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A New General Forward Difference Operator and Some Applications

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Abstract. In this study, the forward difference operator is defined in the most general form. As an application we give some criteria on the behavior of solutions of some first-order difference equations involving this operator.To do this, we use a lemma firstly constructed here that gives the relationship between ordinary difference operator and our new operator. Our main theorem improves the known results in the literature, since the potential function in this equation is of a wider function class, including potential functions in equivalent equations existing in the literature. Also some examples are provided to illustrate our main results.

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1. Introduction

Differential and difference equations have many applications in some fields such as economics, mathematical biology and many areas in applied sciences, we refer the reader to the monographs [1–4] and the references cited therein. The study of solutions of ordinary difference equations has attracted the attention of many authors for many years. In last ten years they defined a generalized difference operator Δ_a, where a is a real number, and by using lemma that gives the relationship between the ordinary difference operator Δ and generalized difference operator Δ_a, gave a number of criteria on the behavior of the difference equations involving operator Δ_a. Some of them are given below: Second order non-linear difference equation involving the generalized difference operators Δ_a and Δ_b of the form

$$\Delta_a(p_n\Delta_a x_n) + q_n\Delta_a x_n = f(n, x_n, \Delta_b x_n)$$

has been investigated by Tan and Yangs [19] and obtained some oscillatory and nonoscillatory results. After that a class of nonlinear third order difference equations involving the generalized difference operator Δ_a that is the more general case of the above equation

$$\Delta_a(p_n\Delta_a^2 y_n) + q_n\Delta_a^2 y_n = f(n, y_n, \Delta_a y_n, \Delta_a^2 y_n).$$

This last equation has been considered by Parhi and Panda [16] and obtained sufficient conditions for being oscillatory or nonoscillatory. In [15], Parhi has investigated second order difference equations involving generalized difference operator of the forms

$$\Delta_a(p_{n-1}\Delta_a y_{n-1}) + q_n y_n = 0, n \geq 1$$

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and

$$\Delta_a(p_{n-1}\Delta_a y_{n-1}) + q_n y_n = f_n, n \geq 1,$$

and obtained some oscillatory and nonoscillatory criteria.

In [6], Bolat and Akin have investigated the oscillatory properties of solutions of the difference equation of

$$\Delta_b(p_n(\Delta_b^{m-1}x_n)^\alpha) + q_n x_{n-\sigma}^\beta = 0,$$

and they obtained some new results.

Bolat [5] has given some results on the trichotomy of nonoscillatory solutions for the difference equation involving the generalized difference operator Δ_a of the form

$$\Delta_a(p_n(\Delta_a(x_n - a^{1-n}x_{n-1}))^\alpha) + q_n x_{n-\sigma}^\beta = 0, n \in \mathbb{N}_{\max\{1,\sigma\}}.$$

In the present work, we define a new generalized difference operator Δ_{a_n} that we give "the variable operator" name, in the form $\Delta_{a_n}y_n = y_{n+1} - a_n y_n$, where a_n is a sequence of real numbers, and consider delay difference equation involving this new generalized difference operator Δ_{a_n} of the form

$$\Delta_{a_n}y_n + p_n y_{n-k} = 0, \quad n, k \in \mathbb{N}, \tag{1.1}$$

where a_n and p_n are real sequences.

The organization of this paper is as follows. Firstly we give the solutions of some classes of the first order ordinary difference equations and the first order generalized difference equations involving generalized difference operator, and secondly we give some results on behavior of their solutions in the second section. In the third section we give some lemmas to prove the main results and to study behavior of solutions of the first order delay difference equations involving variable difference operator, and give a main result. In the last section some examples are provided to illustrate our main results.

By a solution of Eq. (1.1) we mean a sequence (y_n) which satisfies Eq. (1.1) for sufficiently large n.We consider only such solutions which are nontrivial for all large n. A solution of Eq. (1.1) is called nonoscillatory if it is eventually positive or eventually negative. Otherwise it is called oscillatory.

2. On The Behaviour of Solutions of The First Order Difference Equations

Firstly, let's consider the simplest ordinary difference equation of the form

$$\Delta y_n = 0. \tag{2.1}$$

The solution of equation (2.1) is $y_n = c$, where $c = y_0$ is a constant. Secondly, let's consider the simplest difference equation involving generealized difference operator Δ_a of the form

$$\Delta_a y_n = 0, \tag{2.2}$$

where a is nonzero real number. The solution of equation (2.2) is

$$y_n = ca^n, \tag{2.3}$$

where $c = y_0$ is an arbitrary constant. The behavior of the solution (2.3) is very strongly determined by the value and sign of the constant a. We can say: $|a| > 1$ gives growing behavior (exponential growth), $|a| = 1$ gives stabil behavior, $|a| < 1$ gives decaying behavior (exponential decay), $0 < a < 1$ gives stable behavior, $a < 0$ gives oscillating behavior. Note that, while the solution of the equation (2.1) shows a fixed sign behavior, the behavior of the solution (2.3) of the equation (2.2) can be in five different ways according to the state of a.

Finally, let us consider the simplest difference equation involving variable difference operator Δ_{a_n} of the form

$$\Delta_{a_n}y_n = 0. \tag{2.4}$$

The solution of equation (2.4) is

$$y_n = cb_n, \tag{2.5}$$

where $c = y_0$ is an arbitrary constant and $b_n = \prod_{i=0}^{n-1} a_i$. The results on the behavior of the solution (2.5) of the equation (2.4) can be given as similar to the conclusion of Parhi below.

Parhi [14] considered the difference equations of the form

$$y_{n+1} + a_n y_n = 0 \tag{E_1}$$

and the associated nonhomogeneous equation

$$y_{n+1} + a_n y_n = b_n,$$

and gave the following criteria on the behavior of solutions of the equation (E_1):

Theorem 2.1 ([14]). ***i.*** *If $a_n < 0$; $n \geq 0$, then equation (E_1) is disconjugate on $(0, \infty)$.*
ii. *If $a_n > 0$; $n \geq 0$, then equation (E_1) is oscillatory.*
iii. *If $\{a_n\}$ is oscillatory in the sense that for every $N \geq 0$ there exists $n \geq N$ such that $a_{n-1}a_n < 0$, then equation (E_1) is oscillatory.*

Although all of the above equations are in the first order, we can easily see that each of these equations has different solutions and different behaviors.

Many authors studied the oscillation of the delay difference equation with constant coefficient of the form

$$\Delta x_n + p x_{n-k} = 0, \quad n = 0, 1, 2, ..., \tag{E_2}$$

where p is a constant number, and the delay difference equation of the form

$$\Delta x_n + p_n x_{n-k} = 0, \quad n = 0, 1, 2, ..., \tag{E_3}$$

where $\Delta x_n = x_{n+1} - x_n$, $\{p_n\}$ is a sequence of nonnegative real numbers and k is a positive integer. For example; In [12, p.167, Theorem 7.2.1] been showed that for $p \in \mathbb{R}$ and $k \in \mathbb{Z}$, every solution of (E_2) is oscillatory if and only if the following conditions holds:
(a) $k = -1$ and $p \leq -1$;
(b) $k = 0$ and $p \geq 1$;
(c) $k \in \{..., -3, -2\} \cup \{1, 2, ...\}$ and $p\frac{(k+1)^{k+1}}{k^k} > 1$.

In [10] Domshlak has studied the difference equation when $k = 1$ in Equation (E_3) in the form

$$\Delta x_n + p_n x_{n-1} = 0, n = 0, 1, 2, ...,$$

and has given some oscillation results.

Then in [11] Erbe and Zhang showed that if

$$\liminf_{n\to\infty} p_n > \frac{k^k}{(k+1)^{k+1}} \tag{2.6}$$

or

$$\limsup_{n\to\infty} \sum_{i=n-k}^{n} p_i > 1$$

then all the solutions of the equation (E_3) are oscillatory.

In [13] Ladas, Philos and Sficas improved the condition (2.6) and showed that

$$\liminf_{n\to\infty} \left[\frac{1}{k} \sum_{i=n-k}^{n-1} p_i \right] > \frac{k^k}{(k+1)^{k+1}}$$

is a sufficient condition for the oscillation of every solution of the delay difference equation (E_3).

In [17] Stavroulakis proved that if

$$0 < \alpha_0 := \liminf_{n\to\infty} \sum_{i=n-k}^{n-1} p_i \leq \left(\frac{k}{k+1} \right)^{k+1} \tag{2.7}$$

and

$$\limsup_{n\to\infty} p_n > 1 - \frac{\alpha_0^2}{4}$$

are satisfied then all the solutions of equation (E_3) oscillate.

In [9] Chen and Yu obtained that if (2.7) holds and that

$$\limsup_{n\to\infty}\sum_{i=n-k}^{n} p_i > 1 - \frac{1-\alpha_0-\sqrt{1-2\alpha_0-\alpha_0^2}}{2} \tag{2.8}$$

is satisfied then every solution of (E_3) oscillates.

In [18] Stavroulakis improved the condition (2.8) as

$$\limsup_{n\to\infty}\sum_{i=n-k}^{n} p_i > 1 - \frac{\alpha_0^2}{4}$$

or

$$\limsup_{n\to\infty}\sum_{i=n-k}^{n} p_i > 1 - \frac{\alpha_0^k}{4}.$$

In [8] Chatzarakis and Stavroulakis obtained the following condition for the oscillation of every solution of equation (E_3)

$$\limsup_{n\to\infty}\sum_{i=n-k}^{n-1} p_i > 1 - \frac{\alpha_0^2}{2(2-\alpha_0)}. \tag{2.9}$$

In [7] Chatzarakis, Koplatadze and Stavroulakis improved the condition (2.9) as

$$\limsup_{n\to\infty}\sum_{i=n-k}^{n} p_i > 1 - \left(1-\sqrt{1-\alpha_0}\right)^2. \tag{2.10}$$

Now we consider a class of first order linear difference equations with constant coefficients and involving generalized difference operator Δ_a, which is the more general form of (E_2) and (E_3) of the form

$$\Delta_a y_n + p y_{n-k} = 0, \;\; n = 0, 1, 2, ... \tag{2.11}$$

where $k \in \mathbb{Z}$ and $a, p \in \mathbb{R}$. The characteristic equation for equation (2.11) is

$$\lambda^{k+1} - a\lambda^k + p = 0. \tag{2.12}$$

Now similar as in the above results $(a) - (c)$, we can give immediately the following results according to the characteristic equation (2.12).

Theorem 2.2. *Every solution of (2.11) is oscillatory if and only if the following conditions hold:*
(c_1) $k = -1$ *and* $\frac{a}{1+p} < 0$*;*
(c_2) $k = 0$ *and* $a - p < 0$*;*
(c_3) $k = 1$ *and* $a^2 - 4p < 0$ *or* $a < 0$ *and* $p > 0$.

Theorem 2.3. *Let* $a < 0$*. Every solution of (2.11) is oscillatory if and only if the following conditions hold:*

(c_4) k is a positive odd integer and $p > 0$;
(c_5) k is a positive even integer and $p < 0$;
(c_6) k is a negative odd integer such that $k < -1$ and $p < 0$;
(c_7) k is a negative even integer such that $k < -1$ and $p > 0$.

Theorem 2.4. *Let* $a > 0$*. Every solution of (2.11) is oscillatory if and only if the following condition holds:*

(c_8) $k > 0$ is odd integer and $p > \left(\frac{a}{k+1}\right)^{k+1} k^k$.

The proof of Theorem 2.2. According to the characteristic equation (2.12), the proof can easily be made. We omit it in here. □

The proof of Theorem 2.3. In here we will prove the condition (c_4). Proves of the other conditions of the theorem can be made in a similar manner. Set $f(\lambda) := \lambda^{k+1} - a\lambda^k + p$ in (2.12). We have $f^{'}(\lambda) = (k+1)\lambda^k - ak\lambda^{k-1}$. From here we obtain nonzero critical point $\lambda_0 = \frac{ak}{k+1} < 0$. The other hand we have $f^{''}(\lambda_0 = \frac{ak}{k+1}) = \left(\frac{ak}{k+1}\right)^k \frac{(k+1)^2}{ak} > 0$. That is the function f has its minimum value on the point $\lambda_0 = \frac{ak}{k+1}$ as $f_{\min}(\frac{ak}{k+1}) = -\left(\frac{a}{k+1}\right)^{k+1} k^k + p$. Here it must be that $-\left(\frac{a}{k+1}\right)^{k+1} k^k + p > 0$, that is $p > \left(\frac{a}{k+1}\right)^{k+1} k^k$. Since $a < 0$ and k is an odd integer, $p > 0$. That is $f(+0) > 0$, $f(-0) > 0$, $f(\infty) = \infty$ and $f(-\infty) = \infty$. Therefore the characteristic equation (2.12) has no real roots. Hence every solution of (2.11) is oscillatory. The proof is completed. □

The proof of Theorem 2.4. The proof is completely similar to the proof of Theorem 2.3. □

Secondly consider the first order linear difference equation involving generalized difference operator Δ_a, which is the more general form of the equation (2.11) of the form

$$\Delta_a y_n + p_n y_{n-k} = 0, \; n = 0, 1, 2, ... \tag{2.13}$$

Here we can give immediately the following results similar to the above results.

Theorem 2.5. *Suppose that*

$$\liminf_{n\to\infty} p_n = p. \tag{2.14}$$

Every solution of (2.13) *is oscillatory if and only if the conditions* $(c_1) - (c_3)$ *hold.*

Theorem 2.6. *Let be* $a < 0$ *and satisfy* (2.14)*. Every solution of* (2.13) *is oscillatory if and only if the conditions* $(c_4) - (c_7)$ *hold.*

Theorem 2.7. *Let be* $a > 0$ *and satisfy* (2.14)*. Every solution of* (2.13) *is oscillatory if and only if the condition* (c_8) *holds.*

Proofs of Theorem 2.5, Theorem 2.6 and Theorem 2.7. The proofs can be done similarly as Theorems 2.2, 2.3 and 2.4, respectively. □

Finally consider a class of first order linear difference equations involving variable difference operator Δ_{a_n}, which is the more general form of (2.13) of the form

$$\Delta_{a_n} y_n + p_n y_n = 0, n \in \mathbb{N}, \tag{2.15}$$

where a_n and p_n are real sequences, $n \geq 0$. One can easily see that the solution of equation (2.15) is

$$y_n = c \prod_{i=0}^{n-1} (a_i - p_i),$$

where $c = y_0$ is any real constant.

Here we can give immediately the following the results.

Theorem 2.8. *Let* a_n *and* p_n *be rael sequences.When each of the following cases is satisfied, then all the solutions of equation* (2.15) *are oscillatory.*

(i) $(a_n - p_n) < 0$ *for all* $n \in \mathbb{N}$,

(ii) $(a_n - p_n)$ *is oscillatory for all* $n \in \mathbb{N}$.

Proof. The proof is obvious. Therefore we omit it in here. □

3. Main Results

Now let's examine the behavior of the solutions of the first-order delay difference equation involving new generalized difference operator Eq. (1.1), which is the more general form of equation (2.15).

In this section, we consider Eq. (1.1) and give some results for the oscillation of its all solutions. For the investigation of oscillatory of solutions of Eq. (1.1), we need to use the following Lemma which gives relation between the variable forward difference operator Δ_{a_n} and the ordinary forward difference operator Δ.

Lemma 3.1. *Let* (a_n) *be a real valued sequence. Then for any real valued sequence* (y_n)

$$\Delta_{a_n} y_n = b_n \Delta(\frac{y_n}{b_{n-1}})$$

and

$$\Delta_{a_n}^m y_n = b_n[\Delta a_n(]^{(m-1)}\Delta(\frac{y_n}{b_{n-1}}))...), \quad n \in \mathbb{N} \tag{3.1}$$

are satisfied. Where $b_n = \prod_{i=0}^n a_i$ *and* $[\Delta(a_n]^{(m-1)} = \Delta(a_n\,[\Delta(a_n]^{m-2}$.

Proof. The proof can be easly done by the mathematical induction method and according to the definition of variable difference operator Δ_{a_n}. □

Corollary 3.2. *If* $a_n \equiv 1$ *in the Lemma 3.1, then* (3.1) *be*

$$\Delta_1^m y_n = \Delta^m y_n,$$

where $\Delta_1 = \Delta$.

If $a_n \equiv a$*, then* (3.1) *be*

$$\Delta_a^m y_n = a^{n+m}\Delta^m(\frac{y_n}{a^{n+m-1}}).$$

By using Lemma 3.1, we can rewrite Eq. (1.1) in the form

$$\Delta\left(\frac{y_n}{\prod_{i=0}^{n-1} a_i}\right) + \frac{p_n}{\prod_{i=0}^{n} a_i} y_{n-k} = 0$$

or

$$\Delta\left(\frac{y_n}{b_{n-1}}\right) + p_n \frac{y_{n-k}}{b_n} = 0 \tag{3.2}$$

In equation (3.2) by putting up $\frac{y_n}{b_{n-1}} = z_n$, we reach the equation of the form

$$\Delta z_n + p_n^* z_{n-k} = 0, \tag{3.3}$$

where $p_n^* = p_n \prod_{i=n-k}^n \frac{1}{a_i}$.

Now we can state the following oscillation results for the Eq. (1.1).

Theorem 3.3. *Let* p_n^* *be nonnegative for* $n \geq 0$*. Assume that*

$$\alpha_0 := \liminf_{n\to\infty} \sum_{i=n-k}^{n-1} p_i^* \leq \left(\frac{k}{k+1}\right)^{k+1} \tag{3.4}$$

and

$$\limsup_{n\to\infty} \sum_{i=n-k}^{n} p_i^* > 1 - \left(1 - \sqrt{1-\alpha_0}\right)^2 \tag{3.5}$$

are satisfied then every solution of Eq. (1.1) oscillates.

Proof. If we adapt conditions (3.4) and (3.5) to the conditions (2.7) and (2.10) respectively the proof can be done as in [7, 17]. Therefore we can say that all solutions of equation (3.3) are oscillatory. Hence, since $\frac{y_n}{b_{n-1}} = z_n$ and a_n is a positive (or negative) real sequence for $n \in \mathbb{N}$, all solutions y_n of Eq. (1.1) are oscillatory too. This completes the proof. □

4. Examples

Example 4.1. Consider the generalized delay difference equation

$$\Delta_n x_n + p_n x_{n-4} = 0, \ n = 0, 1, 2, ... \tag{4.1}$$

where $p_n = \begin{cases} \frac{1}{20}\frac{n!}{(n-5)!} & , n \neq 0(mod\ 5) \\ \frac{79}{100}\frac{n!}{(n-5)!} & , n = 0(mod\ 5) \end{cases}$.

Here $a_n = n$ and $k = 4$. We see that

$$\alpha_0 := \liminf_{n\to\infty} \sum_{i=n-k}^{n-1} p_i \prod_{j=n-k}^{i} \frac{1}{a_j} = \frac{1}{5} < \left(\frac{4}{5}\right)^4 \approx 0,32768$$

and

$$\limsup_{n\to\infty} \sum_{i=n-k}^{n} p_i \prod_{j=n-k}^{i} \frac{1}{a_j} = 0,99 > 1 - \left(1 - \sqrt{1-\alpha_0}\right)^2 = 0.96.$$

Since the conditions (3.4) and (3.5) are satisfied, then every solution of equation (4.1) oscillates.

Example 4.2. Consider a delay difference equation with generalized variable difference operator of the form

$$\Delta_{\left(\frac{1}{7}+\frac{4}{3^{2n}}\right)} y_n + \left(\frac{9}{56} + \frac{1}{3^{2n}}\right) y_{n-2} = 0, \ n \in \mathbb{N}, \tag{4.2}$$

here $a_n = (\frac{1}{7} + \frac{4}{3^{2n}})$ and $p_n = \left(\frac{9}{56} + \frac{1}{3^{2n}}\right)$ and $k = 2$. Since

$$\alpha_0 := \liminf_{n\to\infty} \sum_{i=n-2}^{n-1} \left(\frac{9}{56} + \frac{1}{3^{2i}}\right) \prod_{j=n-2}^{i} \frac{7(3^{2j})}{28 + 3^{2j}} = 0,011775 < \frac{8}{27}$$

and

$$\limsup_{n\to\infty} \sum_{i=n-2}^{n-1} \left(\frac{9}{56} + \frac{1}{3^{2i}}\right) \prod_{j=n-2}^{i} \frac{7(3^{2j})}{28 + 3^{2j}} = 9 > 0,999,$$

all the conditions of Theorem 3.1 are satisfied . Therefore all solutions of equation (4.2) are oscillatory. One of such solutions is $y_n = \left(-\frac{1}{2}\right)^n$. Whereas in the case $a_n = 1$, equation (4.2) is not oscillatory.

Example 4.3. Consider a delay difference equation with generalized variable difference operator of the form

$$\Delta_{a_n} y_n + p_n y_{n-2} = 0, \ n \in \mathbb{N}, \tag{4.3}$$

here $a_n = (\frac{1}{14} + \frac{16}{3^{2n}})$ and $p_n = \left(\frac{1}{14} + \frac{1}{3^{2n}}\right)$ and $k = 2$. Since

$$\alpha_0 := \liminf_{n\to\infty} \sum_{i=n-2}^{n-1} \frac{3^{2i} + 14}{14(3^{2i})} \prod_{j=n-2}^{i} \frac{14(3^{2j})}{224 + (3^{2j})} = 0.0634 < \frac{8}{27}$$

and

$$\limsup_{n\to\infty} \sum_{i=n-2}^{n-1} \frac{3^{2i} + 14}{14(3^{2i})} \prod_{j=n-2}^{i} \frac{14(3^{2j})}{224 + (3^{2j})} = 15 > 1 - (1 - \sqrt{1 - 0.0634})^2 = 0.999,$$

the conditions of Theorem 3.1. are satisfied. Therefore the equation (4.3) is oscillatory. One of such solutions is $y_n = \left(-\frac{1}{2}\right)^n$. When $a_n = 1$ equation (4.3) becomes

$$\Delta y_n + p_n y_{n-2} = 0, \ n \in \mathbb{N},$$

which is also oscillatory.

Example 4.4. Consider delay difference equation with generalized variable difference operator of the form

$$\Delta_{a_n} y_n + p_n y_{n-1} = 0, \ n \in \mathbb{N}. \tag{4.4}$$

Taking $a_n = \frac{10n^2+n+9}{9(n^2-1)}$ and $p_n = \frac{n+9}{9n}$, we see that the conditions of Theorem 3.1

$$\alpha_0 := \liminf_{n\to\infty} \sum_{i=n-k}^{n-1} p_i \prod_{j=n-k}^{i} \frac{1}{a_j} = 0.1 < \frac{1}{4}$$

is satisfied, but

$$\limsup_{n\to\infty} \sum_{i=n-k}^{n} p_i \prod_{j=n-k}^{i} \frac{1}{a_j} = 0.324 < 1$$

is not satisfied. Hence All solutions of equation (4.4) are not oscillatory. One of such solutions is $y_n = \frac{1}{n}$. Whereas in the case $a_n = 1$, equation (4.4) becomes

$$\Delta y_n + p_n y_{n-1} = 0, \; n \in \mathbb{N},$$

which is oscillatory. Because $\alpha_0 := \liminf_{n\to\infty} \sum_{i=n-k}^{n-1} p_i = \frac{1}{9} < \frac{1}{4}$ and $\limsup_{n\to\infty} \sum_{i=n-k}^{n} p_i = 1.1 > 0.956$.

Corollary 4.5. *When $a_n \equiv 1$, conditions (3.4) and (3.5) are reduced to the conditions (2.7) and (2.10) respetctively. When $b \equiv a$, conditions (3.4) and (3.5) are corresponded one or more of appropriate in the conditions ($c_1 - c_8$).*

5. Conclusion

In equation (E_3), potential function p_n does not satisfy the oscillation conditions for Eq. (1.1) (or equation (3.3)) to be oscillatory of its every solution, see above examples. That is, the function p_n in equation (E_3) can not be taken of place of p_n in Eq. (1.1) (or p_n^* in equation (3.3)). However it can easily be seen that the potential function p_n in Eq. (1.1) satisfies the oscillation conditions in equation (E_3), see above examples. Therefore our results are new and improve previous results.

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