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# Introduction to Timelike Uniform B-splineCurves in Minkowski-3 Space

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**Abstract**

The intention of this article is to study on timelike uniform B-spline curves in Minkowski-3 space. In our paper, we take the control points of uniform B-spline curves as a timelike point in Minkowski-3 space. Then we calculate some geometric elements for this new curve in Minkowski-3 space.

## 1. Introduction

B-spline curves were described by Schoenberg who was worked on B-spline curves for statistical data collection in [1]. The B-spline curves was constructed for computing a convolution of some probability distributions. Moreover, de Boor and Hollig considered a different approach to B-spline curves in [2]. Recently, in Computer Aided Geometric Design (CAGD), B-spline curves have been commonly used for designing an automobile, a boat, an aircraft, [3] and [4]. There are many studies on the B-spline curves, see some of them in [2], [5], [6]. Although degree $d$ of a Bezier curve has $d+1$ control points, degree $d$ of a B-spline curves can have any number of control points supplied a sufficient number of knots are defined in [7] and [8]. In addition, the control points of the Bezier curves provide a global change on the curve, while the control points of the B-spline curves provide a local change on the curve. For this reason, B-spline curves can be given additional freedom by increasing the number of control points in order to define complex curve shapes without increasing the degree of the curve, [9]. Minkowski space was introduced by H. Minkowski. In our paper, we try to investigate some geometric properties of the B-spline curves in Minkowski 3-space. We present the curvature and torsion of the B-spline curves in Minkowski 3-space.

## 2. Preliminaries

In this section the B-spline curves are defined and some preliminaries are given. Then some basics of Minkowski space is given.

**Definition 2.1.** *Let $t_0, t_1, \ldots, t_m$ be knot vectors of the B-spline basis function of degree $d$. The B-spline basis function denoted $N_{i,d}(t)$ is defined by*

$$N_{i,0}(t) = \begin{cases} 1, & if\ t \in [t_i, t_{i+1}) \\ 0, & otherwise \end{cases} \tag{2.1}$$

$$N_{i,d}(t) = \frac{t - t_i}{t_{i+d} - t_i} N_{i,d-1}(t) + \frac{t_{i+d+1} - t}{t_{i+d+1} - t_{i+1}} N_{i+1,d-1}(t) \tag{2.2}$$

*for $i = 0, \ldots, n$ and $d \geq 1$.*

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**Definition 2.2.** *If the B-spline curve of degree d with control points $b_0,...,b_n$ and knots $t_0,t_1,...,t_m$ is defined on the interval $[a,b]=[t_d,t_{m-d}]$, then the curve can be written in the form*

$$B(t)=\sum_{i=0}^{n} b_i N_{i,d}(t).$$

*When the B-spline curves are in the rational form, they are often called integral B-spline curves. Moreover, if the knots are equally spaced, then a B-spline curve is called uniform.*

On the other hand, Minkowski 3-space $\mathbb{R}_1^3$ is a vector space $\mathbb{R}^3$ provide with the Lorentzian inner product $g$ given by

$$\mathrm{g}(\nu,\lambda)=\nu_1\lambda_1+\nu_2\lambda_2-\nu_3\lambda_3,$$

where $\nu=(\nu_1,\nu_2,\nu_3)$ and $\lambda=(\lambda_1,\lambda_2,\lambda_3)\in\mathbb{R}_1^3$. A vector in Minkowski 3-space $\lambda=(\lambda_1,\lambda_2,\lambda_3)\in\mathbb{R}_1^3$ is called spacelike if $\mathrm{g}(\lambda,\lambda)>0$ or $\lambda=0$; timelike if $\mathrm{g}(\lambda,\lambda)<0$; lightlike if $\mathrm{g}(\lambda,\lambda)=0$ and $\lambda\neq 0$. The vectors $\nu$ and $\lambda$ are ortogonal if and only if $\mathrm{g}(\nu,\lambda)=0$. The norm of a vector $\nu$ on Minkowski space $\mathbb{R}_1^3$ is defined by $\|\nu\|_{\mathbb{L}}=\sqrt{|\mathrm{g}(\nu,\nu)|}$. If the vector is timelike, then the form will be $\|\nu\|_{\mathbb{L}}=\sqrt{-\mathrm{g}(\nu,\nu)}$. Let $(c)$ be curve in $\mathbb{R}_1^3$. We say that $(c)$ is timelike curve (resp. spacelike, lightlike) at $t$ if the tangent vector $(c)'(t)$ is a timelike (resp. spacelike, lightlike) vector. The vector fields of the moving Serret-Frenet from along the curve $(c)$ are denoted by $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ where $\mathbf{T}$, $\mathbf{N}$ and $\mathbf{B}$ are called with the tangent, the principal normal and the binormal vector of the curve $(c)$, respectively. If the curve $(c)$ is time-like curve, then $\mathbf{T}$ is timelike vector, $\mathbf{N}$ and $\mathbf{B}$ are spacelike vectors which satisfy $\mathbf{T}\wedge_{\mathbb{L}}\mathbf{N}=-\mathbf{B}$, $\mathbf{N}\wedge_{\mathbb{L}}\mathbf{B}=\mathbf{T}$, $\mathbf{B}\wedge_{\mathbb{L}}\mathbf{T}=-\mathbf{N}$. The derivative of Serret-Frenet frame equations for a timelike curve is

$$\begin{aligned}\mathbf{T}'&=\kappa\mathbf{N}\\ \mathbf{N}'&=\kappa\mathbf{T}+\tau\mathbf{B}\\ \mathbf{B}'&=-\tau\mathbf{N}.\end{aligned}$$

## 3. Main result

**Definition 3.1.** *Let $X=\{b_0,b_1,...,b_n\}$ be a timelike points set in $\mathbb{R}_1^3$. The*

$$TCH\{X\}=\left\{\lambda_0 b_0+...+\lambda_n b_n \middle| \sum_{i=0}^{n}\lambda_i=1,\lambda_i\geq 0\right\}$$

*set formed by these X points are called timelike convex hull of a timelike uniform B-spline curve.*

**Definition 3.2.** *If the control points $b_0,...,b_n\in TCH\{X\}$ are timelike and the knots $t_0,t_1,...,t_m$ on the interval $[a,b]=[t_d,t_{m-d}]$ are equally spaced, then the timelike uniform B-spline curve of degree d in Minkowski 3-space is defined by*

$$B(t)=\sum_{i=0}^{n} b_i N_{i,d}(t),$$

*where $N_{i,d}(t)$ are the basis functions.*

**Example:** Lets consider the timelike uniform B-spline curve $B(t)$ of degree $d=2$ defined on the knots $t_0=0,t_1=1,t_2=2,t_3=3,t_4=4,t_5=5,t_6=6,t_7=7$ and with control points $b_0(2,3),b_1(-1,7),b_2(2,5),b_3(4,5),b_4(1,3)$. The basis graphic and the curve shape are in the following figures.

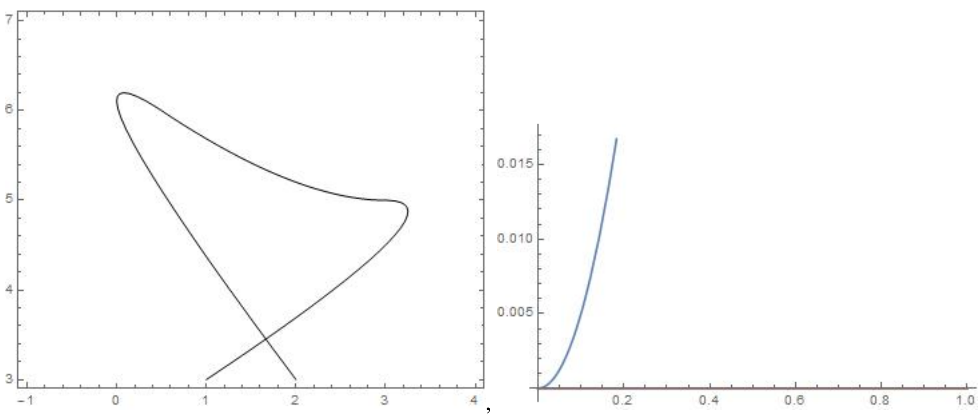


**Figure 3.1:** a) Basis function graphic b) A timelike uniform B-spline curve

**Theorem 3.3.** *Let $B(t)$ be a timelike uniform B-spline curve of degree d with the knot vector $t_0,...,t_m$ in Minkowski 3-space. If $t\in[t_r,t_{r+1})$ $(d\leq r\leq m-d-1)$ then $B(t)=\sum_{i=r-d}^{r} b_i N_{i,d}(t)$. Therefore to compute $B(t)$ its sufficient to compute $N_{r-d,d}(t),...,N_{r,d}(t)$. This shows us that the B-spline curve is achieved by the local control. If $t\in[t_r,t_{r+1})$ $(d\leq r\leq m-d-1)$ then $B(t)\in TCH\{b_{r-d},...,b_r\}$. This means that B-spline curve has an convex hull. If $p_i$ is the multiplicity of the breakpoint $t=u_i$ then $B(t)$ is $C^{d-p_i}$ (or greater) at*

*$t=u_i$ and $C^\infty$ elsewhere. Thus, it is seen that the B-spline curve is satisfied the continuity property. Let $T$ be an affine transformation. If $T(\sum_{i=0}^{n} b_i N_{i,d}(t)) = \sum_{i=0}^{n} T(b_i) N_{i,d}(t)$, the B-spline curve is invariant under affine transformations.*

**Theorem 3.4.** *Let $B(t)$ be a timelike uniform B-spline curve of degree $d$ with the knot vector $t_0,\ldots,t_m$ in Minkowski 3-space. The second and third derivative of the control points $b_i$ are calculated by*

$$\begin{aligned} b_i^{(2)} &= (d-1).m_i.\Delta b_i^{(1)} \\ b_i^{(3)} &= (d-1)(d-2).p_i.(n_i.\Delta b_{i+1}^{(1)} - m_i \Delta b_i^{(1)}) \end{aligned}$$

*where $m_i, n_i, p_i$ are some constants of $t_i$.*

*Proof.* Using the Eq.(2.1) and Eq.(2.2) the control points can be written as

$$\begin{aligned} b_i^{(2)} &= (d-1)\frac{b_{i+1}^{(1)} - b_i^{(1)}}{t_{i+d+1} - t_{i+2}} \\ &= (d-1).m_i.\Delta b_i^{(1)}, \\ b_i^{(3)} &= \frac{(d-2)}{t_{i+d+1} - t_{i+3}}\left(b_{i+1}^{(2)} - b_i^{(2)}\right) \\ &= \frac{(d-2)}{t_{i+d+1} - t_{i+3}}\left((d-1).n_i.(b_{i+2}^{(2)} - b_{i+1}^{(1)}) - (d-1).m_i.(b_{i+1}^{(1)} - b_i^{(1)})\right) \\ &= \frac{(d-1)(d-2)}{t_{i+d+1} - t_{i+3}}\left(.n_i.(b_{i+2}^{(2)} - b_{i+1}^{(1)}) - m_i.(b_{i+1}^{(1)} - b_i^{(1)})\right) \\ &= (d-1)(d-2).p_i.(n_i.\Delta b_{i+1}^{(1)} - m_i \Delta b_i^{(1)}) \end{aligned}$$

where $m_i = \frac{1}{t_{i+d+1}-t_{i+2}}$, $n_i = \frac{1}{t_{i+d+2}-t_{i+3}}$ and $p_i = \frac{1}{t_{i+d+1}-t_{i+3}}$. □

**Theorem 3.5.** *Let $B(t)$ be a timelike uniform B-spline curve of degree $d$ with the knot vector $t_0,\ldots,t_m$ in Minkowski 3-space. The derivatives of B-spline curve is computed by*

$$\begin{aligned} B^{(1)}(t) &= \sum_{i=0}^{n-1} b_i{}^{(1)} N^{(1)}{}_{i,d-1}(t) \\ B^{(2)}(t) &= (d-1)\sum_{i=0}^{n-2} m_i.\Delta b_i^{(1)}.N_{i,d-2}^{(2)} \\ B^{(3)}(t) &= (d-1)(d-2)\sum_{i=0}^{n-3} p_i.\left(n_i.\Delta b_{i+1}^{(1)} - m_i \Delta b_i^{(1)}\right).N_{i,d-3}^{(3)}. \end{aligned}$$

*Proof.* Substituting the above results in Eq.(2.2), the proof is obvious. □

**Theorem 3.6.** *Let $B(t)$ be an arbitrary timelike uniform B-spline curve and $\{T,N,B\}|_{t=0}$ be the Serret-Frenet frame of $B(t)$, where $T$ is timelike, $N$ and $B$ are spacelike. Then the following conditions are satisfied*

$$\begin{gathered} g(T,T) = -1, g(N,N) = 1, g(B,B) = 1 \\ g(T,N) = 0, g(T,B) = 0, g(N,B) = 0. \end{gathered}$$

*The Serret-Frenet frame of the timelike uniform B-spline curve $B(t)$ is obtained by*

$$\begin{aligned} T &= \frac{\sum_{i=0}^{n-1} b_i{}^{(1)} N^{(1)}{}_{i,d-1}(t)}{\left\|\sum_{i=0}^{n-1} b_i{}^{(1)} N^{(1)}{}_{i,d-1}(t)\right\|} \\ B &= \frac{\sum_{i=0}^{n-1} b_i{}^{(1)} N_{i,d-1}^{(1)}(t) \wedge \sum_{i=0}^{n-2} m_i.\Delta b_i^{(1)}.N_{i,d-2}^{(2)}}{\left\|\sum_{i=0}^{n-1} b_i{}^{(1)} N_{i,d-1}^{(1)}(t) \wedge \sum_{i=0}^{n-2} m_i.\Delta b_i^{(1)}.N_{i,d-2}^{(2)}\right\|} \\ N &= -\frac{\begin{array}{l} -g\left(\sum_{i=0}^{n-1} b_i{}^{(1)} N_{i,d-1}^{(1)}(t), \sum_{i=0}^{n-1} b_i{}^{(1)} N^{(1)}{}_{i,d-1}(t)\right)\left(\sum_{i=0}^{n-2} m_i.\Delta b_i^{(1)}.N_{i,d-2}^{(2)}\right) \\ +g\left(\sum_{i=0}^{n-2} m_i.\Delta b_i^{(1)}.N_{i,d-2}^{(2)}, \sum_{i=0}^{n-1} b_i{}^{(1)} N_{i,d-1}^{(1)}(t)\right)\left(\sum_{i=0}^{n-1} b_i{}^{(1)} N_{i,d-1}^{(1)}(t)\right) \end{array}}{\left\|\sum_{i=0}^{n-1} b_i{}^{(1)} N_{i,d-1}^{(1)}(t) \wedge \sum_{i=0}^{n-2} m_i.\Delta b_i^{(1)}.N_{i,d-2}^{(2)}\right\| \left\|\sum_{i=0}^{n-1} b_i{}^{(1)} N^{(1)}{}_{i,d-1}(t)\right\|} \end{aligned}$$

*Proof.* Let consider the B-spline curve $B(t)$ is non unit speed curve in Minkowski 3-space. Using the scalar and vector product in Minkowski 3-space, the tangent vector of the timelike uniform B-spline curve $B(t)$ is calculated as

$$\begin{aligned} T &= \frac{B^{(1)}(t)}{\left\|B^{(1)}(t)\right\|} \\ &= \frac{\sum\limits_{i=0}^{n-1} b_i{}^{(1)} N^{(1)}{}_{i,d-1}(t)}{\left\|\sum\limits_{i=0}^{n-1} b_i{}^{(1)} N^{(1)}{}_{i,d-1}(t)\right\|}, \end{aligned}$$

and the binormal vector of the timelike B-spline curve is

$$\begin{aligned} B &= \frac{B^{(1)}(t)\wedge B^{(2)}(t)}{\left\|B^{(1)}(t)\wedge B^{(2)}(t)\right\|} \\ &= \frac{\sum\limits_{i=0}^{n-1} b_i{}^{(1)} N^{(1)}_{i,d-1}(t)\wedge (d-1)\sum\limits_{i=0}^{n-2} m_i.\Delta b_i^{(1)}.N^{(2)}_{i,d-2}}{\left\|\sum\limits_{i=0}^{n-1} b_i{}^{(1)} N^{(1)}_{i,d-1}(t)\wedge (d-1)\sum\limits_{i=0}^{n-2} m_i.\Delta b_i^{(1)}.N^{(2)}_{i,d-2}\right\|} \\ &= \frac{\sum\limits_{i=0}^{n-1} b_i{}^{(1)} N^{(1)}_{i,d-1}(t)\wedge \sum\limits_{i=0}^{n-2} m_i.\Delta b_i^{(1)}.N^{(2)}_{i,d-2}}{\left\|\sum\limits_{i=0}^{n-1} b_i{}^{(1)} N^{(1)}_{i,d-1}(t)\wedge \sum\limits_{i=0}^{n-2} m_i.\Delta b_i^{(1)}.N^{(2)}_{i,d-2}\right\|}. \end{aligned}$$

The principal normal can be obtained as

$$\begin{aligned} N &= -B\wedge T \\ &= -\frac{\sum\limits_{i=0}^{n-1} b_i{}^{(1)} N^{(1)}_{i,d-1}(t)\wedge \sum\limits_{i=0}^{n-2} m_i.\Delta b_i^{(1)}.N^{(2)}_{i,d-2}}{\left\|\sum\limits_{i=0}^{n-1} b_i{}^{(1)} N^{(1)}_{i,d-1}(t)\wedge \sum\limits_{i=0}^{n-2} m_i.\Delta b_i^{(1)}.N^{(2)}_{i,d-2}\right\|}\wedge\frac{\sum\limits_{i=0}^{n-1} b_i{}^{(1)} N^{(1)}{}_{i,d-1}(t)}{\left\|\sum\limits_{i=0}^{n-1} b_i{}^{(1)} N^{(1)}{}_{i,d-1}(t)\right\|} \\ &= -\frac{\left(\sum\limits_{i=0}^{n-1} b_i{}^{(1)} N^{(1)}_{i,d-1}(t)\wedge \sum\limits_{i=0}^{n-2} m_i.\Delta b_i^{(1)}.N^{(2)}_{i,d-2}\right)\wedge\sum\limits_{i=0}^{n-1} b_i{}^{(1)} N^{(1)}{}_{i,d-1}(t)}{\left\|\sum\limits_{i=0}^{n-1} b_i{}^{(1)} N^{(1)}_{i,d-1}(t)\wedge \sum\limits_{i=0}^{n-2} m_i.\Delta b_i^{(1)}.N^{(2)}_{i,d-2}\right\|\left\|\sum\limits_{i=0}^{n-1} b_i{}^{(1)} N^{(1)}{}_{i,d-1}(t)\right\|} \\ &= -\frac{\begin{array}{l} -g\left(\sum\limits_{i=0}^{n-1} b_i{}^{(1)} N^{(1)}_{i,d-1}(t),\sum\limits_{i=0}^{n-1} b_i{}^{(1)} N^{(1)}{}_{i,d-1}(t)\right)\left(\sum\limits_{i=0}^{n-2} m_i.\Delta b_i^{(1)}.N^{(2)}_{i,d-2}\right) \\ +g\left(\sum\limits_{i=0}^{n-2} m_i.\Delta b_i^{(1)}.N^{(2)}_{i,d-2},\sum\limits_{i=0}^{n-1} b_i{}^{(1)} N^{(1)}_{i,d-1}(t)\right)\sum\limits_{i=0}^{n-1} b_i{}^{(1)} N^{(1)}_{i,d-1}(t)\end{array}}{\left\|\sum\limits_{i=0}^{n-1} b_i{}^{(1)} N^{(1)}_{i,d-1}(t)\wedge \sum\limits_{i=0}^{n-2} m_i.\Delta b_i^{(1)}.N^{(2)}_{i,d-2}\right\|\left\|\sum\limits_{i=0}^{n-1} b_i{}^{(1)} N^{(1)}{}_{i,d-1}(t)\right\|}. \end{aligned}$$

□

**Theorem 3.7.** *If the B-spline curve of degree d with control points $b_0,...,b_n$ and knots $t_0,t_1,...,t_m$ is defined on the interval $[a,b]=[t_d,t_{m-d}]$, the curvature of timelike uniform B-spline curve $B(t)$ is found as*

$$\kappa = |d-1|\frac{\left\|\sum\limits_{i=0}^{n-1} b_i{}^{(1)} N^{(1)}_{i,d-1}(t)\wedge \sum\limits_{i=0}^{n-2} m_i.\Delta b_i^{(1)}.N^{(2)}_{i,d-2}\right\|}{\left\|\sum\limits_{i=0}^{n-1} b_i{}^{(1)} N^{(1)}_{i,d-1}(t)\right\|^3}$$

*Proof.* From the definition of curvature of the non-unit speed curve, we have

$$\begin{aligned} \kappa &= \frac{\left\|B^{(1)}(t)\wedge B^{(2)}(t)\right\|}{\left\|B^{(1)}(t)\right\|^3} \\ &= \frac{\left\|\sum\limits_{i=0}^{n-1} b_i{}^{(1)} N^{(1)}_{i,d-1}(t)\wedge (d-1)\sum\limits_{i=0}^{n-2} m_i.\Delta b_i^{(1)}.N^{(2)}_{i,d-2}\right\|}{\left\|\sum\limits_{i=0}^{n-1} b_i{}^{(1)} N^{(1)}_{i,d-1}(t)\right\|^3} \\ &= |d-1|\frac{\left\|\sum\limits_{i=0}^{n-1} b_i{}^{(1)} N^{(1)}_{i,d-1}(t)\wedge \sum\limits_{i=0}^{n-2} m_i.\Delta b_i^{(1)}.N^{(2)}_{i,d-2}\right\|}{\left\|\sum\limits_{i=0}^{n-1} b_i{}^{(1)} N^{(1)}_{i,d-1}(t)\right\|^3}. \end{aligned}$$

□

**Theorem 3.8.** *If $B(t)$ is a timelike uniform B-spline curve of degree d with the knot vector $t_0,...,t_m$ in Minkowski 3-space, the torsion of a timelike uniform B-spline curve $B(t)$ is computed by*

$$\tau=-(d-2)\frac{\det\left(\sum_{i=0}^{n-1} b_i{}^{(1)}N^{(1)}_{i,d-1}(t),\ \sum_{i=0}^{n-2} m_i.\Delta b_i^{(1)}.N^{(2)}_{i,d-2},\ \sum_{i=0}^{n-3} p_i.\left(n_i.\Delta b_{i+1}^{(1)}-m_i\Delta b_i^{(1)}\right).N^{(3)}_{i,d-3}\right)}{\left\|\sum_{i=0}^{n-1} b_i{}^{(1)}N^{(1)}_{i,d-1}(t)\wedge\sum_{i=0}^{n-2} m_i.\Delta b_i^{(1)}.N^{(2)}_{i,d-2}\right\|^2}$$

*Proof.* Using the definition of torsion, we have the following equations:

$$\begin{aligned}\tau &= \frac{\left(B^{(1)}(t)\ B^{(2)}(t)\ B^{(3)}(t)\right)}{\left\|B^{(1)}(t)\wedge B^{(2)}(t)\right\|^2}\\ &= \frac{\left(\sum_{i=0}^{n-1} b_i{}^{(1)}N^{(1)}_{i,d-1}(t)\quad (d-1)\sum_{i=0}^{n-2} m_i.\Delta b_i^{(1)}.N^{(2)}_{i,d-2}\quad (d-1)(d-2)\sum_{i=0}^{n-3} p_i.\left(n_i.\Delta b_{i+1}^{(1)}-m_i\Delta b_i^{(1)}\right).N^{(3)}_{i,d-3}\right)}{\left\|\sum_{i=0}^{n-1} b_i{}^{(1)}N^{(1)}_{i,d-1}(t)\wedge(d-1)\sum_{i=0}^{n-2} m_i.\Delta b_i^{(1)}.N^{(2)}_{i,d-2}\right\|^2}\\ &= -(d-2)\frac{\det\left(\sum_{i=0}^{n-1} b_i{}^{(1)}N^{(1)}_{i,d-1}(t),\ \sum_{i=0}^{n-2} m_i.\Delta b_i^{(1)}.N^{(2)}_{i,d-2},\ \sum_{i=0}^{n-3} p_i.\left(n_i.\Delta b_{i+1}^{(1)}-m_i\Delta b_i^{(1)}\right).N^{(3)}_{i,d-3}\right)}{\left\|\sum_{i=0}^{n-1} b_i{}^{(1)}N^{(1)}_{i,d-1}(t)\wedge\sum_{i=0}^{n-2} m_i.\Delta b_i^{(1)}.N^{(2)}_{i,d-2}\right\|^2}\end{aligned}$$

□

## 4. Conclusion

In this paper, we present a theoretical work about the timelike uniform B-spline curves in Minkowski-3 space. The timelike B-spline curve in Minkowski 3-space at first time is introduced. The derivatives of control points are calculated. Later Serret-Frenet frame of the timelike uniform B-spline curve is given. Moreover, the curvature and torsion of the B-spline curve are computed.

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