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Research Article

Weighted composition operators between Besov-type spaces

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Abstract

In this paper, we study the boundedness and the compactness of weighted composition operators between Besov-type spaces. Also, we give a Carleson measure characterization of weighted composition operators on Besov spaces.

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1. Introduction

Let $\mathbb{D}$ be the open unit disc in the complex plane $\mathbb{C}$. Denote by $H(\mathbb{D})$ the class of all complex-valued functions analytic on $\mathbb{D}$. Suppose φ and ψ are holomorphic functions defined on $\mathbb{D}$ such that $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. The *weighted composition operator* $W_{\varphi,\psi}$ induced by φ and ψ on $H(\mathbb{D})$ is defined by

$$W_{\varphi,\psi}f(z) = \psi(z)f(\varphi(z)) = \psi(z)C_\varphi(f),$$

for all $f \in H(\mathbb{D})$ and $z \in \mathbb{D}$. When $\psi(z) \equiv 1$, the *composition operator* $W_{\varphi,1}$ is denoted by C_φ, i.e.,

$$W_{\varphi,1}f(z) = f(\varphi(z)) = C_\varphi(f),$$

for all $f \in H(\mathbb{D})$ and $z \in \mathbb{D}$. For the study of composition operators one can refer to [7] and [11].

Fix any $a \in \mathbb{D}$ and let $\sigma_a(z)$ be the Mobius transform defined by

$$\sigma_a(z) = \frac{a-z}{1-\overline{a}z}, z \in \mathbb{D}.$$

We denote the set of all Mobius transformations on $\mathbb{D}$ by G. The inverse of σ_a under composition is again σ_a for $a \in \mathbb{D}$. Further, we have

$$|\sigma_a'(z)| = \frac{1-|a|^2}{|1-\overline{a}z|^2} \tag{1.1}$$

and

$$1-|\sigma_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\overline{a}z|^2} = (1-|z|^2)|\sigma_a'(z)|, \tag{1.2}$$

for every $a, z \in \mathbb{D}$.

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For $1 \le p < \infty$, $L^p(\mathbb{D}, dA)$ will denote the Banach space of Lebesgue measurable functions f on $\mathbb{D}$ with

$$\|f\|_p = \left(\int_{\mathbb{D}} |f(z)|^p dA(z)\right)^{\frac{1}{p}} < +\infty,$$

where $dA(z)$ denote the Lebesgue area measure on $\mathbb{D}$.
For $p = +\infty$, $L^\infty(\mathbb{D}, dA)$ will denote the Banach space of Lebesgue measurable functions f on $\mathbb{D}$ with

$$\|f\|_\infty = ess \sup\{|f(z)| : z \in \mathbb{D}\} < +\infty.$$

For $1 \le p < \infty$, the Bergman space A^p, is defined to be the subspace of $L^p(\mathbb{D}, dA)$ consisting of analytic functions, i.e. $A^p(\mathbb{D}) = L^p(\mathbb{D}) \cap H^\infty(\mathbb{D})$. The Bergman spaces are Banach spaces.

For $1 \leqslant p < +\infty$ and $-1 < r < +\infty$, the (weighted) Bergman space $A^p_r = A^p_r(\mathbb{D})$ of the disc is the space of analytic functions in $L^p(\mathbb{D}, dA_r)$, where

$$dA_r(z) = (r+1)(1-|z|^2)^r dA(z).$$

If f is in A^p_r, we write

$$\|f\|_{A^p_r} = \left(\int_{\mathbb{D}} |f(z)|^p dA_r(z)\right)^{\frac{1}{p}}.$$

When $1 \leqslant p < +\infty$, the space A^p_r is a Banach space with the above norm.

For $1 < p < +\infty$ and $-1 < r < \infty$, an analytic function f on $\mathbb{D}$ is said to belong to the *Besov-type space* $B_{p,r}$ if

$$\|f\|_{B_{p,r}} = \left(\int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^r dA(z)\right)^{\frac{1}{p}} < \infty, \tag{1.3}$$

where $dA(z)$ denote the Lebegue area measure on $\mathbb{D}$. Also, if we take $1 < p < \infty$ and $r = p-2$ in (1.3), then we get analytic Besov space, simply denoted by B_p. We can see that $|f(0)| + \|f\|_{B_{p,r}}$ is a norm on $B_{p,r}$, that makes it a Banach space. Moreover, we can observe that, for f to be in $B_{p,r}$, it is necessary that the derivative of f belong to the weighted Bergman spaces A^p_r.

Definition 1.1. Let μ be a positive measure on $\mathbb{D}$. Then the space $\mathbb{D}_p(\mu)$ is defined as the space of all holomorphic functions $f \in H(\mathbb{D})$ for which $f' \in L^p(\mathbb{D}, \mu)$. Also, the norm on $\mathbb{D}_p(\mu)$ is defined as

$$\|f\|^p_{\mathbb{D}_p(\mu)} = \int_{\mathbb{D}} |f'(z)|^p d\mu(z).$$

Take $0 < p < \infty$. A positive measure μ on $\mathbb{D}$ is called a *p-Carleson measure* in $\mathbb{D}$ if

$$\sup_{I \subseteq \partial\mathbb{D}} \frac{\mu(S(I))}{|I|^p} < \infty, \tag{1.4}$$

where $|I|$ denotes the arc length of I and $S(I)$ denotes the Carleson square based on I,

$$S(I) = \{z \in \mathbb{D} : 1 - |I| \le |z| < 1, \frac{z}{|z|} \in I\}.$$

Again, μ is called a *vanishing p-Carleson measure* if

$$\lim_{|I| \to 0} \frac{\mu(S(I))}{|I|^p} = 0. \tag{1.5}$$

Take $h \in (0,1)$ and $\theta \in [0, 2\pi)$. If we set

$$S(h, \theta) = \{z \in \mathbb{D} : |z - e^{i\theta}| < h\},$$

then we can see that (1.4) and (1.5) are equivalent to

$$\sup_{h \in (0,1), \theta \in [0,2\pi)} \frac{\mu(S(h,\theta))}{h^p} < \infty \tag{1.6}$$

and

$$\lim_{h\to 0}\sup_{\theta\in[0,2\pi)}\frac{\mu(S(h,\theta))}{h^p}=0, \tag{1.7}$$

respectively.

Suppose φ is a holomorphic mapping defined on $\mathbb{D}$. Let $\varphi(\mathbb{D})\subseteq\mathbb{D}$ and $\psi\in B_{q,r}$ be such that $\psi(z)\varphi'(z)\in L^q(\mathbb{D},dA_r)$. We define the measures $\mu_{q,r}$ and $\nu_{q,r}$ on $\mathbb{D}$ by

$$\mu_{q,r}(E)=\int_{\varphi^{-1}(E)}|\psi(z)\varphi'(z)|^q(1-|z|^2)^r dA(z) \tag{1.8}$$

and

$$\nu_{q,r}(E)=\int_{\varphi^{-1}(E)}|\psi'(z)|^q(1-|z|^2)^r dA(z), \tag{1.9}$$

where E is a measurable subset of the unit disc $\mathbb{D}$.

If $\psi\in A^q_r$, then we can define the measure $\nu_{q,\psi,r}$ on $\mathbb{D}$ by

$$\nu_{q,\psi,r}(E)=\int_{\varphi^{-1}(E)}|\psi(z)|^q(1-|z|^2)^r dA(z). \tag{1.10}$$

Definition 1.2. Take $1<p<+\infty$ and $-1<r<\infty$. Let μ be a positive measure on $\mathbb{D}$. Then the measure μ is (p,r)-Carleson measure for $B_{p,r}$ if there is a constant $K>0$ such that

$$\int_{\mathbb{D}}|f'(w)|^p d\mu(w)\leqslant K\|f\|^p_{B_{p,r}},$$

for all $f\in B_{p,r}$. That is, the inclusion operator $i: B_{p,r}\to\mathbb{D}_p(\mu)$ is bounded. Further, the measure μ is a vanishing p-Carleson measure for $B_{p,r}$ if the inclusion operator $i: B_{p,r}\to\mathbb{D}_p(\mu)$ is compact.

The following characterization of $(p,r)-$Carleson measures can be obtained easily from [1].

Theorem 1.3. *Take $1<p<\infty$ and $-1<r<\infty$. Let μ be a positive measure on $\mathbb{D}$. Then the following statements are equivalent:*

(1) *The measure μ is a (p, r)-Carleson measure for $B_{p,r}$.*
(2) *There exists a constant $K<\infty$ such that*

$$\mu(S(h,\theta))\leq Kh^p$$

for all $\theta\in[0,2\pi)$ and $h\in(0,1)$.

(3) *There exists a constant $C<\infty$ such that*

$$\int_{\mathbb{D}}|\sigma'_a(z)|^p d\mu(z)\leq C$$

for all $a\in\mathbb{D}$.

Using ([6], Lemma 2.1) and ([8], page 163), the following lemma can be proved easily.

Lemma 1.4. *Let φ be a holomorphic mapping defined on $\mathbb{D}$ such that $\varphi(\mathbb{D})\subseteq\mathbb{D}$. Take $\psi\in B_{q,r}$ such that $\psi(z)\varphi'(z)\in L^q(\mathbb{D},dA_r)$. Then*

$$\int_{\mathbb{D}}g\,d\mu_{q,r}=\int_{\mathbb{D}}|\psi(z)\varphi'(z)|^q(g\circ\varphi)(z)(1-|z|^2)^r dA(z) \tag{1.11}$$

and

$$\int_{\mathbb{D}}g\,d\nu_{q,r}=\int_{\mathbb{D}}|\psi'(z)|^q(g\circ\varphi)(z)(1-|z|^2)^r dA(z). \tag{1.12}$$

where g is an arbitrary measurable positive function in $\mathbb{D}$.

We use the following lemma for compactness of the weighted composition operators on Besov-type spaces. The proof of this lemma follows by similar lines as in the case of composition operators on Besov spaces ([12], Lemma 3.8).

Lemma 1.5. *Given $1 \leqslant p, q < \infty$, $-1 < r < \infty$, let φ be a holomorphic mapping defined on $\mathbb{D}$ with $\varphi(\mathbb{D}) \subseteq \mathbb{D}$ and $\psi \in B_{q,r}$ be such that $W_{\varphi,\psi} : B_{p,r} \to B_{q,r}$ is bounded. Then $W_{\varphi,\psi} : B_{p,r} \to B_{q,r}$ is compact (weakly compact) if and only if whenever $\{f_n\}$ is a bounded sequence in $B_{p,r}$ converging to zero uniformly on compact subsets of $\mathbb{D}$, then $\|W_{\varphi,\psi}(f_n)\|_{B_{q,r}} \to 0$ (respectivly, $\{W_{\varphi,\psi}(f_n)\}$ is a weak null sequence in $B_{q,r}$).*

Boundededness and compactness of the weighted composition operators on spaces of analytic functions has been studied by many authors. For example we refer to [2–5, 9, 10, 13].

In this article, by using the Carleson measure, we characterize the boundeness and compactness of $W_{\varphi,\psi}$ on Besov-type spaces, in section 2. The Carleson measure characterization of $W_{\varphi,\psi}$ acing on Besov spaces is given in section 3.

2. Bounded and compact weighted composition operators on Besov-type spaces

In this section, we characterize the boundeness and compactness of $W_{\varphi,\psi}$ on Besov-type spaces by using Carleson measures.

Theorem 2.1. *Take $1 < p \leqslant q < \infty$ and $-1 < r < \infty$. Let $\varphi \in B_{p,r}$ be such that $\varphi(\mathbb{D}) \subseteq \mathbb{D}$, $\psi \in A^q_r$ and the measure $\nu_{q,\psi,r}$ is a vanishing $(q,\ r)$-Carleson measure for $B_{q,r}$. Then $W_{\varphi,\psi}$ defines a bounded operator from $B_{p,r}$ into A^q_r. Moreover, $W_{\varphi,\psi} : B_{p,r} \to A^q_r$ is compact.*

Proof. We prove the compactness only. Let $\{f_n\}$ be a bounded sequence in $B_{p,r}$ such that $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. Since the measure $\nu_{q,\psi,r}$ is a vanishing $(q,r)-$Carleson measure for $B_{q,r}$, the inclusion map $i : B_{q,r} \to L^q(\mathbb{D}, \nu_{q,\psi,r})$ is compact.
Since $B_{p,r} \subset B_{q,r}$, we have $\|f_n\|_{L^q(\mathbb{D},\nu_{q,\psi,r})} \to 0$ as $n \to \infty$. Therefore, by Lemma 1.4, we have

$$\begin{aligned} \|W_{\varphi,\psi}(f_n)\|^q_{A^q_r} &= \int_{\mathbb{D}} |\psi(z)|^q |(f_n \circ \varphi)(z)|^q (1-|z|^2)^r dA(z) \\ &= \int_{\mathbb{D}} |f_n(z)|^q d\nu_{q,\psi,r}(z) \to 0, \quad \text{as} \quad n \to \infty. \end{aligned} \tag{2.1}$$

Thus, $W_{\varphi,\psi} : B_{p,r} \to A^q_r$ is compact. □

Theorem 2.2. *Take $1 < p \leqslant q < \infty$ and $-1 < r < \infty$. Let $\varphi, \psi \in B_{p,r}$ be such that $\varphi(\mathbb{D}) \subseteq \mathbb{D}$ and the measure $\nu_{q,r}$ is a vanishing $(q,\ r)$-Carleson measure for $B_{q,r}$. Then $W_{\varphi,\psi}$ is a bounded operator from $B_{p,r}$ into $B_{q,r}$ if and only if $W_{\varphi,\psi\varphi'}$ is a bounded operator from A^p_r into A^q_r.*

Proof. Suppose that $W_{\varphi,\psi} : B_{p,r} \to B_{q,r}$ is bounded. Then there exists a constant $C > 0$ such that

$$\|W_{\varphi,\psi}(g)\|_{B_{q,r}} \leqslant C\|g\|_{B_{p,r}}$$

for all $g \in B_{p,r}$. Also, by Theorem 2.1, we can find a constant $M > 0$ such that

$$\|W_{\varphi,\psi'}(g)\|_{A^q_r} \leqslant M\|g\|_{B_{p,r}}, \qquad g \in B_{p,r}.$$

Take $f \in A_r^p$ and let the function $g \in B_{p,r}$ be such that $g' = f$ and $g(0) = 0$. Then,

$$\begin{aligned}
\|W_{\varphi,\psi\varphi'}(f)\|_{A_r^q} &= \|\psi\varphi' f \circ \varphi\|_{A_r^q} \\
&= \|\psi\varphi' f \circ \varphi + \psi' g \circ \varphi - \psi' g \circ \varphi\|_{A_r^q} \\
&\leqslant \|(\psi g \circ \varphi)'\|_{A_r^q} + \|\psi' g \circ \varphi\|_{A_r^q} \\
&= \|\psi g \circ \varphi\|_{B_{q,r}} + \|\psi' g \circ \varphi\|_{A_r^q} \\
&\leqslant C\|g\|_{B_{p,r}} + M\|g\|_{B_{p,r}} \\
&= (C+M)\|g\|_{B_{p,r}} \\
&= (C+M)\|f\|_{A_r^p}.
\end{aligned}$$

Thus, $W_{\varphi,\psi\varphi'} : A_r^p \to A_r^q$ is bounded.
Conversely, suppose $W_{\varphi,\psi\varphi'} : A_r^p \to A_r^q$ is bounded. Again, by Theorem 2.1,

$$W_{\varphi,\psi'} : B_{p,r} \to A_r^q$$

is bounded. Take $f \in B_{p,r}$ such that $f(0) = 0$. Then, we have

$$\begin{aligned}
\|W_{\varphi,\psi}(f)\|_{B_{q,r}} &= \|(\psi f \circ \varphi)'\|_{A_r^q} \\
&= \|\psi\varphi' f' \circ \varphi + \psi' f \circ \varphi\|_{A_r^q} \\
&\leqslant \|W_{\varphi,\psi\varphi'}(f')\|_{A_r^q} + \|W_{\varphi,\psi'}(f)\|_{A_r^q} < +\infty.
\end{aligned}$$

The theorem is proved. □

Theorem 2.3. *Take $1 < p \leqslant q < \infty$ and $-1 < r < \infty$. Let $\varphi, \psi \in B_{p,r}$ be such that $\varphi(\mathbb{D}) \subseteq \mathbb{D}$ and the measure $\nu_{q,r}$ is a vanishing (q, r)-Carleson measure for $B_{q,r}$. Then $W_{\varphi,\psi}$ is a compact operator from $B_{p,r}$ into $B_{q,r}$ if and only if $W_{\varphi,\psi\varphi'}$ is a compact operator from A_r^p into A_r^q.*

Proof. Suppose that $W_{\varphi,\psi} : B_{p,r} \to B_{q,r}$ is compact. Let $\{f_n\}$ be a bounded sequence in A_r^p such that $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. For each n, there exists a function $g_n \in B_{p,r}$ such that $g_n' = f_n$ and $g_n(0) = 0$. The sequence $\{g_n\}$ also converges to zero uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$. Further, since $W_{\varphi,\psi} : B_{p,r} \to B_{q,r}$ is compact, so $\|W_{\varphi,\psi}(g_n)\|_{B_{q,r}} \to 0$ as $n \to \infty$. Again, by Theorem 2.1, $W_{\varphi,\psi'} : B_{p,r} \to A_r^q$ is compact, so $\|W_{\varphi,\psi'}(g_n)\|_{A_r^q}$ also converges to zero as $n \to \infty$. We have

$$\begin{aligned}
\|W_{\varphi,\psi\varphi'}(f_n)\|_{A_r^q} &= \|\psi\varphi' f_n \circ \varphi\|_{A_r^q} \\
&\leqslant \|\psi\varphi' f_n \circ \varphi + \psi' g_n \circ \varphi\|_{A_r^q} + \|\psi' g_n \circ \varphi\|_{A_r^q} \\
&= \|(\psi g_n \circ \varphi)'\|_{A_r^q} + \|W_{\varphi,\psi'}(g_n)\|_{A_r^q} \\
&= \|W_{\varphi,\psi}(g_n)\|_{B_{q,r}} + \|W_{\varphi,\psi'}(g_n)\|_{A_r^q} \to 0, \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Therefore, $W_{\varphi,\psi\varphi'} : A_r^p \to A_r^q$ is compact.
Conversely, suppose $W_{\varphi,\psi\varphi'} : A_r^p \to A_r^q$ is compact. Again, by Theorem 2.1, $W_{\varphi,\psi'} : B_{p,r} \to A_r^q$ is compact. Let $\{g_n\}$ be the same sequence as in the direct part. Then,

$$\begin{aligned}
\|W_{\varphi,\psi}(g_n)\|_{B_{q,r}} &= \|(\psi g_n \circ \varphi)'\|_{A_r^q} \\
&= \|\psi\varphi' g_n' \circ \varphi + \psi' g_n \circ \varphi\|_{A_r^q} \\
&\quad \|W_{\varphi,\psi\varphi'}(f_n)\|_{A_r^q} + \|W_{\varphi,\psi'}(g_n)\|_{A_r^q} \to 0, \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Thus, $W_{\varphi,\psi} : B_{p,r} \to B_{q,r}$ is compact. □

Theorem 2.4. *Take $1 < p < \infty$ and $-1 < r < \infty$. Let $\varphi, \psi \in B_{p,r}$ be such that $\varphi(\mathbb{D}) \subseteq \mathbb{D}$ and the measure $\nu_{p,r}$ is a vanishing $(p, r)-$Carleson measure for $B_{p,r}$. Then $W_{\varphi,\psi}$ is a bounded (compact) operator from $B_{p,r}$ into $B_{p,r}$ if and only if $\mu_{p,r}$ is a bounded (vanishing) $(p, r)-$Carleson measure for $B_{p,r}$.*

Proof. We only prove the boundedness. Suppose first that $W_{\varphi,\psi} : B_{p,r} \to B_{p,r}$ is bounded. Then by Theorem 2.2, $W_{\varphi,\psi\varphi'}$ is a bounded operator on A_r^p. Let $f \in B_{p,r}$ be such that

$f(0) = 0$. Then, by using Lemma 1.4, we have

$$\begin{aligned} \|W_{\varphi,\psi\varphi'}(f')\|^p_{A^p_r} &= \int_{\mathbb{D}} |\psi(z)\varphi'(z)|^p |f'(\varphi(z))|^p (1-|z|^2)^r dA(z) \\ &= \int_{\mathbb{D}} |f'(w)|^p d\mu_{p,r}(w). \end{aligned}$$

Since $W_{\varphi,\psi\varphi'}$ is bounded on A^p_r, therefore we can find a constant $C > 0$ such that

$$\|W_{\varphi,\psi\varphi'}(f')\|^p_{A^p_r} \leqslant C\|f'\|^p_{A^p_r}.$$

Hence,

$$\int_{\mathbb{D}} |f'(w)|^p d\mu_{p,r}(w) \leqslant C\|f\|^p_{B_{p,r}}.$$

That is, the inclusion operator $i : B_{p,r} \to \mathbb{D}_{p,r}(\mu)$ is bounded. Thus the measure $\mu_{p,r}$ is a bounded $(p,r)-$Carleson measure for $B_{p,r}$.

Conversely, suppose that $\mu_{p,r}$ is a bounded $(p,r)-$Carleson measure for $B_{p,r}$. We want to show that $W_{\varphi,\psi} : B_{p,r} \to B_{p,r}$ is bounded. We have

$$(\psi(f\circ\varphi)') = \psi\varphi'(f'\circ\varphi) + \psi'(f\circ\varphi). \tag{2.2}$$

Take $f \in B_{p,r}$. So by Lemma 1.4,

$$\int_{\mathbb{D}} |\psi(z)\varphi'(z)|^p |f'(\varphi(z))|^p (1-|z|^2)^r dA(z) = \int_{\mathbb{D}} |f'(w)|^p d\mu_{p,r}(w) < +\infty. \tag{2.3}$$

Also, by using Theorem 2.1, we get

$$\int_{\mathbb{D}} |\psi'(z)|^p |f(\varphi(z))|^p (1-|z|^2)^r dA(z) = \|W_{\varphi,\psi'}(f)\|^p_{A^p_r} < +\infty. \tag{2.4}$$

By using (2.2), (2.3) and (2.4), $W_{\varphi,\psi} : B_{p,r} \to B_{p,r}$ is bounded.

Compactness of $W_{\varphi,\psi}$ can be proved by using the Theorems 2.1 and 2.3, which we omit its proof. □

3. Carleson measure characterization of the weighted composition operators on Besov spaces

In this section, we give a Carleson measure charecterization of $W_{\varphi,\psi}$ on Besov space.

Let $1 < p, q < \infty$, φ be a holomorphic mapping defined on $\mathbb{D}$ such that $\varphi(\mathbb{D}) \subseteq \mathbb{D}$ and $\psi \in B_q$ be such that $\psi(z)\varphi'(z)(1-|z|^2) \in L^q(\mathbb{D}, d\lambda)$ (where $d\lambda(z) = (1-|z|^2)^{-2}dA(z)$ is the Mobius invariant measure on $\mathbb{D}$). For $f \in B_p$ there exists a constant C_q such that

$$\begin{aligned} \|W_{\varphi,\psi}(f)\|^q_{B_q} &= \int_{\mathbb{D}} |(\psi C_\varphi f)'(z)|^q (1-|z|^2)^{q-2} dA(z) \\ &\leqslant C_q \int_{\mathbb{D}} |\psi'(z)|^q |(C_\varphi f)(z)|^q (1-|z|^2)^{q-2} dA(z) \\ &\quad + C_q \int_{\mathbb{D}} |\psi(z)|^q |\varphi'(z)|^q |f'(\varphi(z))|^q (1-|z|^2)^{q-2} dA(z). \end{aligned}$$

By using Lemma 1.4, we have

$$\|W_{\varphi,\psi}(f)\|^q_{B_q} \leqslant C_q \int_{\mathbb{D}} |f(w)|^q d\nu_q(w) + C_q \int_{\mathbb{D}} |f'(w)|^q d\mu_q(w). \tag{3.1}$$

Since $W_{\varphi,\psi} : B_p \to B_q$ is a bounded operator if and only if there exists a positive constant C such that

$$\|W_{\varphi,\psi}(f)\|^q_{B_q} \leq C\|f\|^q_{B_p},$$

so, the following theorem holds.

Theorem 3.1. *Let $1 < p < \infty$, φ be a holomorphic mapping defined on $\mathbb{D}$ such that $\varphi(\mathbb{D}) \subseteq \mathbb{D}$ and $\psi \in B_p$ be such that $\psi(z)\varphi'(z)(1-|z|^2) \in L^p(\mathbb{D}, d\lambda)$. If the measure μ_p is a p-Carleson measure and ν_p is a vanishing p-Carleson measure for B_p, then $W_{\varphi,\psi} : B_p \to B_p$ is a bounded operator.*

Proof. Suppose that μ_p is a p-Carleson measure for B_p. By Definition 1.2, there exisits a constant C_1 such that

$$\int_D |f'(w)|^p d\mu_p(w) \leq C_1 \|f\|^p_{B_p}. \tag{3.2}$$

Let ν_p is a vanishing p-Carleson measure for B_p. By using Theorem 2.1 for $r = p-2$, there exisits a constant C_2 such that

$$\begin{aligned} \|W_{\varphi,\psi'}(f)\|^p_{A^p} &= \int_D |\psi'(z)|^p |f\circ\varphi|^p (1-|z|^2)^{p-2} dAz \\ &= \int_D |f(w)|^p d\nu_p(w) \\ &\leq C_2 \|f\|^p_{B_p}. \end{aligned} \tag{3.3}$$

By using (3.2) and (3.3), from (3.1) the theorem is proved. □

Theorem 3.2. *Suppose $1 < p \leqslant q < \infty$ and φ is a holomorphic mapping defined on $\mathbb{D}$. Let $\varphi(\mathbb{D}) \subseteq \mathbb{D}$ and $\psi \in B_q$ be such that $\psi(z)\varphi'(z)(1-|z|^2) \in L^q(\mathbb{D}, d\lambda)$. If the measures ν_q and μ_q are vanishing q-Carleson measures for B_q, then $W_{\varphi,\psi} : B_p \to B_q$ is a compact operator.*

Proof. Let $\{f_n\}$ be a bounded sequence in B_p such that $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. Then the mean value property for the holomorphic function yields

$$f'_n(w) = \frac{4}{\pi(1-|w|)^2} \int_{|w-z|<\frac{1-|w|}{2}} f'_n(z) dA(z). \tag{3.4}$$

Therefore by Jensen's inequality,

$$|f'_n(w)|^q \leq \frac{4}{\pi(1-|w|)^2} \int_{|w-z|<\frac{1-|w|}{2}} |f'_n(z)|^q dA(z). \tag{3.5}$$

Since the measure ν_q is a vanishing q-Carleson measure for B_q, by using Theorem 2.1(relation (2.1)), we have

$$\int_D |f_n(w)|^q d\nu_q(w) \to 0, \quad \text{as} \quad n \to \infty. \tag{3.6}$$

By using (3.1), (3.5), (3.6) and Fubini's Theorem,

$$\begin{aligned} \|W_{\varphi,\psi}(f_n)\|^q_{B_q} &\leq C_q \int_D |f_n(w)|^q d\nu_q(w) + C_q \int_D |f'_n(w)|^q d\mu_q(w) \\ &\leq C_q \int_D \frac{4}{\pi(1-|w|)^2} \left(\int_{|w-z|<\frac{1-|w|}{2}} |f'_n(z)|^q dA(z) \right) d\mu_q(w) \\ &\leq C_q \frac{4}{\pi} \int_D |f'_n(z)|^q \left(\int_D \frac{1}{(1-|w|)^2} \chi_{\{z:|w-z|<\frac{1-|w|}{2}\}}(z) d\mu_q(w) \right) dA(z). \end{aligned}$$

Note that if $|w-z| < \dfrac{1-|w|}{2}$, then $w \in S(2(1-|z|), \theta)$, where $z = |z|e^{i\theta}$, since

$$|w - e^{i\theta}| \leq |z-w| + |e^{i\theta} - z| < \frac{1-|w|}{2} + |\frac{z}{|z|} - z| < 2(1-|z|).$$

Moreover, if $|w-z| < \dfrac{1-|w|}{2}$ then

$$\frac{1}{(1-|w|)^2} \leq const. \frac{1}{(1-|z|)^2}.$$

Hence,

$$\begin{aligned}\|W_{\varphi,\psi}(f_n)\|^q_{B_q} &\leq const. \int_D \frac{|f_n'(z)|^q}{(1-|z|)^2}\left(\int_{S(2(1-|z|),\theta)} d\mu_q(w)\right) dA(z)\\ &= const. \left(\int_{|z|>1-\frac{\delta}{2}} + \int_{|z|\leq 1-\frac{\delta}{2}} \frac{|f_n'(z)|^q}{(1-|z|)^2}\left(\int_{S(2(1-|z|),\theta)} d\mu_q(w)\right) dA(z)\right)\\ &= I + II \qquad (3.7)\end{aligned}$$

for any $0 < \delta < 1$.

Fix $\epsilon > 0$ and let $\delta > 0$ be such that for any $\theta \in [0, 2\pi]$ and any $h < \delta$,

$$\mu_q(S(h,\theta)) < \epsilon h^q, \qquad (3.8)$$

and so

$$\int_{S(h,\theta)} d\mu_q < \epsilon h^q.$$

By (3.8),

$$\begin{aligned} I &\leq const.\ 2^q \epsilon \int_{|z|>1-\frac{\delta}{2}} \frac{|f_n'(z)|^q}{(1-|z|^2)^2}(1-|z|^2)^q dA(z)\\ &\leq const.\ \epsilon \|f_n\|^q_{B_q} < const.\ \epsilon. \qquad (3.9)\end{aligned}$$

For n large enough, since $f_n' \to 0$ uniformly on compact sets, we have

$$II \leq const. \int_{|z|\leq 1-\frac{\delta}{2}} |f_n'(z)|^q \left(\int_D d\mu_q\right) dA(z) < const.\ \epsilon. \qquad (3.10)$$

Therefore, from (3.7), (3.9) and (3.10) we obtain

$$\|W_{\varphi,\psi}(f_n)\|^q_{B_q} < const.\epsilon.$$

Thus $\|W_{\varphi,\psi}(f_n)\|^q_{B_q} \to 0$ as $n \to \infty$, and from Lemma 1.5, $W_{\varphi,\psi}$ is compact. □

References

[1] J. Arazy, S.D. Fisher and J. Peetre, *Mobius invariant function spaces*, J. Reine Angew. Math. **363**, 110–145, 1985.

[2] D. Blasi and J. Pav, *A Characterization of Besov type spaces and applications to Hankel-type operators*, Michigan Math. J. **56**, 401–418, 2008.

[3] F. Colonna and S. Li, *Weighted Composition Operators from the Besov spaces into the Bloch space*, Bull. Malays. Math. Sci. Soc. **36**, 1027–1039, 2013.

[4] F. Colonna and M. Tjani, *Weighted Composition Operators from the Besov spaces into the weighted type space H^∞_μ*, J. Math. Anal. Appl. **402**, 594–611, 2013.

[5] F. Colonna and M. Tjani, *Weighted Composition Operators from the analytic Besov spaces to BMOA* , Oper. Theory Adv. Appl. **236**, 133–157, 2014.

[6] M.D. Contreras and A.G. Hernandez-Diaz, *Weighted composition operators on Hardy spaces*, J. Math. Anal. Appl. **263**, 224–233, 2001.

[7] C.C. Cowen and B.D. MacCluer, *Composition operators on spaces of analytic functions*, CRC Press, Boca Raton, New York, 1995.

[8] P.R. Halmos, *Measure Theory*, Grad. Texts in Math., 18, Springer-Verlag, New York, 1974.

[9] M. Hassanlou, J. Laitila and H. Vaezi, *Weighted composition operators between weak spaces of vector-valued analytic functions*, Analysis, **37**, 39–45, 2017.

[10] M. Hassanlou, H. Vaezi and M. Wang, *Weighted composition operators on weak vector-valued Bergman spaces and Hardy spaces*, Banach J. Math. Anal. **9** (2), 35–43, 2015.

[11] J.H. Shapiro, *Composition operators and classical function theory*, Berlin, Germany, Springer- Verlag, 1993.

[12] M. Tjani, *Compact composition operators on Besov space*, Trans. Amer. Math. Soc. **355** (11), 4683–4698, 2003.
[13] W. Yang, *Weighted composition operators from Bloch-type spaces to weighted-type spaces*, Ars. Combin. **92**, 415–423, 2009.