0, \quad \gamma_t'(1) = 7t^3 - 8t^2 + 9t - 2 > 0,$$

we conclude that $\gamma_t' > 0$, so γ_t is increasing for $t \in (t_3, 5/6)$. Therefore for $t \in (t_3, 5/6)$ the inequality (4.38) holds. For $t \in (0, t_3]$, the maximal value of the function γ_t equals $\gamma_t(x_2)$. Hence and from (4.35)-(4.38) the inequalities (4.31) and (4.33) follow. □

Remark 4.6. In this paper we deal with the classes $\mathcal{S}^*(m, \Lambda)$. The classes $\mathcal{S}^{*,0}(m, \Lambda)$ have been defined also however they have not been examined. Although the results for the class $\mathcal{S}^*(m, \Lambda)$ are valid for the class $\mathcal{S}^{*,0}(m, \Lambda)$, the detailed study of the class $\mathcal{S}^{*,0}(m, \Lambda)$ seems to be more sophisticated based on knowledge on the class $\mathcal{B}^0$ of bounded non-vanishing analytic functions. Let us recall the famous Krzyż's conjecture [7] for the class $\mathcal{B}^0$. Namely, he supposed that

$$|b_n| \le \frac{2}{\mathrm{e}}, \quad n \in \mathbb{N},$$

for $\varphi \in \mathcal{B}^0$ of the form (4.2) with equality only for the function

$$\varphi_n(z) := \exp\left(\frac{z^n - 1}{z^n + 1}\right) = \frac{1}{\mathrm{e}} + \frac{2}{\mathrm{e}}z + \dots, z \in \mathbb{D},$$

and its rotations (for further details see e.g., [9], [11]).

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