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Differential Equations for a Space Curve According to The Unit Darboux Vector

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Abstract. In this work, the differential equation of a differentiable curve is expressed, by making use of Laplace and normal Laplace operators, as a linear combination of the unit Darboux vector defined as $C = sin\varphi T + cos\varphi B$ of that curve. Later, the necessary and sufficient conditions are given for the space curves to be a 1-type Darboux vector.

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1. Introduction

It is really important to find a relation between a special curve and its curvatures in differential geometry. One of such special curves of this kinds is an helix. It is well-known that the necessary and sufficient condition for a curve to be an helix, in the Euclidean 3-space E^3, is that the ratio of the curvature to the torsion of the given curve must be constant [9]. So many researchers have studied on helices and there are lots of papers focusing exclusively on helices. There have been so many studies in literature, to cite some examples, Chen and Ishikawa classified the biharmonic curves [3, 5]. Later Kocayigit and Hacisalihoglu have studied the space curves and biharmonic curves in the Euclidean 3-space E^3 and Minkowski 3-space E_1^3 [7, 8]. Also Arslan and et al. [2] have given some characterizations of 1-type Darboux vector by using Laplacian and normal Laplacian operators. In this paper, by taking Fenchel's work [4] into account, the differential equation of a space curve, in the Euclidean 3-space, is given first according to the unit Darboux vector and then according to the normal connexion. In the case of helix of the curve, the differential equation obtained from Laplace and normal Laplace operators, is also given.

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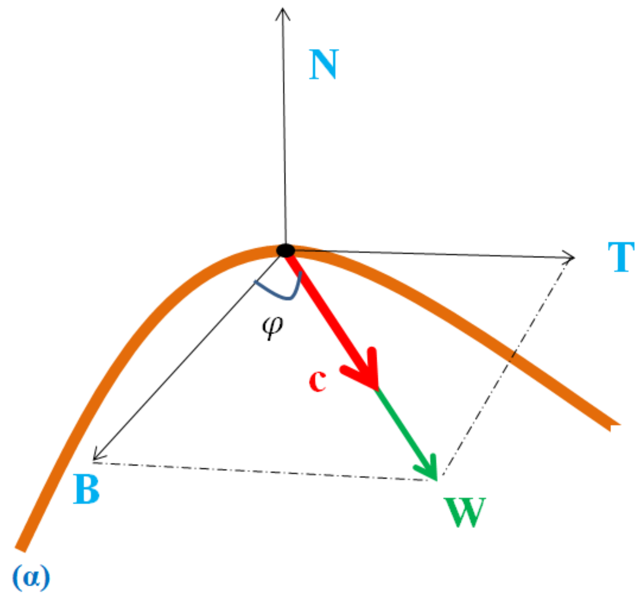

Figure 1. Darboux Vector

2. Preliminaries

Let $\alpha : I \to E^3$, $\alpha(s) = (\alpha_1(s), \alpha_2(s), \alpha_3(s))$ be a differentiable curve with a unit speed. The Frenet frame of this curve is given as

$$T(s) = \alpha^{'}(s), \quad N(s) = \frac{\alpha^{''}(s)}{\| \alpha^{''}(s) \|}, \quad B(s) = T(s) \wedge N(s).$$

If we denote the curvature of the curve α by $\kappa(s)$ and the torsion by $\tau(s)$ then we have

$$\kappa(s) = \| \alpha^{''}(s) \|, \qquad \tau(s) = \frac{\langle \alpha^{'}(s) \wedge \alpha^{''}(s), \alpha^{'''}(s) \rangle}{\| \alpha' \wedge \alpha'' \|^2}.$$

Frenet vectors T, N, B and their derivative vectors satisfy the following Frenet-Serret formulae along the curve α,

$$\begin{aligned} \nabla_{\alpha'} T(s) &= \kappa(s) N(s), \\ \nabla_{\alpha'} N(s) &= -\kappa(s) T(s) + \tau(s) B(s), \\ \nabla_{\alpha'} B(s) &= -\tau(s) N(s) \end{aligned} \tag{2.1}$$

where ∇ is the Levi-Civita connection given by $\nabla_{\alpha'} = \frac{d}{ds}$ and s is the arc length parameter of the curve α [6]. The vector fields T, N, B are called unit tangent vector field, principle normal vector field and binormal vector field of α respectively. The Frenet formulae given in (2.1) may be interpreted as follows: If a moving point traverses the curve in such a way that s is the time parameter, then the moving frame $\{T, N, B\}$ moves according to equations (2.1). This motion contains, apart from an instantaneous translation, instantaneous rotation with angular velocity vector given by the Darboux vector $W = \tau T + \kappa B$, [1]. So the unit Darboux vector is defined as :

$$W = \frac{\tau}{\sqrt{\kappa^2 + \tau^2}} T + \frac{\kappa}{\sqrt{\kappa^2 + \tau^2}} B.$$

If the angle between the Darboux vector W, whose direction is that of instantaneous axis of the rotation and the binormal vector B, is φ, then the unit Darboux vector can be given as

$$C = sin\varphi T + cos\varphi B, \quad sin\varphi = \frac{\tau}{\| W \|}, \; cos\varphi = \frac{\kappa}{\| W \|} \tag{2.2}$$

citefenchel-1951.
Let $\alpha : I \to E^3$ be a differentiable curve then the Laplacian operator of α ,the normal connection of α and the normal Laplacian operator of α are defined as [3, 5]

$$\Delta = -\nabla^2_{\alpha'} = -\nabla_{\alpha'}\nabla_{\alpha'} \quad , \tag{2.3}$$

$$\nabla^{\perp}_{\alpha'}\overrightarrow{\xi} = \nabla_{\alpha'}\overrightarrow{\xi} - \langle \nabla_{\alpha'}\overrightarrow{\xi}, \overrightarrow{T}\rangle \overrightarrow{T}\,, \quad where \;\; \forall \overrightarrow{\xi} \in \chi(\alpha(I))^{\perp}$$

$$\Delta^{\perp} = -\nabla^{\perp(2)}_{\alpha'} = -\nabla^{\perp}_{\alpha'}\nabla^{\perp}_{\alpha'}, \tag{2.4}$$

respectively.

Let C be the unit Darboux vector and Δ be the Laplacian operator of the curve α. Then α is said to be an harmonic Darboux vector if and only if $\Delta C = 0$ and if $\Delta C = \lambda C$ holds this time we call C as an harmonic 1-type provided that λ is constant, [2].

3. Differential Equations for A Space Curve According to The Unit Darboux Vector

In this section, we give the differential equations which characterize a curve α in E^3 , as a linear combination of both the unit Darboux vector C and the normal unit Darboux vector $C^{\perp}$.

Theorem 3.1. *Let $\alpha : I \to E^3$ be a Frenet curve with curvature κ, torsion τ and unit Darboux vector C then the differential equation characterizing the curve α is given by*

$$\nabla^3_{\alpha'}C - \mu_3\nabla^2_{\alpha'}C - \mu_2\nabla_{\alpha'}C - \mu_1 C = 0$$

$$\begin{aligned}
\mu_3 &= \frac{\varphi''}{\varphi'} + \frac{(\varphi' \parallel W \parallel)'}{\parallel W \parallel},\\
\mu_2 &= (\frac{\varphi''}{\varphi'})' - ((\varphi')^2 + \parallel W \parallel^2) - \frac{\varphi''}{\varphi'}\frac{(\varphi' \parallel W \parallel)'}{\parallel W \parallel},\\
\mu_1 &= \frac{\varphi'}{\parallel W \parallel}(\varphi' \parallel W \parallel)' - ((\varphi')^2)'.
\end{aligned}$$

Proof. When we take the derivative of $C = sin\varphi T + cos\varphi B$ with respect to s, we get

$$\nabla_{\alpha'}C = \varphi'(cos\varphi T - sin\varphi B) \tag{3.1}$$

By using the equations (2.2) and (3.1) we can evaluate the values of T and B as

$$T = sin\varphi C + \frac{cos\varphi}{\varphi'}\nabla_{\alpha'}C, \quad B = cos\varphi C - \frac{sin\varphi}{\varphi'}\nabla_{\alpha'}C \tag{3.2}$$

If we take the derivative of equation (3.1) with respect to s, this time we have

$$\nabla^2_{\alpha'}C = \varphi''(cos\varphi T - sin\varphi B) + \varphi'(cos\varphi T - sin\varphi B)'$$

it follows

$$\begin{aligned}
\nabla^2_{\alpha'}C &= \varphi''(cos\varphi T - sin\varphi B) - (\varphi')^2(sin\varphi T + cos\varphi B)\\
&\quad + \varphi'(\kappa cos\varphi + \tau sin\varphi)N
\end{aligned}$$

and taking the equation (2.2) into account with the second derivative of C we obtain

$$\nabla^2_{\alpha'}C = \varphi''(cos\varphi T - sin\varphi B) - (\varphi')^2 C + \varphi' \parallel W \parallel N \tag{3.3}$$

If we put the values of T and B from (3.2) into the equation (3.3) we get

$$\begin{aligned}
\nabla^2_{\alpha'}C &= \varphi'' cos\varphi\Big(sin\varphi C + \frac{\nabla_{\alpha'}C}{\varphi'}cos\varphi\Big) - \varphi'' sin\varphi\Big(cos\varphi C - \frac{\nabla_{\alpha'}C}{\varphi'}sin\varphi\Big)\\
&\quad -(\varphi')^2 C + \varphi' \parallel W \parallel N
\end{aligned}$$

So the second derivative of C is given as

$$\nabla^2_{\alpha'}C = \frac{\varphi''}{\varphi'}\nabla_{\alpha'}C - (\varphi')^2 C + \varphi' \parallel W \parallel N \tag{3.4}$$

and from here we can deduce the normal vector N as

$$N = \frac{1}{(\varphi')^2 \parallel W \parallel}\Big(\varphi'\nabla^2_{\alpha'}C - \varphi''\nabla_{\alpha'}C + (\varphi')^3 C\Big) \tag{3.5}$$

Finally we take the derivative of (3.4) in this case we get

$$\begin{aligned} \nabla^3_{\alpha'}C &= (\frac{\varphi''}{\varphi'})'\nabla_{\alpha'}C + \frac{\varphi''}{\varphi'}\nabla^2_{\alpha'}C - ((\varphi')^2)'C - (\varphi')^2\nabla_{\alpha'}C + (\varphi' \parallel W \parallel)'N \\ &\quad +\varphi' \parallel W \parallel (-\kappa T + \tau B) \end{aligned} \tag{3.6}$$

Now let's put the values of T and B, taken from (3.2), into (3.6) we obtain

$$\begin{aligned} \nabla^3_{\alpha'}C &= \Big((\frac{\varphi''}{\varphi'})' - ((\varphi')^2 + \parallel W \parallel^2)\Big)\nabla_{\alpha'}C + \frac{\varphi''}{\varphi'}\nabla^2_{\alpha'}C \\ &\quad -(\varphi')^2)'C + (\varphi' \parallel W \parallel)'N \end{aligned} \tag{3.7}$$

It is about to finish to obtain the desired differential equation that we substitute (3.5) into the equation (3.7) it becomes

$$\begin{aligned} \nabla^3_{\alpha'}C &= \Big((\frac{\varphi''}{\varphi'})' - ((\varphi')^2 + \parallel W \parallel^2)\nabla_{\alpha'}C + \frac{\varphi''}{\varphi'}\nabla^2_{\alpha'}C - ((\varphi')^2)'C \\ &\quad +(\varphi' \parallel W \parallel)'(\frac{1}{(\varphi')^2 \parallel W \parallel}(\varphi'\nabla^2_{\alpha'}C - \varphi''\nabla_{\alpha'}C + (\varphi')^3 C))\Big) \end{aligned}$$

If we rearrange the above expression we have

$$\begin{aligned} \nabla^3_{\alpha'}C &= \Big(\frac{\varphi''}{\varphi'} + \frac{(\varphi' \parallel W \parallel)'}{\varphi' \parallel W \parallel}\Big)\nabla^2_{\alpha'}C \\ &\quad +\Big((\frac{\varphi''}{\varphi'})' - ((\varphi')^2 + \parallel W \parallel^2) - \frac{\varphi''}{(\varphi')^2 \parallel W \parallel}(\varphi' \parallel W \parallel)'\Big)\nabla_{\alpha'}C \\ &\quad +\Big(\frac{\varphi'}{\parallel W \parallel}(\varphi' \parallel W \parallel)' - ((\varphi')^2)'\Big)C \end{aligned}$$

By writing the coefficients

$$\begin{aligned} \mu_3 &= \Big(\frac{\varphi''}{\varphi'} + \frac{(\varphi' \parallel W \parallel)'}{\varphi' \parallel W \parallel}\Big), \\ \mu_2 &= \Big((\frac{\varphi''}{\varphi'})' - ((\varphi')^2 + \parallel W \parallel^2) - \frac{\varphi''}{(\varphi')^2 \parallel W \parallel}(\varphi' \parallel W \parallel)'\Big), \\ \mu_1 &= \Big(\frac{\varphi'}{\parallel W \parallel}(\varphi' \parallel W \parallel)' - ((\varphi')^2)'\Big) \end{aligned}$$

we get the equation that completes the proof

$$\nabla^3_{\alpha'}C = \mu_3\nabla^2_{\alpha'}C + \mu_2\nabla_{\alpha'}C + \mu_1 C$$

□

Theorem 3.2. *Let $\alpha : I \to E^3$ be a Frenet curve with curvature κ, torsion τ and normal Darboux vector $C^\perp$ then the differential equation characterizing the curve α is given by*

$$\nabla^{\perp(2)}_{\alpha'}C^\perp - \lambda_2(\nabla^\perp_{\alpha'})C^\perp - \lambda_1 C^\perp = 0 \tag{3.8}$$

$$\begin{aligned} \lambda_2 &= \frac{\varphi'\tau sin\varphi - (\tau cos\varphi)'}{\tau cos\varphi}, \\ \lambda_1 &= -\frac{(\varphi'\tau sin\varphi - (\tau cos\varphi)')\varphi' tan\varphi - \tau^3 cos\varphi - \tau(\varphi' sin\varphi)'}{\tau cos\varphi}. \end{aligned}$$

Proof. Let $\alpha : I \to E^3$ be a Frenet curve with curvature κ, torsion τ and normal Darboux vector $C^{\perp}$. We know that

$$C = sin\varphi T + cos\varphi B \qquad and \qquad C^{\perp} = cos\varphi B$$

When we differentiate $C^{\perp} = cos\varphi B$ repeatedly with respect to s, we find that

$$\nabla^{\perp}_{\alpha'} C^{\perp} = \frac{dC^{\perp}}{ds} = -\tau cos\varphi N - \varphi' sin\varphi B \tag{3.9}$$

and

$$\nabla^{\perp(2)}_{\alpha'} C^{\perp} = (\varphi'\tau sin\varphi - (\tau cos\varphi)')N - (\tau^2 cos\varphi + (\varphi' sin\varphi)')B \tag{3.10}$$

Since $C^{\perp} = cos\varphi B$, that is, $B = \frac{1}{cos\varphi} C^{\perp}$. If we put the value of B into the equation (3.9) we get

$$N = -\frac{1}{\tau cos\varphi}(\nabla^{\perp}_{\alpha'} C^{\perp} + \varphi' tan\varphi C^{\perp}).$$

Finally putting the values of $C^{\perp}$ and N into the equality (3.10) we obtain

$$\begin{aligned} \nabla^{\perp(2)}_{\alpha'} C^{\perp} &= \frac{\varphi'\tau sin\varphi - (\tau cos\varphi)'}{\tau cos\varphi}(\nabla^{\perp}_{\alpha'} C^{\perp} + \varphi' tan\varphi C^{\perp}) \\ &\quad -\frac{\tau^2 cos\varphi + (\varphi' sin\varphi)'}{cos\varphi} C^{\perp} \end{aligned}$$

it follows

$$\begin{aligned} \nabla^{\perp(2)}_{\alpha'} C^{\perp} &= \frac{\varphi'\tau sin\varphi - (\tau cos\varphi)'}{\tau cos\varphi}\nabla^{\perp}_{\alpha'} C^{\perp} \\ &\quad -\frac{\Big((\tau cos\varphi)' - \varphi'\tau sin\varphi\Big)\varphi'\tau tan\varphi - \tau^3 cos\varphi - \tau(\varphi' sin\varphi)'}{\tau cos\varphi} C^{\perp} \end{aligned}$$

Setting the coefficients

$$\begin{aligned} \lambda_2 &= \frac{\varphi'\tau sin\varphi - (\tau cos\varphi)'}{\tau cos\varphi}, \\ \lambda_1 &= -\frac{(\varphi'\tau sin\varphi - (\tau cos\varphi)')\varphi' tan\varphi - \tau^3 cos\varphi - \tau(\varphi' sin\varphi)'}{\tau cos\varphi}, \end{aligned}$$

we get required equation which we want to show. □

Corollary 3.3. *Let a differentiable curve α be a circular helix, then the differential equation characterizing the curve according to the normal Darboux vector $C^{\perp}$ is given by*

$$\nabla^{\perp(2)}_{\alpha'} C^{\perp} - \tau^2 C^{\perp} = 0 \tag{3.11}$$

Proof. If α is a circular helix, then we have a constant ratio $\frac{\kappa}{\tau}$. Since
$\frac{\kappa}{\tau} = tan\varphi = const$, so taking derivative of $\frac{\kappa}{\tau}$ with respect to s, yields
$(\frac{\kappa}{\tau})' = \varphi' sec^2\varphi = 0$ or $\varphi' = 0$.
Considering the equation (3.8) we attain $\lambda_2 = 0$ and $\lambda_1 = -\tau^2$. □

Theorem 3.4. *Let $\alpha : I \to E^3$ be a Frenet curve with unit Darboux vector C. Then the unit Darboux vector C is an harmonic Darboux vector if and only if $\varphi' = 0$, where φ is the angle between W and B.*

Proof. Since the unit Darboux vector C is harmonic, that is, $\triangle C = 0$ and also we have (2.3), (3.3) so we can write

$$-\varphi''(cos\varphi T - sin\varphi B) + (\varphi')^2 C - \varphi' \parallel W \parallel N = 0$$

Now it is clear that $\varphi' = 0$. □

Corollary 3.5. *Let $\alpha : I \to E^3$ be a curve with the unit Darboux vector C. Then the curve α is a generalized helix if and only if $\triangle C = 0$, where*
$\triangle$ denotes the Laplacian operator.

Proof. Assume that α is a generalized helix, then the ratio $\frac{\kappa}{\tau}$ is constant along the curve α and we have $\varphi' = 0$.
If we take the equations (2.3) and (3.3) together into account we find that, $\Delta C = 0$.
Conversely, suppose that $\Delta C = 0$ then we have $\varphi' = 0$ that is,
$(\frac{\tau}{\kappa})' = \varphi' sec^2\varphi = 0 \ \Rightarrow \frac{\tau}{\kappa} = constant \ \Rightarrow \ \alpha$ is a generalized helix. □

Corollary 3.6. *Let $\alpha : I \to E^3$ be a generalized helix. Then the unit Darboux vector C belonging to this curve cannot be a 1-type harmonic Darboux vector.*

Proof. By the definition of 1-type harmonic Darboux vector, we write $\Delta C = \lambda C$. Considering the expression (3.3), we have

$$\varphi'' cos\varphi + (\lambda - (\varphi')^2) sin\varphi = 0, \ \ \varphi'' sin\varphi - (\lambda - (\varphi')^2) cos\varphi = 0, \ \ \varphi' \parallel W \parallel = 0$$

it is clear that $\lambda = 0$, that is, C is harmonic so the proof is completed. □

Theorem 3.7. *Let $\alpha : I \to E^3$ be a Frenet curve with normal Darboux vector $C^\perp$ and let $\triangle^\perp$ be a normal Laplacian operator. Then $\Delta^\perp C^\perp = \lambda C^\perp$ holds, provided that $\lambda \in \mathbb{R}$ and*

$$\lambda = \tau^2 + (\frac{\tau'}{2\tau} cos\varphi)' sec\varphi. \tag{3.12}$$

Proof. From (2.4) and (3.10) we can figure out $\Delta^\perp C^\perp$ vector as

$$\Delta^\perp C^\perp = ((\tau cos\varphi)' - \varphi' \tau sin\varphi) N + (\tau^2 cos\varphi + (\varphi' sin\varphi)') B$$

Since we want to reckon $\Delta^\perp C^\perp = \lambda C^\perp$, it follows that

$$(\tau cos\varphi)' - \varphi' \tau sin\varphi = 0 \ \ and \ \ \tau^2 cos\varphi + (\varphi' sin\varphi)' = \lambda cos\varphi \tag{3.13}$$

If $(\tau cos\varphi)' - \varphi' \tau sin\varphi = 0$ then we have

$$\varphi' sin\varphi = \frac{\tau' cos\varphi}{2\tau}$$

Putting this value into (3.13) completes the proof. □

Corollary 3.8. *Let $\alpha : I \to E^3$ be a Frenet curve with normal Darboux vector $C^\perp$ and let $\triangle^\perp$ be a normal Laplacian operator. Then $\Delta^\perp C^\perp = \lambda C^\perp$ holds along the curve, if and only if α is a circular helix, provided that $\tau^2 = \lambda$.*

Proof. Suppose that $\alpha : I \to E^3$ is a circular helix, then we have constant curvatures κ and τ so we can write $\kappa' = \tau' = 0$. It is obvious that from (3.11) we get $\tau^2 = \lambda$.
Conversely assume that $\tau^2 = \lambda = \ const$. Taking account of (3.11) gives us $\nabla^{\perp(2)}_{\alpha'} C^\perp - \tau^2 C^\perp = 0$ or $\Delta^\perp C^\perp = \lambda_1 C^\perp$.
Since $\lambda_1 = \tau^2$, then we get $\Delta^\perp C^\perp = \tau^2 C^\perp$,that is, α is an helix. Therefore $\frac{\tau}{\kappa} = \ const$.
Since $\tau = constant$, so we obtain $\kappa = constant$, that means α is a circular helix. □

Corollary 3.9. *Let $\alpha : I \to E^3$ be a curve with normal Darboux vector $C^\perp$ and let $\triangle^\perp C^\perp = 0$ hold. Then curvatures of the curve entail the following equality*

$$\kappa\tau^2 + \Big(\frac{\tau'\kappa}{2\tau\sqrt{\kappa^2 + \tau^2}}\Big)' \sqrt{\kappa^2 + \tau^2} = 0$$

Proof. If $\triangle^\perp C^\perp = 0$ then because of (3.12) we have

$$\tau^2 + (\frac{\tau'}{2\tau} cos\varphi)' sec\varphi = 0$$

If we substitute $\frac{\kappa}{\|W\|}$ for $cos\varphi$ into (3.12) we obtain desired result and it completes the proof. □

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