

# An Efficient Operational Matrix Method for Solving a Class of Two-Dimensional Singular Volterra Integral Equations

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## Abstract

In this paper, we consider a spectral method to solve a class of two-dimensional singular Volterra integral equations using some basic concepts of fractional calculus. This method uses a modification of hat functions for finding a numerical solution of the considered equation. Some properties of the modification of hat functions are presented. The main contribution of this work is to introduce the fractional order operational matrix of integration for the considered basis functions. Making use of the Riemann-Liouville fractional integral operator helps us to reduce the main problem to a system of linear algebraic equations which can be solved easily. After that, error analysis of the method is discussed. Finally, numerical examples are included to confirm the accuracy and applicability of the suggested method.

## 1. Introduction

Singular integral equations consist a class of integral equations in which the kernel is singular within the range of integration, or one or both limits of integration are infinite [1]. There are some analytical and numerical methods to solve one-dimensional singular integral equations with different kinds of singularity. Ioakimidis [2] has used quadrature methods for obtaining a numerical solution for singular integral equations with singular kernels. In [3], a numerical method has been proposed for the numerical solution of singular integral equations of the Cauchy type via replacing the integral equation by integral relations at a discrete set of points. Gauss-Chebyshev formulae has been used to find the numerical solution of singular integral equations of the Cauchy type in [4]. Monegato and Scuderi in [5] introduced high order methods for the second kind Fredholm integral equations with weakly singular kernels. For more methods on these equations, the interested reader can refer to [6]-[13].

The main aim of this paper is to introduce an application of fractional calculus in solving a class of two-dimensional Volterra integral equations as

$$u(x, t) = f(x, t) + \int_0^t \int_0^x (x-y)^{-\alpha} (t-z)^{-\beta} u(y, z) dy dz, \quad (x, t) \in D, \quad (1.1)$$

where  $u(x, t)$  is the unknown function on  $D := [0, l] \times [0, T]$ ,  $f(x, t)$  is a given known function and  $0 < \alpha, \beta < 1$ . This equation, is a singular integral equation with weakly singular convolution kernel. In recent decades fractional calculus provided a wonderful tool for the explanation of many mathematical models in science and engineering [14, 15]. A general outlook of fractional calculus and its basic theories can be found in [16]-[24].

In this work, we propose a numerical method to solve Eq. (1.1) using the modification of hat functions (MHFs). The MHFs have been employed to obtain numerical solutions of two-dimensional linear Fredholm integral equations [25], nonlinear Stratonovich Volterra integral equations [26], Volterra-Fredholm integral equations [27] and systems of linear Stratonovich Volterra integral equations [28]. The operational matrix technique is used to reduce the main problem to a system of algebraic equations. It should be noted that any other well-known basis functions that their operational matrix of fractional integration are known such as Legendre polynomials [29], Chebyshev polynomials [30], Haar wavelet functions [31] and hat functions [32] could be employed in our new approach to solve Eq. (1.1).

This paper is organized as follows: Some preliminaries in fractional calculus and properties of the MHFs are given in Section 2. Section 3 is committed to introducing the operational matrix of fractional integration of the MHFs. In Section 4, a numerical method is given to solve Eq.

(1.1). Error analysis of the method is discussed in Section 5. Numerical examples are given in Section 6 to demonstrate the accuracy and applicability of the method. Concluding remarks are presented in Section 7.

## 2. Basic concepts

In this section, we give some definitions which will be used further in this paper.

There are different definitions for fractional integral in literature (see [19]). Here, we consider the Riemann-Liouville fractional integral operator  $I_x^\alpha$  to reach our aim.

**Definition 2.1.** The Riemann-Liouville integral operator  $I_x^\alpha$  of order  $\alpha > 0$  is given by [17]

$$I_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-y)^{\alpha-1} f(y) dy,$$

where  $\Gamma(\alpha)$  is the gamma function; as

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt.$$

**Definition 2.2.** Let  $a = (a_1, a_2) \in (0, \infty) \times (0, \infty)$ ,  $\theta = (0, 0)$  and  $u \in L^1(D)$ . The left-sided mixed Riemann-Liouville integral of order  $a$  of  $u$  is defined by [33]

$$I_{\theta}^a u(x, t) = \frac{1}{\Gamma(a_1)\Gamma(a_2)} \int_0^t \int_0^x (x-y)^{a_1-1} (t-z)^{a_2-1} u(y, z) dy dz.$$

Hat functions are defined on the interval  $[0, 1]$  and are linear piecewise continuous functions with shape hats [32]. Here we consider the MHFs which are quadratic piecewise continuous functions with shape hats and replace the domain of the definition to  $[0, l]$ .

**Definition 2.3.** A set of  $(n + 1)$ -MHFs consists of  $n + 1$  functions which are defined on the interval  $[0, l]$  as follows [25]:

$$\psi_{0,l}(x) = \begin{cases} \frac{1}{2h^2} (x-h)(x-2h), & 0 \leq x \leq 2h, \\ 0, & \text{otherwise,} \end{cases}$$

when  $i$  is odd and  $1 \leq i \leq n - 1$ :

$$\psi_{i,l}(x) = \begin{cases} \frac{-1}{h^2} (x-(i-1)h)(x-(i+1)h), & (i-1)h \leq x \leq (i+1)h, \\ 0, & \text{otherwise,} \end{cases}$$

when  $i$  is even and  $2 \leq i \leq n - 2$ :

$$\psi_{i,l}(x) = \begin{cases} \frac{1}{2h^2} (x-(i-1)h)(x-(i-2)h), & (i-2)h \leq x \leq ih, \\ \frac{1}{2h^2} (x-(i+1)h)(x-(i+2)h), & ih \leq x \leq (i+2)h, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\psi_{n,l}(x) = \begin{cases} \frac{1}{2h^2} (x-(l-h))(x-(l-2h)), & l-2h \leq x \leq l, \\ 0, & \text{otherwise.} \end{cases}$$

where  $h = \frac{l}{n}$  and  $n \geq 2$  is an even integer number. These functions are linearly independent functions in  $L^2[0, l]$ .

Using Definition 2.3, the MHFs satisfy the following properties:

$$\psi_{i,l}(jh) = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases} \tag{2.1}$$

$$\sum_{i=0}^n \psi_{i,l}(x) = 1,$$

$$\psi_{i,l}(x)\psi_{j,l}(x) = \begin{cases} 0, & \text{if } i \text{ is even and } |i-j| \geq 3, \\ 0, & \text{if } i \text{ is odd and } |i-j| \geq 2. \end{cases}$$

An arbitrary function  $f(x) \in L^2[0, l]$  may be approximated in terms of the MHFs as

$$f(x) \simeq f_n(x) = \sum_{i=0}^n f_i \psi_{i,l}(x) = F^T \psi_l(x) = \psi_l^T(x) F,$$

where

$$\Psi_l(x) = [\Psi_{0,l}(x), \Psi_{1,l}(x), \dots, \Psi_{n,l}(x)]^T, \quad (2.2)$$

and

$$F = [f_0, f_1, \dots, f_n]^T,$$

in which  $f_i = f(ih)$ .

**Definition 2.4.** A  $(n+1) \times (m+1)$ -set of two-dimensional modification of hat functions (2DMHF) includes  $(n+1) \times (m+1)$  functions which are defined on  $D$  as

$$\phi_{i,j}(x,t) = \Psi_{i,l}(x) \Psi_{j,T}(t).$$

A function  $u(x,t)$  in  $L^2(D)$  can be approximated in terms of the 2DMHF as

$$u(x,t) \simeq \sum_{i=0}^n \sum_{j=0}^m u_{i,j} \phi_{i,j}(x,t) = U^T \phi(x,t),$$

where

$$\phi(x,t) = \Psi_l(x) \otimes \Psi_T(t), \quad (2.3)$$

in which  $\otimes$  denotes the Kronecker product and

$$U = [u_{0,0}, u_{0,1}, \dots, u_{0,m}, \dots, u_{n,0}, u_{n,1}, \dots, u_{n,m}]^T,$$

such that  $u_{i,j} = u(ih_1, jh_2)$  with  $h_1 = \frac{l}{n}$  and  $h_2 = \frac{T}{m}$ .

From (2.1), it is seen that

$$\phi_{i,j}(ph_1, qh_2) = \begin{cases} 1, & p=i \ \& \ q=j, \\ 0, & \text{otherwise.} \end{cases} \quad (2.4)$$

### 3. Operational matrix of fractional integration

In this section, the fractional order operational matrix of integration of the MHFs is introduced.

**Theorem 3.1.** Let  $\Psi(x)$  be the MHFs vector given by (2.2) and  $\alpha > 0$ , then

$$I_x^\alpha \Psi_l(x) \simeq P_l^{(\alpha)} \Psi_l(x),$$

where  $P_l^{(\alpha)}$  is the  $(n+1) \times (n+1)$  operational matrix of fractional integration of order  $\alpha$  in the Riemann-Liouville sense and is defined as follows

$$P_l^{(\alpha)} = \frac{h^\alpha}{2\Gamma(\alpha+3)} \begin{bmatrix} 0 & \beta_1 & \beta_2 & \beta_3 & \beta_4 & \dots & \beta_{n-1} & \beta_n \\ 0 & \eta_0 & \eta_1 & \eta_2 & \eta_3 & \dots & \eta_{n-2} & \eta_{n-1} \\ 0 & \xi_{-1} & \xi_0 & \xi_1 & \xi_2 & \dots & \xi_{n-3} & \xi_{n-2} \\ 0 & 0 & 0 & \eta_0 & \eta_1 & \dots & \eta_{n-4} & \eta_{n-3} \\ 0 & 0 & 0 & \xi_{-1} & \xi_0 & \dots & \xi_{n-5} & \xi_{n-4} \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & \eta_0 & \eta_1 \\ 0 & 0 & 0 & 0 & 0 & \dots & \xi_{-1} & \xi_0 \end{bmatrix},$$

where

$$\beta_1 = \alpha(3+2\alpha),$$

$$\beta_k = k^{\alpha+1}(2k-6-3\alpha) + 2k^\alpha(1+\alpha)(2+\alpha) - (k-2)^{\alpha+1}(2k-2+\alpha), \quad k=2,3,\dots,n,$$

$$\eta_0 = 4(1+\alpha),$$

$$\eta_k = 4[(k-1)^{\alpha+1}(k+1+\alpha) - (k+1)^{\alpha+1}(k-1-\alpha)], \quad k=1,2,\dots,n-1,$$

$$\xi_{-1} = -\alpha,$$

$$\xi_0 = 2^{\alpha+1}(2-\alpha),$$

$$\xi_1 = 3^{\alpha+1}(4-\alpha) - 6(2+\alpha),$$

$$\xi_k = (k+2)^{\alpha+1}(2k+2-\alpha) - 6k^{\alpha+1}(2+\alpha) - (k-2)^{\alpha+1}(2k-2+\alpha), \quad k=2,3,\dots,n-2.$$

**Proof.** For proof see [34], Theorem 3.1.

## 4. Numerical method

In this section, a numerical method is proposed to solve Eq. (1.1) using the properties of the MHFs. To this aim, we need the following theorem.

**Theorem 4.1.** Let  $\phi(x, t)$  be the 2DMHFs vector defined by (2.3) and  $0 < \alpha, \beta < 1$ , then

$$\int_0^t \int_0^x (x-y)^{-\alpha} (t-z)^{-\beta} \phi(y, z) dy dz \simeq \chi_{\alpha, \beta} Q^{(\alpha, \beta)} \phi(x, t), \quad (4.1)$$

where

$$\chi_{\alpha, \beta} = \Gamma(1-\alpha)\Gamma(1-\beta), \quad (4.2)$$

and

$$Q^{(\alpha, \beta)} = P_l^{(1-\alpha)} \otimes P_T^{(1-\beta)}. \quad (4.3)$$

**Proof.** By considering the Riemann-Liouville integral operator in Definition 2.2 and after some manipulation, we obtain

$$\begin{aligned} \int_0^t \int_0^x (x-y)^{-\alpha} (t-z)^{-\beta} \phi(y, z) dy dz &= \frac{\Gamma(1-\alpha)\Gamma(1-\beta)}{\Gamma(1-\alpha)\Gamma(1-\beta)} \times \int_0^t \int_0^x (x-y)^{(1-\alpha)-1} (t-z)^{(1-\beta)-1} \psi_l(y) \otimes \psi_T(z) dy dz \\ &= \Gamma(1-\alpha)\Gamma(1-\beta) \left( \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-y)^{(1-\alpha)-1} \psi_l(y) dy \right) \\ &\quad \otimes \left( \frac{1}{\Gamma(1-\beta)} \int_0^t (t-z)^{(1-\beta)-1} \psi_T(z) dz \right) \\ &= \Gamma(1-\alpha)\Gamma(1-\beta) \left( I_x^{1-\alpha} \psi_l(x) \right) \otimes \left( I_t^{1-\beta} \psi_T(t) \right) \\ &\simeq \Gamma(1-\alpha)\Gamma(1-\beta) \left( P_l^{(1-\alpha)} \psi_l(x) \right) \otimes \left( P_T^{(1-\beta)} \psi_T(t) \right) \\ &= \Gamma(1-\alpha)\Gamma(1-\beta) \left( P_l^{(1-\alpha)} \otimes P_T^{(1-\beta)} \right) (\psi_L(x) \otimes \psi_T(t)) \\ &= \Gamma(1-\alpha)\Gamma(1-\beta) \left( P_l^{(1-\alpha)} \otimes P_T^{(1-\beta)} \right) \phi(x, t). \end{aligned} \quad (4.4)$$

Therefore, taking into account Eqs. (4.2)–(4.4) the proof is completed.  $\square$

Now, by employing Theorem 4.1 we suggest our numerical method for solving Eq. (1.1). To do this, we expand the functions  $u(x, t)$  and  $f(x, t)$  in terms of the 2DMHFs by the way mentioned in Section 2, respectively, as follows

$$u(x, t) \simeq U^T \phi(x, t), \quad (4.5)$$

$$f(x, t) \simeq F^T \phi(x, t), \quad (4.6)$$

where  $U$  is the unknown vector. Substituting Eqs. (4.5) and (4.6) into Eq. (1.1), we have

$$U^T \phi(x, t) = F^T \phi(x, t) + \int_0^t \int_0^x (x-y)^{-\alpha} (t-z)^{-\beta} U^T \phi(y, z) dy dz. \quad (4.7)$$

Utilizing Eq. (4.1) in (4.7) gives

$$U^T \phi(x, t) = F^T \phi(x, t) + \chi_{\alpha, \beta} U^T Q^{(\alpha, \beta)} \phi(x, t).$$

Finally, we obtain the following system

$$U^T - F^T - \chi_{\alpha, \beta} U^T Q^{(\alpha, \beta)} = 0,$$

that can be rewritten in a matrix form equation as

$$(I - \chi_{\alpha, \beta} Q^{(\alpha, \beta)T}) U = F, \quad (4.8)$$

where  $I$  is the identity matrix of order  $(n+1)(m+1) \times (n+1)(m+1)$ .

In our implementation, we have used the Mathematica function *Solve* for solving the final system in (4.8).

### 5. Error analysis

The purpose of this section is to introduce an error estimate for the numerical solution of Eq. (1.1) obtained by the presented method. For convenience, suppose that  $l = T = 1$  and  $h := \frac{1}{n}$ . Then we get the following results.

**Theorem 5.1.** Assume that  $u(x, t) \in C^3(D)$ , and

$$u_n(x, t) = \sum_{i=0}^n \sum_{j=0}^n u(ih, jh) \phi_{i,j}(x, t)$$

is the approximation of  $u(x, t)$  by the 2DMHFs. Then, we have

$$|u(x, t) - u_n(x, t)| = O(h^3). \tag{5.1}$$

**Proof.** Suppose that  $D_{i,j} = [x_i, x_{i+2}] \times [t_j, t_{j+2}]$ ,  $i, j = 0, 1, \dots, n - 2$ , therefore we have  $D = \cup D_{i,j}$ . From Eq. (2.4) it is seen that  $u_n(x, t)$  is a quadratic polynomial which interpolates  $u(x, t)$  at  $(x, t) = (ph, qh)$ ,  $p = i, i + 1, i + 2$ ,  $q = j, j + 1, j + 2$  on  $D_{i,j}$ . So for the interpolation error on  $D_{i,j}$  we have [35]

$$\begin{aligned} u(x, t) - u_n(x, t) &= \frac{1}{6} \frac{\partial^3 u(\xi, t)}{\partial x^3} \prod_{p=i}^{i+2} (x - ph) + \frac{1}{6} \frac{\partial^3 u(x, \eta)}{\partial t^3} \prod_{q=j}^{j+2} (t - qh) \\ &\quad - \frac{1}{36} \frac{\partial^6 u(\xi', \eta')}{\partial x^3 \partial t^3} \prod_{p=i}^{i+2} (x - ph) \prod_{q=j}^{j+2} (t - qh), \end{aligned}$$

where  $\xi, \xi' \in [x_i, x_{i+2}]$  and  $\eta, \eta' \in [t_j, t_{j+2}]$ . Therefore

$$\begin{aligned} |u(x, t) - u_n(x, t)| &\leq \frac{1}{6} \max_{(x,t) \in D_{i,j}} \left| \frac{\partial^3 u(\xi, t)}{\partial x^3} \right| \left| \prod_{p=i}^{i+2} (x - ph) \right| \\ &\quad + \frac{1}{6} \max_{(x,t) \in D_{i,j}} \left| \frac{\partial^3 u(x, \eta)}{\partial t^3} \right| \left| \prod_{q=j}^{j+2} (t - qh) \right| \\ &\quad + \frac{1}{36} \max_{(x,t) \in D_{i,j}} \left| \frac{\partial^6 u(\xi', \eta')}{\partial x^3 \partial t^3} \right| \left| \prod_{p=i}^{i+2} (x - ph) \right| \left| \prod_{q=j}^{j+2} (t - qh) \right|. \end{aligned} \tag{5.2}$$

There are real numbers  $M_1, M_2$  and  $M_3$ , such that

$$\max_{(x,t) \in D_{i,j}} \left| \frac{\partial^3 u(\xi, t)}{\partial x^3} \right| \leq M_1, \tag{5.3}$$

$$\max_{(x,t) \in D_{i,j}} \left| \frac{\partial^3 u(x, \eta)}{\partial t^3} \right| \leq M_2, \tag{5.4}$$

$$\max_{(x,t) \in D_{i,j}} \left| \frac{\partial^6 u(\xi', \eta')}{\partial x^3 \partial t^3} \right| \leq M_3. \tag{5.5}$$

On the other hand, we know that

$$\left| \prod_{p=i}^{i+2} (x - ph) \right| \leq \frac{2\sqrt{3}}{9} h^3, \tag{5.6}$$

$$\left| \prod_{q=j}^{j+2} (t - qh) \right| \leq \frac{2\sqrt{3}}{9} h^3. \tag{5.7}$$

Using (5.3)–(5.7) in (5.2) gives

$$|u(x, t) - u_n(x, t)| \leq \frac{\sqrt{3}M_1}{27} h^3 + \frac{\sqrt{3}M_2}{27} h^3 + \frac{M_3}{243} h^6,$$

which completes the proof.  $\square$

**Theorem 5.2.** Let  $u(x, t) \in C^3(D)$  be the exact solution of Eq. (1.1) and  $u_n(x, t)$  be its approximation obtained by the proposed method in the previous section, then

$$|u(x, t) - u_n(x, t)| = O(h^3).$$

**Proof.** Using Definition 2.2 and (4.2) we rewrite Eq. (1.1) as

$$u(x, t) = f(x, t) + \chi_{\alpha, \beta} I_{\theta}^{\alpha} u(x, t), \tag{5.8}$$

where  $a_1 = 1 - \alpha$  and  $a_2 = 1 - \beta$ . Similarly, by neglecting the error of the operational matrix, it is seen from (4.7) that

$$u_n(x, t) = f_n(x, t) + \chi_{\alpha, \beta} I_{\theta}^{\alpha} u_n(x, t). \tag{5.9}$$

Subtracting (5.9) from (5.8) yields

$$|u(x, t) - u_n(x, t)| \leq |f(x, t) - f_n(x, t)| + \chi_{\alpha, \beta} |I_{\theta}^{\alpha}(u) - I_{\theta}^{\alpha}(u_n)|. \tag{5.10}$$

By employing (5.1), we obtain the following estimates

$$|f(x, t) - f_n(x, t)| = O(h^3), \tag{5.11}$$

$$|I_{\theta}^{\alpha}(u) - I_{\theta}^{\alpha}(u_n)| = O(h^3). \tag{5.12}$$

Therefore, using (5.10)–(5.12), the proof is completed.□

### 6. Numerical examples

In this section, four examples are included to show the applicability, efficiency and accuracy of the proposed method. In all the examples, we consider  $l = T = 1$ ,  $n = m$  and  $h = \frac{1}{n}$ . In order to demonstrate the error of the method we introduce the notations

$$e_n = \max_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}} |u(ih, jh) - u_n(ih, jh)|, \\ \epsilon_n = \log_2 \left( \frac{e_n}{e_{2n}} \right),$$

where  $u(x, t)$  is the exact solution and  $u_n(x, t)$  is the computed solution obtained by the present method. The computations were performed on a personal computer using a 2.60 GHz processor and the codes were written in Mathematica 11.

**Example 6.1.** As the first example, consider Eq. (1.1) with  $\alpha = \beta = \frac{1}{2}$  and the function  $f(x, t)$  is such that the exact solution of the problem is  $u(x, t) = \sin(xt)$ .

We have solved the considered equation in this example with different values of  $n$  and reported the numerical results in Table 1, Table 2 and Fig. 6.1. The numerical results confirm that the convergence order of the proposed method is  $O(h^3)$ .

**Table 1:** The absolute error at  $x = 0.5$  and some selected values of  $t$  with  $n = 2, 4, 8, 16$  for Example 6.1.

$t$	$n = 2$	$n = 4$	$n = 8$	$n = 16$
0.1	2.8061E-02	1.6256E-04	1.2950E-05	1.7363E-06
0.2	5.4755E-02	1.2044E-04	1.2440E-05	5.7903E-07
0.3	8.0205E-02	1.7403E-06	1.8686E-05	1.3683E-06
0.4	1.0453E-01	8.0012E-05	1.9832E-06	2.1673E-06
0.5	1.2786E-01	8.6827E-06	9.2267E-07	1.1182E-08
0.6	1.5032E-01	1.4418E-04	1.5849E-05	1.4427E-06
0.7	1.7203E-01	1.6602E-04	3.5720E-06	6.4436E-08
0.8	1.9310E-01	1.9260E-04	3.2879E-05	2.2763E-06
0.9	2.1364E-01	3.4021E-04	2.8298E-05	3.9246E-06

**Table 2:** Numerical results for Example 6.1.

$n$	4	8	16	32
$e_n$	4.2610E-03	3.5930E-03	2.6193E-05	2.1096E-06
$\epsilon_n$	3.57	3.78	3.63	—

**Example 6.2.** Consider Eq. (1.1) with  $\alpha = \frac{2}{3}$ ,  $\beta = \frac{1}{3}$  and

$$f(x, t) = x^6 \left( t^{\frac{4}{3}} + t^3 \right) - \frac{729}{6086080} x^{\frac{19}{3}} t^2 \left( 6561 t^{\frac{5}{3}} + 1760 \sqrt{3} \pi \right),$$

which has the exact solution  $u(x, t) = x^6 \left( t^{\frac{4}{3}} + t^3 \right)$ .

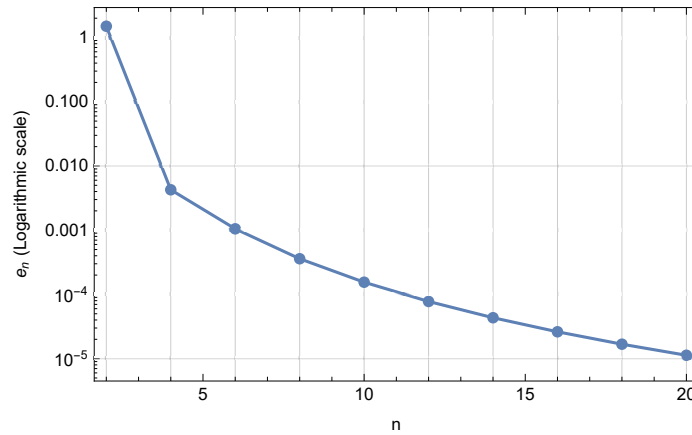


Figure 6.1:  $e_n$  on logarithmic scale for Example 6.1.

Table 3: The absolute error at  $x = 0.5$  and some selected values of  $t$  with  $n = 2, 4, 8, 16$  for Example 6.2.

$t$	$n = 2$	$n = 4$	$n = 8$	$n = 16$
0.1	4.5191E-03	5.8186E-05	2.2606E-05	4.1468E-06
0.2	1.4798E-02	1.4233E-04	1.6604E-05	3.7513E-06
0.3	3.0914E-02	3.2751E-04	3.4959E-05	5.9114E-06
0.4	5.2822E-02	5.7154E-04	7.5115E-05	9.1917E-06
0.5	8.0458E-02	8.0863E-04	1.0657E-04	1.5264E-05
0.6	1.1374E-01	1.3920E-03	1.3247E-04	2.1537E-05
0.7	1.5260E-01	1.7557E-03	1.6985E-04	2.5701E-05
0.8	1.9695E-01	1.8155E-03	1.5472E-04	2.7789E-05
0.9	2.4669E-01	1.4851E-03	1.1840E-04	2.7401E-05

Table 4: Numerical results for Example 6.2.

$n$	4	8	16	32
$e_n$	2.1663E-01	2.4800E-02	2.0087E-03	1.9377E-04
$\epsilon_n$	3.12	3.63	3.37	—

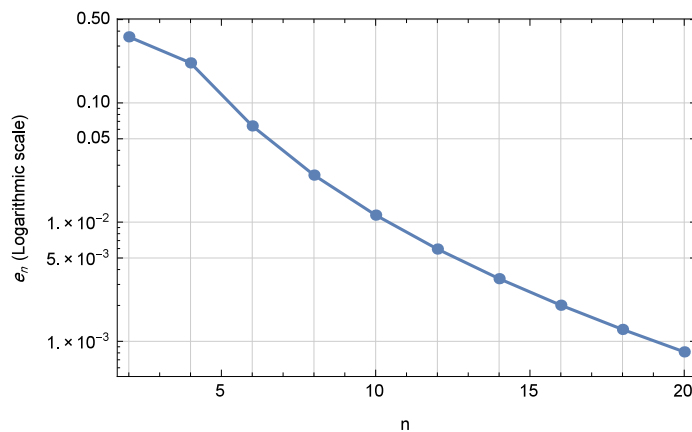


Figure 6.2:  $e_n$  on logarithmic scale for Example 6.2.

The presented method has been applied to this equation with different values of  $n$ . The numerical results for this example are seen in Table 3, Table 4 and Fig. 6.2. Table 3 shows the absolute error at some selected grid points with different  $n$ . The values of  $\epsilon_n$  in Table 4 confirm that the error is  $O(h^3)$  and Fig. 6.2 displays the convergence of the method.

**Example 6.3.** Consider Eq. (1.1) with  $\alpha = \frac{1}{2}$ ,  $\beta = \frac{1}{3}$  and

$$f(x, t) = x^2 t^4 - \frac{972}{1925} x^{\frac{5}{2}} t^{\frac{14}{3}}.$$

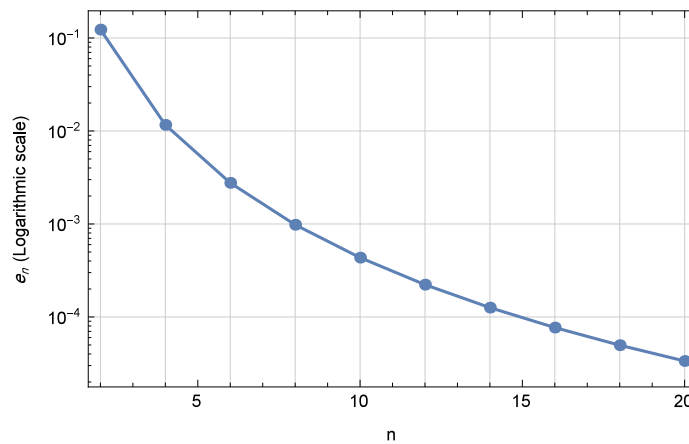
**Table 5:** The absolute error at  $x = 0.5$  and some selected values of  $t$  with  $n = 2, 4, 8, 16$  for Example 6.3.

$t$	$n = 2$	$n = 4$	$n = 8$	$n = 16$
0.1	$1.9964E - 02$	$1.5950E - 03$	$6.0907E - 05$	$6.5053E - 06$
0.2	$2.9903E - 02$	$1.1848E - 03$	$1.0031E - 04$	$7.2572E - 06$
0.3	$3.0716E - 02$	$3.3073E - 03$	$2.9834E - 04$	$1.7434E - 05$
0.4	$2.3903E - 02$	$1.4516E - 03$	$9.4651E - 05$	$4.2587E - 05$
0.5	$1.1564E - 02$	$7.7908E - 05$	$1.1807E - 05$	$1.3121E - 06$
0.6	$3.6003E - 03$	$5.2420E - 03$	$3.1794E - 04$	$5.0484E - 05$
0.7	$1.8291E - 02$	$3.7167E - 03$	$4.2978E - 04$	$2.5486E - 05$
0.8	$2.8607E - 02$	$7.5402E - 04$	$7.2285E - 04$	$4.6354E - 05$
0.9	$3.0050E - 02$	$3.6701E - 03$	$2.0164E - 04$	$9.3176E - 05$

Here, the exact solution is  $u(x, t) = x^2 t^4$ . The numerical results for this example can be observed in Table 5, Table 6 and Fig. 6.3.

**Table 6:** Numerical results for Example 6.3.

$n$	4	8	16	32
$e_n$	$1.1627E - 02$	$9.8060E - 04$	$7.6913E - 05$	$5.8282E - 06$
$\epsilon_n$	3.57	3.67	3.72	—



**Figure 6.3:**  $e_n$  on logarithmic scale for Example 6.3.

**Example 6.4.** In Eq. (1.1), consider  $\alpha = \beta = \frac{1}{2}$  and

$$f(x, t) = -\frac{4}{105} t^{\frac{1}{2}} x^{\frac{1}{2}} (56t^2 + 48x^3 + 105) + t^2 + x^3 + 1,$$

which its exact solution is  $u(x, t) = x^3 + t^2 + 1$ .

The numerical results for Example 6.4 are displayed in Table 7, Table 8 and Fig. 6.4.

**Table 7:** The absolute error at  $t = 0.5$  and some selected values of  $x$  with  $n = 4, 8, 16, 32$  for Example 6.4.

$x$	$n = 4$	$n = 8$	$n = 16$	$n = 32$
0.1	$1.3321E - 02$	$1.1337E - 03$	$5.7134E - 05$	$1.6212E - 06$
0.2	$1.5102E - 02$	$9.5963E - 05$	$1.9323E - 05$	$1.5189E - 05$
0.3	$1.1342E - 02$	$1.5772E - 03$	$1.2592E - 04$	$7.2678E - 06$
0.4	$8.0404E - 03$	$8.8473E - 04$	$1.6926E - 04$	$1.3029E - 05$
0.5	$1.1197E - 02$	$8.0388E - 04$	$5.5802E - 05$	$3.8313E - 06$
0.6	$2.6582E - 02$	$2.1404E - 03$	$2.0027E - 05$	$4.2261E - 06$
0.7	$3.0561E - 02$	$1.0987E - 03$	$1.0930E - 04$	$2.2050E - 05$
0.8	$2.9134E - 02$	$2.9554E - 03$	$2.3026E - 04$	$6.9424E - 07$
0.9	$2.8302E - 02$	$2.4703E - 03$	$2.8957E - 04$	$2.2221E - 05$



**Table 8:** Numerical results for Example 6.4.

$n$	4	8	16	32
$e_n$	$2.8859E-01$	$2.1523E-02$	$1.5844E-03$	$1.1787E-04$
$\epsilon_n$	3.75	3.76	3.75	—

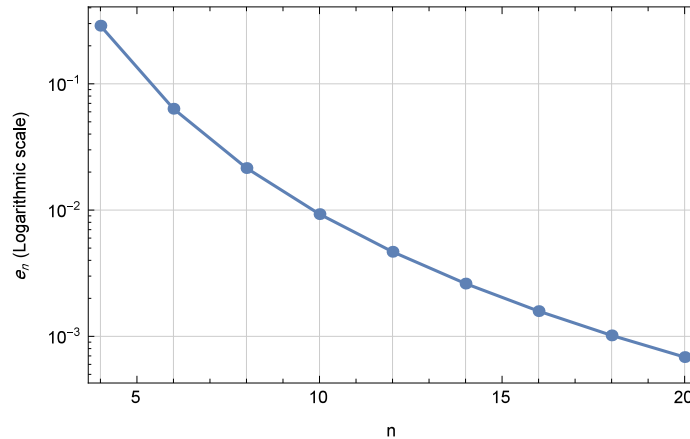
**Figure 6.4:**  $e_n$  on logarithmic scale for Example 6.4.

Table 9 reports the computing time (in seconds) for solving the final system in Eq. (4.8) with different values of  $n$  for Examples 6.1–6.4.

**Table 9:** The computing time (in seconds) for Examples 6.1–6.4.

$n$	2	4	8	16	32
Example 6.1	0.000	0.015	0.032	0.188	1.875
Example 6.2	0.000	0.000	0.031	0.141	1.875
Example 6.3	0.000	0.000	0.016	0.141	1.984
Example 6.4	0.000	0.000	0.000	0.078	1.407

## 7. Conclusion

In this paper, the MHFs have been used to solve the two-dimensional Volterra integral equations with weakly singular kernels. The operational matrix of fractional integration was obtained which helped us to reduce the main problem to a system of algebraic equations. The error analysis verified that the convergence order is  $O(h^3)$  and also the numerical results in Section 6 (Tables 2, 4, 6 and 8) confirmed this convergence order. Compared to the other piecewise functions such as block-pulse functions ( $O(h)$ ), Haar wavelet functions ( $O(h^2)$ ) and hat functions ( $O(h^2)$ ), the MHFs have higher order of convergence. The fractional order operational matrix of integration of the MHFs has a large number of zeros and it makes the proposed method computationally attractive. Table 9 shows the high performance of the method even when we have a large system of equations with 1089 unknown parameters (for  $n = 32$ ).

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