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ON THE CLASSICAL PRIME SPECTRUM OF LATTICE MODULES

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Abstract. Let M be a lattice module over a C-lattice L. A proper element P of M is said to be classical prime if for $a, b \in L$ and $X \in M, abX \leq P$ implies that $aX \leq P$ or $bX \leq P$. The set of all classical prime elements of M, $Spec^{cp}(M)$ is called as classical prime spectrum. In this article, we introduce and study a topology on $Spec^{cp}(M)$, called as Zariski-like topology of M. We investigate this topological space from the point of view of spectral spaces. We show that if M has ascending chain condition on classical prime radical elements, then $Spec^{cp}(M)$ with the Zariski-like topology is a spectral space.

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1. Introduction

Zariski-like topology on the classical prime spectrum of a module is being introduce and studied by M. Behboodi and M. J. Noori in [7]. There are many generalizations of the Zariski topology over the set of all prime submodules of a R-module M (see [5], [6], [14]). As a generalization of most of the results in [7], we introduce and study the Zariski-like topology on the classical prime spectrum of a lattice module M over a C-lattice L.

A lattice L is said to be *complete*, if for any subset S of L, we have $\vee S, \wedge S \in L$. A complete lattice L with least element 0_L and greatest element 1_L is said to be a *multiplicative lattice*, if there is defined a binary operation "." called multiplication on L satisfying the following conditions:

(1) $a.b = b.a$, for all $a, b \in L$;
(2) $a.(b.c) = (a.b).c$, for all $a, b, c \in L$;
(3) $a.(\vee_\alpha b_\alpha) = \vee_\alpha (a.b_\alpha)$, for all $a, b_\alpha \in L$;
(4) $a.1_L = a$, for all $a \in L$.

Henceforth, $a.b$ will be simply denoted by ab.

An element a in L is called compact if $a \leq \vee_{\alpha \in I} b_\alpha$ (I is an indexed set) implies $a \leq b_{\alpha_1} \vee b_{\alpha_2} \vee \cdots \vee b_{\alpha_n}$ for some subset $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ of I. By a *C-lattice*, we mean a multiplicative lattice L, with least element 0_L and greatest element 1_L which is compact as well as multiplicative identity, that is generated under joins by a multiplicatively closed subset C of compact elements of L. An element $a \in L$ is said to be *proper*, if $a < 1_L$.

A proper element p of a multiplicative lattice L is said to be *prime* if $ab \leq p$ implies $a \leq p$ or $b \leq p$ for $a, b \in L$. The collection of all prime elements of L is denoted by $Spec(L)$.

The Zariski topology on the set $Spec(L)$ of all prime elements in multiplicative lattices is being studied in [21], by Thakare, Manjarekar and Maeda and in [20], by Thakare and Manjarekar as a generalization of the Zariski topology of a commutative ring with unity.

A proper element m of a multiplicative lattice L is said to be *maximal* if for every $x \in L$ with $m < x \leq 1_L$ implies $x = 1_L$.

A complete lattice M with smallest element 0_M and greatest element 1_M is said to be a *lattice module* over the multiplicative lattice L or L-module if there is a multiplication between elements of M and L, denoted by $aN \in M$, for $a \in L$ and $N \in M$, which satisfies the following properties:

(1) $(ab)N = a(bN)$;
(2) $(\vee_\alpha a_\alpha)(\vee_\beta N_\beta) = (\vee_{\alpha\beta} a_\alpha N_\beta)$;
(3) $1_L N = N$;
(4) $0_L N = 0_M$; for all $a, b, a_\alpha \in L$, and for all $N, N_\beta \in M$.

Let M be a lattice module over a C-lattice L. For $N \in M, b \in L$, denote $(N : b) = \vee\{K \in M | bK \leq N\}$. If $a, b \in L$, we write $(a : b) = \vee\{x \in L | bx \leq a\}$. If $A, B \in M$, then $(A : B) = \vee\{x \in L | Bx \leq A\}$. An element $N \in M$ is said to be compact if $N \leq \vee_{\alpha \in I} A_\alpha$ (I is an indexed set) implies $N \leq A_{\alpha_1} \vee A_{\alpha_2} \vee \cdots \vee A_{\alpha_n}$ for some subset $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ of I.

An element $N \in M$ is said to be *meet principal* (respectively *join principal*) if it satisfies the identity $A \wedge aN = (a \wedge (A : N))N$ (respectively $((aN \vee A) : N) = (a \vee (A : N)))$ for all $a \in L$ and for all $A \in M$. Also N is said to be *principal* if it is both join as well as meet principal. If each element of M is the join of principal (compact) elements of M, then M is called the *principally generated (compactly generated)* lattice module.

An element $N \in M$ is said to be proper, if $N < 1_M$. A proper element N of a lattice module M is said to be *prime* if $aX \leq N$ implies $X \leq N$ or $a1_M \leq N$, i.e., $a \leq (N : 1_M)$ for every $a \in L$ and $X \in M$. The prime spectrum of a lattice module

M is the set of all prime elements of M and it is denoted by $Spec(M)$. In [4], S. Ballal and V. Kharat studied the Zariski topology over $Spec(M)$ as a generalization of the results carried out in [20], [21]. Also in [10], F. Callialp et al. studied the Zariski topology over $Spec(M)$ over multiplicative lattice L.

A non-zero element $N \in M$ is said to be *second*, if for $a \in L$, either $aN = N$ or $aN = 0_M$. The Zariski topology on the second spectrum of a lattice module is studied by N. Phadatare et al. in [17]. In [18], N. Phadatare and V. Kharat introduced and studied the concept of second radical elements of a lattice module M over a C-lattice L as a generalization of second socle of a submodule of R-module M. An element $N < 1_M$ of M is said to be *maximal* if $N \leq B$ implies either $N = B$ or $B = 1_M, B \in M$. A non-zero element $K \neq 1_M$ of M is said to be *minimal* if $0_M \leq N < K$ implies $N = 0_M, N \in M$.

Further all these concepts and for more information on multiplicative lattices, lattice modules and topology, the reader may refer ([1], [2], [13], [16], [19]).

2. Zariski-like topology on $Spec^{cp}(M)$

Let M be a lattice module over a C-lattice L. A proper element $P \in M$ is said to be classical prime if for $a, b \in L$ and $X \in M, abX \leq P$ implies that $aX \leq P$ or $bX \leq P$ ([15]). The classical prime spectrum $Spec^{cp}(M)$ is defined to be the set of all classical prime elements of M, i.e., $Spec^{cp}(M) = \{P \in M \mid P \text{ is a classical prime element of } M\}$. Let N be any element of M. Let $F^c(N)$ be the set of all classical prime elements of M which contains N, i.e., $F^c(N) = \{P \in Spec^{cp}(M) | N \leq P\}$. Note that, $F^c(0_M) = Spec^{cp}(M)$ and $F^c(1_M)$ is an empty set.

Proposition 2.1. *Let M be a lattice module over a C-lattice L and $N, N_i, K \in M$ $(i \in I)$. Then the following statements holds.*

(1) $\cap_{i\in I} F^c(N_i) = F^c(\vee_{i\in I} N_i)$ *for any index set I.*
(2) $F^c(N) \cup F^c(K) \subseteq F^c(N \wedge K)$.

Proof. (1) Since for each $i \in I$, $N_i \leq \vee N_i$, therefore for $P \in F^c(\vee_{i\in I} N_i)$, we have $\vee_{i\in I} N_i \leq P$ and hence $N_i \leq P$ and $P \in F^c(N_i)$. Therefore $F^c(\vee_{i\in I} N_i) \subseteq \cap_{i\in I} F^c(N_i)$.

Conversely, suppose that $P \in \cap_{i\in I} F^c(N_i)$, then $P \in F^c(N_i)$ for each $i \in I$, therefore $N_i \leq P$ for each $i \in I$. Hence $\vee_{i\in I} N_i \leq P$ and so $P \in F^c(\vee_{i\in I} N_i)$. Consequently, $\cap_{i\in I} F^c(N_i) \subseteq F^c(\vee_{i\in I} N_i)$. Thus $\cap_{i\in I} F^c(N_i) = F^c(\vee_{i\in I} N_i)$.
(2) Since $N \wedge K \leq N, K$, we have $F^c(N), F^c(K) \subseteq F^c(N \wedge K)$ and therefore $F^c(N) \cup F^c(K) \subseteq F^c(N \wedge K)$. □

Let $\xi^c(M) = \{F^c(N)|N \in M\}$, then $\xi^c(M)$ contains empty set and $Spec^{cp}(M)$. By Proposition 2.1, $\xi^c(M)$ is closed under arbitrary intersections. In general $\xi^c(M)$ is not closed under finite union. A lattice module M over a C-lattice L is called top lattice module if $\xi^c(M)$ is closed under finite union. In this case, $\xi^c(M)$ induces a topology τ^c on $Spec^{cp}(M)$, we call it the Zariski topology.

Let M be a lattice module over a C-lattice L. For each element N of M, let $G^c(N) = Spec^{cp}(M) - F^c(N)$ and $\mathbb{G}^c(N) = \{G^c(N)|N \in M\}$. Let $\psi^c(M)$ be the collection of all unions of finite intersections of elements of $\mathbb{G}^c(N)$, then $\psi^c(M)$ is a topology on $Spec^{cp}(M)$ by the sub-basis $\mathbb{G}^c(N)$. We call the topology $\psi^c(M)$, a Zariski-like topology.

Note that, the set $\{G^c(N_1) \cap G^c(N_2) \cap \cdots \cap G^c(N_k)|N_i \in M, 1 \leq i \leq k, k \in \mathbb{N}\}$ is a basis for the Zariski-like topology of M.

Let M be a lattice module over a C-lattice L and let $Spec^{cp}(M)$ be equipped with the Zariski-like topology. Let $Y \subseteq Spec^{cp}(M)$. The closure of Y in $Spec^{cp}(M)$ is denoted by $Cl(Y)$ and meet of all elements of Y by $Z(Y)$. Note that, if $Y = \emptyset$, then $Z(Y) = 1_M$.

Lemma 2.2. *Let M be a lattice module over a C-lattice L and let Y be a non-empty subset of $Spec^{cp}(M)$. Then $Cl(Y) = \cup_{P\in Y} F^c(P)$.*

Proof. Suppose that Y is a non-empty subset of $Spec^{cp}(M)$. Clearly $Y \subseteq \cup_{P\in Y} F^c(P)$. Suppose D is any closed subset of $Spec^{cp}(M)$ such that $Y \subseteq D$. Thus $D = \cap_{k\in J}(\cup_{l=1}^{n_k} F^c(N_{kl}))$, for some $N_{kl} \in M$, $k \in J$ and $n_k \in \mathbb{N}$. Let $P_1 \in \cup_{P\in Y} F^c(P)$, then there exists $P_0 \in Y$ such that $P_1 \in F^c(P_0)$ and so $P_0 \leq P_1$. Since $P_0 \in Y \subseteq D$, therefore for each $k \in J$ there exists $l \in \{1, 2, \cdots, n_k\}$ such that $N_{kl} \leq P_0$ and hence $N_{kl} \leq P_0 \leq P_1$. Therefore $P_1 \in F^c(N_{kl})$ for each $k \in J$ and hence $P_1 \in \cap_{k\in J}(\cup_{l=1}^{n_k} F^c(N_{kl})) = D$. It follows that $\cup_{P\in Y} F^c(P) \subseteq D$. Thus $\cup_{P\in Y} F^c(P)$ is the smallest closed set in $Spec^{cp}(M)$ which contains Y. Consequently, $Cl(Y) = \cup_{P\in Y} F^c(P)$. □

Corollary 2.3. *Let M be a lattice module over a C-lattice L. Then*

(1) *$Cl(\{P\}) = F^c(P)$, for each $P \in Spec^{cp}(M)$.*

(2) *$P_0 \in Cl(\{P\})$ if and only if $P \leq P_0$ if and only if $F^c(P_0) \subseteq F^c(P)$, for $P_0 \in Spec^{cp}(M)$.*

(3) *The set $\{P\}$ is closed in $Spec^{cp}(M)$ if and only if P is a maximal classical prime element of M.*

Proof. (1) By Lemma 2.2, for $Y \subseteq Spec^{cp}(M)$, $Cl(Y) = \cup_{P\in Y} F^c(P)$. Assume that $Y = \{P\}$, then $\cup_{P\in Y} F^c(P) = F^c(P)$. Hence $Cl(\{P\}) = F^c(P)$.

(2) Suppose that $P_0 \in Cl(\{P\})$. Then $P_0 \in Cl(\{P\}) = F^c(P)$, by part (1) and

therefore $P \leq P_0$ which implies that $F^c(P_0) \subseteq F^c(P)$. Conversely, suppose that $F^c(P_0) \subseteq F^c(P)$. Since $P_0 \in F^c(P_0) \subseteq F^c(P)$, $P \leq P_0$ and hence $P_0 \in F^c(P) = Cl(\{P\})$.
(3) Suppose that the set $\{P\}$ is closed in $Spec^{cp}(M)$ and P is not maximal, then there exists Q such that $P \leq Q$, which implies that $Q \in Cl(\{P\})$ by part (2). But $\{P\}$ is closed, therefore $Q \in \{P\}$ and so $P = Q$. Consequently, P is a maximal classical prime element. Conversely, suppose that P is a maximal classical prime element of M. Let $Q \in Cl(\{P\})$. Then by part (1), $Q \in F^c(P)$, therefore $P \leq Q$. But P is maximal, hence $P = Q$ and therefore $Cl(\{P\}) = \{P\}$. Consequently, {P} is closed in $Spec^{cp}(M)$. □

Lemma 2.4. *Let M be a lattice module over a C-lattice L and let Y be a non-empty closed subset of $Spec^{cp}(M)$, then $Y = \cup_{P \in Y} F^c(P)$.*

Proof. Let Y be a non-empty closed subset of $Spec^{cp}(M)$. It is clear that, $Y \subseteq \cup_{P \in Y} F^c(P)$. By Corollary 2.3(1), for each $P \in Y$, $Cl(\{P\}) = F^c(P)$, therefore $F^c(P) = Cl(\{P\}) \subseteq Cl(Y) = Y$. Hence $\cup_{P \in Y} F^c(P) \subseteq Y$. Consequently, $Y = \cup_{P \in Y} F^c(P)$. □

A topological space X is called irreducible if $X \neq \emptyset$ and every finite intersection of non-empty open sets of X is non-empty. A non-empty subset Y of a topological space X is called an irreducible set if the subspace Y of X is irreducible, equivalently, for any two closed sets Y_1 and Y_2 of X, $Y \subseteq Y_1 \cup Y_2$ implies either $Y \subseteq Y_1$ or $Y \subseteq Y_2$ ([8]).

Lemma 2.5. *Let M be a lattice module over a C-lattice L. Then for each $P \in Spec^{cp}(M)$, $F^c(P)$ is irreducible.*

Proof. Suppose that X_1 and X_2 are two closed subsets of $Spec^{cp}(M)$ and $F^c(P) \subseteq X_1 \cup X_2$. Since $P \in F^c(P)$, therefore $P \in X_1 \cup X_2$ which implies either $P \in X_1$ or $P \in X_2$. Suppose that $P \in X_1$. Since X_1 is closed in $Spec^{cp}(M)$, we have $X_1 = \cap_{k \in J}(\cup_{l=1}^{n_k} F^c(N_{kl}))$, for some $N_{kl} \in M$, $k \in J, n_k \in \mathbb{N}$. Thus $P \in \cup_{l=1}^{n_k} F^c(N_{kl})$, for each $k \in J$. It follows that $F^c(P) \subseteq \cup_{l=1}^{n_k} F^c(N_{kl})$, for each $k \in J$. Therefore $F^c(P) \subseteq \cap_{k \in J}(\cup_{l=1}^{n_k} F^c(N_{kl})) = X_1$. Consequently, $F^c(P)$ is irreducible. □

Theorem 2.6. *Let M be a lattice module over a C-lattice L and $Y \subseteq Spec^{cp}(M)$. Then*

(1) *If Y is irreducible, then $Z(Y)$ is a classical prime element.*
(2) *If $Z(Y)$ is a classical prime element and $Z(Y) \in Cl(Y)$, then Y is irreducible.*

Proof. (1) Suppose that Y is an irreducible subset of $Spec^{cp}(M)$. Clearly, $Z(Y) = \wedge_{P\in Y} P < 1_M$ and $Y \subseteq F^c(Z(Y))$. Let $abX \leq Z(Y)$, for $a, b \in L$ and $X \in M$. Now for $P \in Y$, $P \in F^c(Z(Y))$, hence $Z(Y) \leq P$ and therefore $abX \leq Z(Y) \leq P$. Since P is classical prime, either $aX \leq P$ or $bX \leq P$, which implies that $P \in F^c(aX)$ or $P \in F^c(bX)$ and hence $P \in F^c(aX) \cup F^c(bX)$. Therefore $Y \subseteq F^c(aX) \cup F^c(bX)$. Since Y is irreducible, $Y \subseteq F^c(aX)$ or $Y \subseteq F^c(bX)$. If $Y \subseteq F^c(aX)$, then $aX \leq P$, for all $P \in Y$ and hence $aX \leq Z(Y)$. If $Y \subseteq F^c(bX)$, then $bX \leq P$, for all $P \in Y$, hence $bX \leq Z(Y)$. Consequently, Z(Y) is a classical prime element of M.
(2) Suppose that $Z(Y)$ is a classical prime element and $Z(Y) \in Cl(Y)$. We have $Z(Y) \leq P$, for each $P \in Y$, therefore $F^c(P) \subseteq F^c(Z(Y))$, for each $P \in Y$ by Corollary 2.3(2). Thus $Cl(Y) = \cup_{P\in Y} F^c(P) \subseteq F^c(Z(Y))$, by Lemma 2.2. On the other hand, since $Z(Y)$ is a classical prime element and $Z(Y) \in Cl(Y)$, $F^c(Z(Y)) \subseteq Cl(Y)$. Consequently, $Cl(Y) = F^c(Z(Y))$. Now suppose that $Y \subseteq Y_1 \cup Y_2$, where Y_1 and Y_2 are closed subsets of $Spec^{cp}(M)$, then $Cl(Y) \subseteq Y_1 \cup Y_2$ and hence $F^c(Z(Y)) \subseteq Y_1 \cup Y_2$. Since $Z(Y)$ is a classical prime element, $F^c(Z(Y))$ is irreducible by Lemma 2.5. Therefore we have $F^c(Z(Y)) \subseteq Y_1$ or $F^c(Z(Y)) \subseteq Y_2$. It follows that $Y \subseteq Y_1$ or $Y \subseteq Y_2$. Consequently, Y is irreducible. □

Definition 2.7. Let M be a lattice module over a C-lattice L. Let N be an element of M. Then the classical prime radical $\sqrt[cp]{N}$ of N is the meet of all classical prime elements of M containing N, i.e., $\sqrt[cp]{N} = \wedge\{P \in Spec^{cp}(M) | N \leq P\}$.

$\sqrt[cp]{N} = 1_M$, if there is no classical prime element which contains N. An element N is said to be classical prime radical element if $N = \sqrt[cp]{N}$. Note that, $N \leq \sqrt[cp]{N}$ and $F^c(N) = F^c(\sqrt[cp]{N})$.

Corollary 2.8. *Let M be a lattice module over a C-lattice L and let N be any element of M. Then the following are equivalent:*

(1) *The subset $F^c(N)$ of $Spec^{cp}(M)$ is irreducible.*
(2) *$\sqrt[cp]{N}$ is a classical prime element.*

Proof. (1)⇒(2) Suppose that $F^c(N)$ is an irreducible subset of $Spec^{cp}(M)$, then $Z(F^c(N))$ is classical prime element of M, by Theorem 2.6(1). Now, $\sqrt[cp]{N} = \wedge\{P \in Spec^{cp}(M) | N \leq P\} = \wedge\{P \in F^c(N)\} = Z(F^c(N))$. Consequently, $\sqrt[cp]{N}$ is a classical prime element.
(2)⇒(1) Suppose that $\sqrt[cp]{N}$ is a classical prime element, then $F^c(\sqrt[cp]{N})$ is irreducible by Lemma 2.5. Since, for each $N \in M$, $F^c(N) = F^c(\sqrt[cp]{N})$, therefore $F^c(N)$ is an irreducible subset of $Spec^{cp}(M)$. □

Let Y be a closed subset of a topological space. An element $y \in Y$ is called a generic point of Y if $Y = Cl(\{y\})$ (see [3]). Note that, a generic point of the irreducible closed subset Y of a topological space is unique if the topological space is a T_0-space.

Theorem 2.9. *Let M be a lattice module over a C-lattice L. Then*

(1) *$Spec^{cp}(M)$ is always a T_0-space.*
(2) *Every $P \in Spec^{cp}(M)$ is a generic point of the irreducible closed subset $F^c(P)$.*
(3) *Every finite irreducible closed subset of $Spec^{cp}(M)$ has a generic point.*

Proof. (1) Suppose that $P_1, P_2 \in Spec^{cp}(M)$. Then by Corollary 2.3(1), $Cl(\{P_1\}) = F^c(P_1), Cl(\{P_2\}) = F^c(P_2)$ and therefore $Cl(\{P_1\}) = Cl(\{P_2\})$ if and only if $F^c(P_1) = F^c(P_2)$ if and only if $P_1 = P_2$ by Corollary 2.3(2). Now, by the fact that a topological space is a T_0-space if the closures of distinct points are distinct, we conclude that, $Spec^{cp}(M)$ is a T_0-space.
(2) For each $P \in Spec^{cp}(M)$, $F^c(P) = Cl(\{P\})$ by Corollary 2.3(1). Hence P is a generic point of the irreducible closed subset $F^c(P)$.
(3) Suppose that Y is an irreducible closed subset of $Spec^{cp}(M)$ and $Y = \{P_1, P_2, \cdots, P_k\}$, where $P_i \in Spec^{cp}(M)$, $k \in \mathbb{N}$. By Lemma 2.4, $Y = Cl(Y) = F^c(P_1) \cup F^c(P_2) \cup \cdots \cup F^c(P_k)$. Since Y is irreducible, $Y = F^c(P_i)$, for some $i (1 \leq i \leq k)$. By part(2), P_i is a generic point of $F^c(P_i) = Y$. □

Theorem 2.10. *Let M be a lattice module over a C-lattice L such that M has ascending chain condition on classical prime radical elements. Then $Spec^{cp}(M)$ with the Zariski-like topology is a quasi-compact space.*

Proof. Let M be a lattice module over a C-lattice L and suppose that M has ascending chain condition on classical prime radical elements. Let $\mathcal{B}$ be a family of open sets covering $Spec^{cp}(M)$ and suppose that no finite subfamily of $\mathcal{B}$ covers $Spec^{cp}(M)$. Since $F^c(\sqrt[cp]{0_M}) = F^c(0_M) = Spec^{cp}(M)$, we may use the ascending chain condition on classical prime radical elements to choose an element N maximal with respect to the property that no finite subfamily of $\mathcal{B}$ covers $F^c(N)$(we may assume $N = \sqrt[cp]{N}$, because $F^c(N) = F^c(\sqrt[cp]{N})$).

Suppose that N is not classical prime element of M. Then there exists $X \in M$ and $a, b \in L$, such that $abX \leq N$, $aX \nleq N$ and $bX \nleq N$. Thus $N < N \vee aX \leq \sqrt[cp]{N \vee aX}$ and $N < N \vee bX \leq \sqrt[cp]{N \vee bX}$. Hence, without loss of generality, there must exists a finite subfamily $\mathcal{B}'$ of $\mathcal{B}$ that covers both $F^c(N \vee aX)$ and $F^c(N \vee bX)$. Let $P \in F^c(N)$. Since $abX \leq N$, therefore $abX \leq P$ and since P is classical prime, we have $aX \leq P$ or $bX \leq P$. Thus $P \in F^c(N \vee aX)$ or $P \in F^c(N \vee bX)$ and therefore $F^c(N) \subseteq F^c(N \vee aX) \cup F^c(N \vee bX)$. Thus $F^c(N)$ is covered with the finite subfamily $\mathcal{B}^{'}$, which is contradiction. Therefore N is a classical prime element of M.

Now choose $U \in \mathcal{B}$ such that $N \in U$. Thus N must have a neighborhood $\cap_{i=1}^{n} G^c(K_i)$, for some $K_i \in M$ and $n \in \mathbb{N}$, such that $\cap_{i=1}^{n} G^c(K_i) \subseteq U$. Suppose that for each $i(1 \leq i \leq n)$, $P \in G^c(K_i \vee N) \cap F^c(N)$, then $K_i \vee N \nleq P$ and $N \leq P$. Thus $K_i \nleq P$ and $P \in G^c(K_i)$. Consequently, $N \in [G^c(K_i \vee N) \cap F^c(N)] \subseteq G^c(K_i)$ and hence for each $i(1 \leq i \leq n)$, $N \in \cap_{i=1}^{n}[G^c(K_i \vee N) \cap F^c(N)] \subseteq \cap_{i=1}^{n} G^c(K_i) \subseteq U$. Thus $[\cap_{i=1}^{n} G^c(K_i \vee N)] \cap F^c(N)$, where $N < K_i \vee N$ is a neighborhood of N, with $[\cap_{i=1}^{n} G^c(K_i \vee N)] \cap F^c(N) \subseteq U$.

Since for each $i(1 \leq i \leq n)$, $N < K_i \vee N$, $F^c(K_i \vee N)$ can be covered by some finite subfamily $\mathcal{B}_i^{'}$ of $\mathcal{B}$. Now $F^c(N) - [\cup_{i=1}^{n} F^c(K_i \vee N)] = F^c(N) - [\cap_{i=1}^{n} G^c(K_i \vee N)]^{'} = \cap_{i=1}^{n} G^c(K_i \vee N) \cap F^c(N) \subseteq U$(here $'$ denotes complement). Therefore $F^c(N)$ can be covered by $\mathcal{B}_1^{'} \cup \mathcal{B}_2^{'} \cup \cdots \cup \mathcal{B}_n^{'} \cup \{U\}$, which is contradiction to our choice of N. Thus there must exists a finite subfamily of $\mathcal{B}$ which covers $Spec^{cp}(M)$. Therefore, $Spec^{cp}(M)$ is a quasi-compact space. □

A topological space X is a spectral space if X is homeomorphic to $Spec(S)$, with Zariski topology, for some commutative ring S. Spectral spaces have been characterized by Hochster ([12]) as the topological spaces X which satisfy the following conditions.

(1) X is a T_0-space.
(2) X is a quasi-compact.
(3) The quasi-compact open subsets of X are closed under finite intersection and form an open basis.
(4) Each irreducible closed subset of X has a generic point.

Theorem 2.11. *Let M be a lattice module over a C-lattice L with finite spectrum. Then $Spec^{cp}(M)$ is a spectral space.*

Proof. Since $Spec^{cp}(M)$ is finite, by Theorem 2.9, $Spec^{cp}(M)$ is a T_0-space and every irreducible closed subset of $Spec^{cp}(M)$ has a generic point. Also, since $Spec^{cp}(M)$ is finite, it is quasi-compact and the quasi-compact open subsets of $Spec^{cp}(M)$ are closed under finite intersections and form an open basis ([9]). Hence, by Hochster's characterization, $Spec^{cp}(M)$ is a spectral space. □

3. Patch-like topology on $Spec^{cp}(M)$

Let X be a topological space. By the patch topology on X we mean the topology which has as a sub-basis for its closed sets the closed sets and compact open sets of the original space. By a patch we mean set closed in the patch topology ([11], [12]).

Definition 3.1. Let M be a lattice module over a C-lattice L. Let $E(M)$ be the family of all subsets of $Spec^{cp}(M)$ of the form $F^c(N) \cap G^c(K)$, where $N, K \in M$. Clearly $E(M)$ contains both $Spec^{cp}(M) = F^c(0_M) \cap G^c(1_M)$ and empty set $\emptyset = F^c(1_M) \cap G^c(0_M)$. Let $T_p(M)$ be the collection U of all unions of finite intersections of elements of $E(M)$. Then $T_p(M)$ is a topology on $Spec^{cp}(M)$ and is called the patch-like topology of M. In fact $E(M)$ is a sub-basis for the patch-like topology of M.

Note that, the patch-like topology on $Spec^{cp}(M)$ is finer than the Zariski-like topology on $Spec^{cp}(M)$.

Theorem 3.2. *Let M be a lattice module over a C-lattice L. Then $Spec^{cp}(M)$ with the patch-like topology is a Hausdorff space.*

Proof. Suppose that P_1, $P_2 \in Spec^{cp}(M)$ and $P_1 \neq P_2$. Since $P_1 \neq P_2$, so either $P_1 \nleq P_2$ or $P_2 \nleq P_1$. Suppose that $P_1 \nleq P_2$. By Definition 3.1, $U_1 = G^c(1_M) \cap F^c(P_1)$ is a patch-like-neighborhood of P_1 and $U_2 = G^c(P_1) \cap F^c(P_2)$ is a patch-like-neighborhood of P_2. Clearly, $G^c(P_1) \cap F^c(P_1) = \emptyset$ and hence $U_1 \cap U_2 = \emptyset$. Therefore, $Spec^{cp}(M)$ is a Hausdorff space. □

Theorem 3.3. *Let M be a lattice module over a C-lattice L such that M has ascending chain condition on classical prime radical elements. Then $Spec^{cp}(M)$ with the patch-like topology is a compact space.*

Proof. Let M be a lattice module over a C-lattice L and suppose that M has ascending chain condition on classical prime radical elements. Let $\mathcal{A}$ be a family of open sets covering $Spec^{cp}(M)$ and suppose that no finite subfamily of $\mathcal{A}$ covers $Spec^{cp}(M)$. Since $F^c(\sqrt[cp]{0_M}) = F^c(0_M) = Spec^{cp}(M)$, we may use the ascending chain condition on classical prime radical elements to choose an element N maximal with respect to the property that no finite subfamily of $\mathcal{A}$ covers $F^c(N)$ (we may assume $N = \sqrt[cp]{N}$, because $F^c(N) = F^c(\sqrt[cp]{N})$).

Suppose that N is not classical prime element of M. Then there exists $X \in M$ and $a, b \in L$, such that $abX \leq N$, $aX \nleq N$ and $bX \nleq N$. Thus $N < N \vee aX \leq \sqrt[cp]{N \vee aX}$ and $N < N \vee bX \leq \sqrt[cp]{N \vee bX}$. Hence, without loss of generality, there must exists a finite subfamily $\mathcal{A}'$ of $\mathcal{A}$ that covers both $F^c(N \vee aX)$ and $F^c(N \vee bX)$. Let $P \in F^c(N)$. Since $abX \leq N$ and $N \leq P$, we have $abX \leq P$. Since P is classical prime, we have either $aX \leq P$ or $bX \leq P$. Thus $N \vee aX \leq P$ or $N \vee bX \leq P$. Therefore, either $P \in F^c(N \vee aX)$ or $P \in F^c(N \vee bX)$ and hence $F^c(N) \subseteq F^c(N \vee aX) \cup F^c(N \vee bX)$. Thus $F^c(N)$ is covered with the finite subfamily $\mathcal{A}^{'}$, which is contradiction. Therefore N is a classical prime element of M.

Now choose $U \in \mathcal{A}$ such that $N \in U$. Thus N must have a patch-like-neighborhood $\cap_{i=1}^{n}[G^c(K_i) \cap F^c(N_i)]$ for some $K_i, N_i \in M, n \in \mathbb{N}$ such that $\cap_{i=1}^{n}[G^c(K_i) \cap F^c(N_i)] \subseteq U$. Suppose that for each $i(1 \leq i \leq n)$, $P \in [G^c(K_i \vee N) \cap F^c(N)]$. Then $P \in G^c(K_i \vee N)$ and $P \in F^c(N)$ and so that $K_i \vee N \nleq P$ and $N \leq P$. Thus $K_i \nleq P$, i.e., $P \in G^c(K_i)$. On the other hand, $N \in F^c(N_i)$, therefore $N_i \leq N$ and $N_i \leq N, N \leq P$ implies $N_i \leq P$, hence $P \in F^c(N_i)$. Consequently, $N \in [G^c(K_i \vee N) \cap F^c(N)] \subseteq [G^c(K_i) \cap F^c(N_i)]$ and hence $N \in \cap_{i=1}^{n}[G^c(K_i \vee N) \cap F^c(N)] \subseteq \cap_{i=1}^{n}[G^c(K_i) \cap F^c(N_i)] \subseteq U$. Thus $[\cap_{i=1}^{n} G^c(K_i \vee N)] \cap F^c(N)$, where $N < K_i \vee N$, is a neighborhood of N, with $[\cap_{i=1}^{n} G^c(K_i \vee N)] \cap F^c(N) \subseteq U$.
Since for each $i(1 \leq i \leq n)$, $N < K_i \vee N$, $F^c(K_i \vee N)$ is covered by some finite subfamily $\mathcal{A}_i^{'}$ of $\mathcal{A}$. Now $F^c(N) - [\cup_{i=1}^{n} F^c(K_i \vee N)] = F^c(N) - [\cap_{i=1}^{n} G^c(K_i \vee N)]^{'} = [\cap_{i=1}^{n} G^c(K_i \vee N)] \cap F^c(N) \subseteq U$(here $'$ denotes complement). Hence $F^c(N)$ can be covered by $\mathcal{A}_1^{'} \cup \mathcal{A}_2^{'} \cup \cdots \cup \mathcal{A}_n^{'} \cup \{U\}$, which is contradiction to our choice of N. Thus there must exists a finite subfamily of $\mathcal{A}$ which covers $Spec^{cp}(M)$. Therefore, $Spec^{cp}(M)$ is compact in the patch-like topology of M. □

We require the following evident Lemma.

Lemma 3.4. *Let τ_1 and τ_2 be two topologies on X such that $\tau_1 \subseteq \tau_2$. If X is quasi-compact (i.e. any open cover of it has a finite subcover) in τ_2, then X is also quasi-compact in τ_1.*

Theorem 3.5. *Let M be a lattice module over a C-lattice L such that M has ascending chain condition on classical prime radical elements. Then for each $n \in \mathbb{N}$ and elements $N_i(1 \leq i \leq n)$ of M, $G^c(N_1) \cap G^c(N_2) \cap \cdots \cap G^c(N_n)$ is a quasi-compact subset of $Spec^{cp}(M)$ with the Zariski-like topology. Consequently, $Spec^{cp}(M)$ has a basis of quasi-compact open subsets.*

Proof. By Definition 3.1, for each element N of M, $F^c(N) = F^c(N) \cap G^c(1_M)$ is an open subset of $Spec^{cp}(M)$ with the patch-like topology and $G^c(N)$ is a complement of $F^c(N)$. Therefore for each $N \in M$, $G^c(N)$ is a closed subset in $Spec^{cp}(M)$. Thus for each $n \in \mathbb{N}$ and elements $N_i(1 \leq i \leq n)$ of M, $G^c(N_1) \cap G^c(N_2) \cap \cdots \cap G^c(N_n)$ is also a closed subset in $Spec^{cp}(M)$ with the patch-like topology. Since every closed subset of a compact space is compact, therefore $G^c(N_1) \cap G^c(N_2) \cap \cdots \cap G^c(N_n)$ is compact in $Spec^{cp}(M)$ with the patch-like topology and by Lemma 3.4, it is quasi-compact in $Spec^{cp}(M)$ with the Zariski-like topology. Now, $Spec^{cp}(M) = G^c(1_M)$ and $\mathbb{B} = \{G^c(N_1) \cap G^c(N_2) \cap \cdots \cap G^c(N_n) \mid N_i \in M, 1 \leq i \leq n, n \in \mathbb{N}\}$ is a basis for the Zariski-like topology of M. Consequently, $Spec^{cp}(M)$ is quasi-compact and has a basis of quasi-compact open subsets. □

Corollary 3.6. *Let M be a lattice module over a C-lattice L such that M has ascending chain condition on classical prime radical elements. Then quasi-compact open sets of $Spec^{cp}(M)$ (with the Zariski-like topology) are closed under finite intersections.*

Proof. Let U_1 and U_2 be two quasi-compact open sets of $Spec^{cp}(M)$ and let $U = U_1 \cap U_2$. Each of U_1 and U_2 is a finite union of members of $\mathbb{B} = \{G^c(N_1) \cap G^c(N_2) \cap \cdots \cap G^c(N_n) \mid N_i \in M, 1 \leq i \leq n, n \in \mathbb{N}\}$ and hence $U = \cup_{i=1}^{m}(\cap_{j=1}^{n_i} G^c(N_j))$. Let Π be any open cover of U. Then Π also covers each $\cap_{j=1}^{n_i} G^c(N_j)$ which is quasi-compact by Theorem 3.5. Hence each $\cap_{j=1}^{n_i} G^c(N_j)$ has a finite subcover of Π and therefore U has also a finite subcover of Π. Thus U is quasi-compact, as required. □

Lemma 3.7. *Let M be a lattice module over a C-lattice L such that M has ascending chain condition on classical prime radical elements. Then every irreducible closed subset of $Spec^{cp}(M)$ (with the Zariski-like topology) has a generic point.*

Proof. Suppose that Y is an irreducible closed subset of $Spec^{cp}(M)$ with the Zariski-like topology. By Lemma 2.4, we have $Y = \cup_{P \in Y} F^c(P)$. By Definition 3.1, for each $P \in Y$, $F^c(P)$ is an open subset of $Spec^{cp}(M)$ with the patch-like topology. Now, since $Y \subseteq Spec^{cp}(M)$ is closed with the Zariski-like topology, the complement of Y is open by this topology and therefore the complement of Y is open with the patch-like topology. Hence $Y \subseteq Spec^{cp}(M)$ is closed with the patch-like topology. By Theorem 3.3, $Spec^{cp}(M)$ is compact with the patch-like topology and since $Y \subseteq Spec^{cp}(M)$ is closed, therefore Y is also compact. Now, since $Y = \cup_{P \in Y} F^c(P)$ and each $F^c(P)$ is patch-like open, therefore there exists a finite subset X of Y such that $Y = \cup_{P \in X} F^c(P)$. Since Y is irreducible, $Y = F^c(P)$ for some $P \in X$. Therefore, we have $Y = F^c(P) = Cl(\{P\})$ for some $P \in Y$. Consequently, P is a generic point for Y. □

We conclude this section by proving the main theorem .

Theorem 3.8. *Let M be a lattice module over a C-lattice L such that M has ascending chain condition on classical prime radical elements. Then $Spec^{cp}(M)$ with the Zariski-like topology is a spectral space.*

Proof. By Theorem 2.9, $Spec^{cp}(M)$ is a T_0-space. Since M satisfies ascending chain condition on classical prime radical elements, therefore by Theorem 2.10,

$Spec^{cp}(M)$ is quasi-compact. By Theorem 3.5, $Spec^{cp}(M)$ has a basis of quasi-compact open subsets and by Corollary 3.6, the family of quasi-compact open subsets of $Spec^{cp}(M)$ are closed under finite intersections. Finally, by Lemma 3.7, each irreducible closed subset of $Spec^{cp}(M)$ has a generic point. Thus, by Hochster's characterization, $Spec^{cp}(M)$ is a spectral space. □

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References

[1] F. Alarcon, D. D. Anderson and C. Jayaram, *Some results on abstract commutative ideal theory*, Period. Math. Hungar., 30(1) (1995), 1-26.

[2] E. A. Al-Khouja, *Maximal elements and prime elements in lattice modules*, Damascus Univ. J. Basic Sci., 19(2) (2003), 9-21.

[3] M. F. Atiyah and I. G. MacDonald, Introduction to Commutative Algebra, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont. 1969.

[4] S. Ballal and V. Kharat, *Zariski topology on lattice modules*, Asian-Eur. J. Math., 8(4) (2015), 1550066 (10 pp).

[5] M. Behboodi and M. R. Haddadi, *Classical Zariski topology of modules and spectral spaces I*, Int. Electron. J. Algebra, 4 (2008), 104-130.

[6] M. Behboodi and M. R. Haddadi, *Classical Zariski topology of modules and spectral spaces II*, Int. Electron. J. Algebra, 4 (2008), 131-148.

[7] M. Behboodi and M. J. Noori, *Zariski-like topology on the classical prime spectrum of a module*, Bull. Iranian Math. Soc., 35(1) (2009), 253-269.

[8] N. Bourbaki, Algebre Commutative, Chap 1-2, Hermann, Paris, 1961.

[9] N. Bourbaki, Elements of Mathematics, General topology, Part 1, Hermann, Paris; Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont. 1966.

[10] F. Callialp, G. Ulucak and U. Tekir, *On the Zariski topology over an L-module M*, Turkish J. Math., 41(2) (2017), 326-336.

[11] K. R. Goodearl and R. B. Warfield, Jr., An Introduction to Noncommutative Noetherian Rings, Second edition, London Math. Soc. Student Texts, 16, Cambridge Univ. Press, Cambridge, 2004.

[12] M. Hochster, *Prime ideal structure in commutative rings*, Trans. Amer. Math. Soc., 142 (1969), 43-60.

[13] J. A. Johnson, *A-adic completions of Noetherian lattice modules*, Fund. Math., 66 (1970), 347-373.

[14] C. P. Lu, *The Zariski topology on the prime spectrum of a module*, Houston J. Math., 25(3) (1999), 417-432.

[15] C. S. Manjarekar and A. V. Bingi, *Absorbing elements in lattice modules*, Int. Electron. J. Algebra, 19 (2016), 58-76.

[16] J. R. Munkres, Topology: a First Course, Prentice-Hall, Inc. Eglewood Cliffs, New Jersey, 1975.

[17] N. Phadatare, S. Ballal and V. Kharat, *On the second spectrum of lattice modules*, Discuss. Math. Gen. Algebra Appl., 37(1) (2017), 59-74.

[18] N. Phadatare and V. Kharat, *On the second radical elements of lattice modules*, Tbilisi Math. J., 11(4) (2018), 165-173.

[19] N. Phadatare, V. Kharat and S. Ballal, *On the maximal spectrum of lattice modules*, Southeast Asian Bull. Math., Accepted.

[20] N. K. Thakare and C. S. Manjarekar, *Abstract spectral theory: Multiplicative lattices in which every character is contained in a unique maximal character*, Algebra and its applications (New Delhi, 1981), Lecture Notes in Pure and Appl. Math., 91, Dekker, New York, (1984), 265-276.

[21] N. K. Thakare, C. S. Manjarekar and S. Maeda, *Abstract spectral theory II: minimal characters and minimal spectrums of multiplicative lattices*, Acta Sci. Math. (Szeged), 52(1-2) (1988), 53-67.

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