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Research Article

Marshall-Olkin Half Logistic Distribution with Theory and Applications

Diren Yeğen
M.Sc., Ankara, Turkey, direnyegen@gmail.com

Gamze Özel, Ph.D.
Assoc. Prof., Department of Statistics, Faculty of Science, Hacettepe University, Ankara, Turkey, gamzeozl@hacettepe.edu.tr

* Hacettepe Üniversitesi, Fen Fakültesi, İstatistik Bölümü, 06800-Beytepe, Ankara, Türkiye

ABSTRACT

In this study we present a new distribution named as Marshall-Olkin Half Logistic (MOHL) by extending half logistic distribution to increase the flexibility. We derive some fundamental properties including survival function, hazard rate function, skewness, kurtosis, order statistics and entropy function. Parameters are estimated using maximum likelihood estimation method to fit new model. Then, a simulation study is conducted to show the performance of the proposed model

Keywords: Half Logistic Distribution, Maximum Likelihood Estimation, Simulation, Entropy, Order Statistics.

Teori ve Uygulamaları ile Marshall-Olkin Yarı Lojistik Dağılımı

ÖZ

Bu çalışmada esnekliği arttırmak için yarı lojistik dağılımı genişletilerek yeni bir dağılım olarak Marshall-Olkin Yarı Lojistik (MOYL) dağılımı önerilmiştir. Yaşam fonksiyonu, tehlike oranı fonksiyonu, çarpıklık, basıklık, sıralı istatistikler ve entropi fonksiyonu gibi temel özellikler elde edilmiştir. Yeni modelin uyumu için en çok olabilirlik yöntemi kullanılmıştır Daha sonra önerilen modelin performansı için bir simülasyon çalışması yapılmıştır.

Anahtar Kelimeler: Yarı Lojistik Dağılımı, En Çok Olabilirlik Tahmini, Simülasyon, Entropi, Sıralı İstatistik

1. Introduction

The half-logistic (HL) distribution is one of the probability distributions which is a member of the family of logistic distribution. Its probability density function (pdf) is given by

$$g(x)=\frac{2\lambda e^{-\lambda x}}{\left(1+e^{-\lambda x}\right)^2} \quad x>0,\quad \lambda>0 \tag{1}$$

The corresponding cumulative distribution function (cdf) is given by

$$G(x)=\frac{1-e^{-\lambda x}}{1+e^{-\lambda x}} \quad x>0,\quad \lambda>0 \tag{2}$$

Balakrishnan (1985) studied order statistics from the HL distribution. Balakrishnan and Puthenpura (1986) obtained the best unbiased estimates of the location and scale parameter of the distribution while Olapade (2003) presented some theorems to characterize the distribution. Balakrishnan and Wong (1991) obtained maximum likelihood estimates (MLEs) for the location and scale parameters of the HL distribution. Torabi and Bagheri (2010) presented an extended generalized HL distribution and studied different methods for estimating its parameters based on complete and censored data.

In lifetime study more distributions with flexibility required that can accommodate different kind of data set in practice. In consideration of these kinds of problems, several authors give an important attention to HL distribution in recent years and proposed various extensions and new forms of the HL distribution such as the generalized HL (GHL), power HL (PwHL) by Krishnarani (2016), generalized HL (OGHL) by Olapade (2014), exponentiated HL family of distributions (EHL-G) by Cordeiro et al. (2014), type I half-logistic family by Cordeiro et al. (2016).

Adding parameters to well-established distribution is a time honored device for obtaining more flexible new families of distributions. Marshall and Olkin (1997) introduced an interesting method of adding a new parameter to an existing distribution. The resulting distribution, known as Marshall-Olkin (MO) extended distribution, includes the baseline distribution as a special case and gives more flexibility to model various types of data. The MO family of distributions is also known as the proportional odds family (proportional odds model) or family with tilt parameter (Marshall and Olkin 1997). The probability density function (pdf) and cumulative density function (cdf) are given by respectively;

$$F(x)=\frac{G(x)}{G(x)+b\left(1-G(x)\right)},\ b>0 \tag{3}$$

$$f(x)=\frac{bF(x)}{\left[1-(1-b)\left(1-F(x)\right)\right]^2},\ b>0 \tag{4}$$

Some special cases discussed in the literature include the MO extensions of the Weibull distribution (Ghitany et al., 2005a, Zhang and Xie, 2007), Pareto distribution (Ghitany et al., 2005b), gamma distribution (Ristic´ et al., 2007), Lomax distribution (Ghitany et al., 2007) and linear failure-rate distribution (Ghitany and Kotz, 2007). More recently, Gómez-Déniz (2010) presented a new generalization of the geometric distribution using the MO scheme. Economou and Caroni (2007) showed that the MO extended distributions have a proportional odds property and Caroni (2010) presented some Monte Carlo simulations considering hypothesis testing on the parameter α for the extended Weibull distribution. The maximum likelihood estimation for the MO family is given in Lam and Leung (2001) and Gupta and Peng (2009). Gupta et al. (2010) compared this family and the original distribution with respect to some stochastic orderings and also investigate thoroughly the monotonicity of the failure rate of the resulting distribution when the baseline distribution is Weibull. Nanda and Das (2012) investigated the tilt parameter of the M-O extended family.

The main purposes of the study are to develop MOHL distribution and investigates its fundamental properties including moments, moment generating function (mgf), entropies, mode etc. Moreover, estimate the parameters by using maximum likelihood estimation (MLE) with an application.

2. Marshall Olkin Half Logistic Distribution

In this section, we introduce MOHL distribution and obtain cdf, pdf, survival, hazard and quantile functions. Using (2) in (3), the cdf of the MOHL is given by

$$\begin{aligned} F_{MOHL}(x) &= \frac{\dfrac{1-e^{-\lambda x}}{1+e^{-\lambda x}}}{\dfrac{1-e^{-\lambda x}}{1+e^{-\lambda x}} + b\left(1-\dfrac{1-e^{-\lambda x}}{1+e^{-\lambda x}}\right)} \\ &= \frac{1-e^{-\lambda x}}{1-e^{-\lambda x} + b\left(1-\dfrac{1-e^{-\lambda x}}{1+e^{-\lambda x}}\right)} \end{aligned} \tag{5}$$

The pdf corresponding to Eq. (5) is obtained as

$$f_{MOHL}(x) = \frac{2\lambda \cdot b \cdot e^{\lambda x}}{\left(e^{\lambda x} + 2b - 1\right)} \tag{6}$$

where x>0, λ>0 and b>0. From Eq. (5), we obtain survival function of the MOHL distribution as follows:

$$\begin{aligned} S_{MOHL}(x) &= 1 - F(x) \\ &= \frac{2be^{-\lambda x}}{1-e^{-\lambda x} + 2be^{-\lambda x}} \end{aligned} \tag{7}$$

The MOHL distribution can be applied in survival analysis, hydrology, economics. The other characteristic of the random variable is the hazard function. It is an important quantity characterizing life phenomenon. It can be loosely interpreted as the conditional probability of failure, given it has survived to time t. Then, the hazard function is given by

$$\begin{aligned} h_{MOHL}(x) &= \frac{f(x)}{1-F(x)} \\ &= \frac{\lambda e^{2\lambda x}}{\left(e^{\lambda x} + 2b - 1\right)^2 + \left(1 - e^{-\lambda x} + 2be^{-\lambda x}\right)} \end{aligned} \tag{8}$$

Plots of the probability density of MOHL distribution for the several values of the parameters in Figure 1.

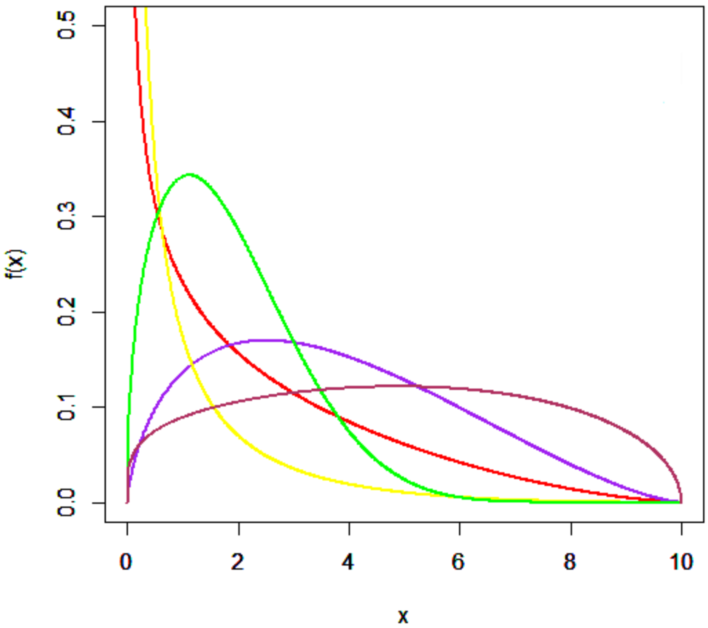

Figure 1. Plot for density function of the MOHL distribution for some selected parameter values

Figure 1 shows that the plots of pdf for MOHL distribution for several values of parameters. The density function can take various forms depending on the parameter values. Both unimodal and monotonically decreasing, bathtub, skewed shapes appear to be possible. It is evident that the MOHL distribution is very flexible.

Figure 2 explains the behavior of hrf of the MOHL distribution for several parameter values. Figure 2 shows that the distribution has bathtub behavior and seems much flexible in explaining the death rate and existence rate for the lifetime of the certain product. This attractive flexibility makes the hrf of the MOHL useful and suitable for non-monotone empirical hazard behaviours which are more likely to be encountered or observed in real life situations.

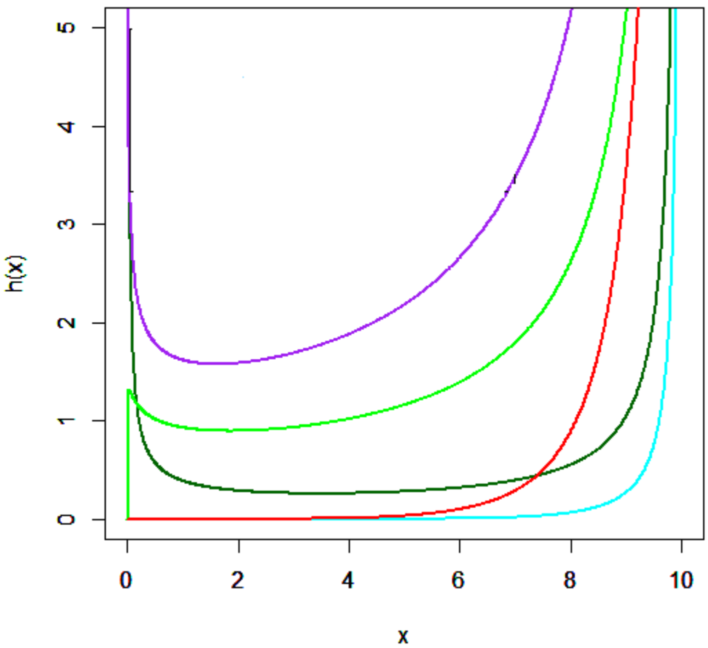

Figure 2. Plot for hazard function of the MOHL distribution for some selected parameter values

3. Properties of MOHL Distribution

3.1. Quantile Function

Quantile functions are in widespread use in general statistics and often find representations in terms of lookup tables for key percentiles. Now consider the quantile function of MOHL distribution. By inverting Eq. (5), we obtain the quantile function of X, say $Q(u) = F_{MOHL}^{-1}(u) = x$, as

$$x = -\frac{1}{\lambda}\log\left(\frac{1-u}{1-u+2bu}\right) \tag{9}$$

So, the MOHL distribution is easily simulated as follows: if U has a uniform $U(0, 1)$ distribution, then $X = Q(U)$ has the density function (6).

Let us point out that the skewness and kurtosis of the MOHL distribution can be derived by quantiles (Kenney, 1939). The Bowleys' skewness (B) based on Eq. (9) is given by

$$B = \frac{Q\left(\frac{3}{4}\right) + 2Q\left(\frac{2}{4}\right) + Q\left(\frac{1}{4}\right)}{Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right)} \tag{10}$$

and the Moors' kurtosis (M) is obtained as

$$M = \frac{Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right) - Q\left(\frac{3}{8}\right) + Q\left(\frac{1}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{2}{8}\right)} \tag{11}$$

where Q(.) represents the quantile function of the MOHL distribution (Moors, 1988). If the distribution is symmetric, we have S=0 and if the distribution is left (or right) skewed S>0 or (S<0). If K increases, the tail of the distribution becomes heavier. The plots for the skewness and kurtosis for the MOHL distribution for the several values of parameters are presented in Figure 3.

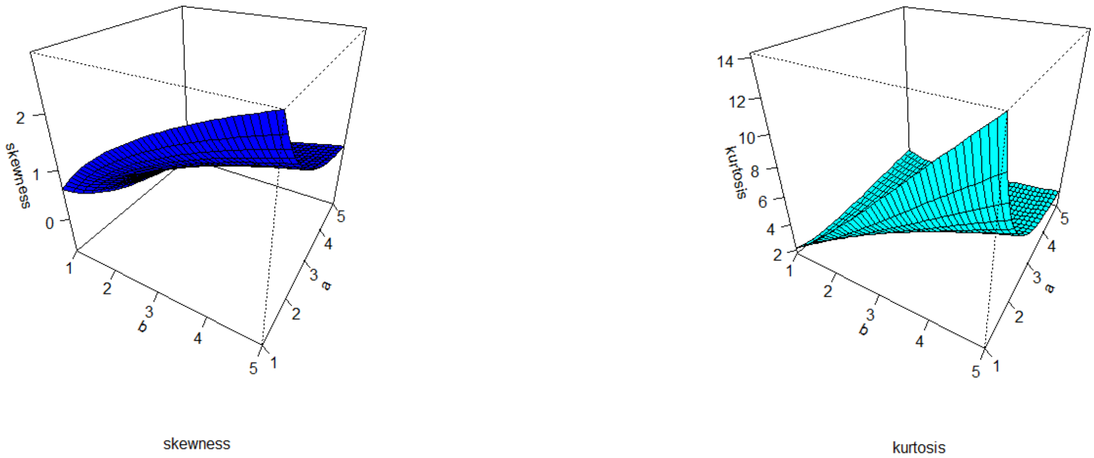

Figure 4. Plots of the skewness and kurtosis for the MOHL distribution

3.2. Order Statistics

Now, we obtain order statistics of the MOHL distribution. Let $x_{1,n} \le x_{2:n}, \ldots \le x_{n:n}$ be the ordered sample from a population. The cdf of $X_{n:n}$, the n[th] order statistics, is given by

$$F_{n:n}(x) = \left[F(x)\right]^n = \left[\frac{1-e^{-\lambda x}}{1-e^{-\lambda x}+2be^{-\lambda x}}\right]^n \tag{12}$$

with F(x) is the cdf of the MOHL distribution. Then, the pdf the n[th] order statistics for the MOHL random variable $X_{n:n}$ can be obtained by using Eqs.(5) and (6) in above equation to be

$$\begin{aligned} f_{n:n}(x) &= n\left[F(x)\right]^{n-1} f(x) \\ &= n\left[\frac{1-e^{-\lambda x}}{1-e^{-\lambda x}+2be^{-\lambda x}}\right]^{n-1} \cdot \frac{2\lambda be^{\lambda x}}{\left(e^{\lambda x}+2b-1\right)^2} \end{aligned} \tag{13}$$

with F(x) and f(x) are the cdf and pdf of the MOHLL distribution, respectively.

The cdf of $X_{1:n}$, the first order statistics, is given by

$$f_{1:n}(x) = \left[1-F(x)\right]^n = \left[\frac{2be^{-\lambda x}}{1-e^{-\lambda x}+2be^{-\lambda x}}\right]^n \tag{14}$$

and the pdf of $X_{1:n}$, the first order statistics, is obtained as

$$f_{1:n}(x) = n\left[1-F(x)\right]^{n-1} \cdot -f(x)$$
$$= n\left[\frac{2be^{-\lambda x}}{1-e^{-\lambda x}+2be^{-\lambda x}}\right]^{n} \cdot \frac{-2\lambda be^{\lambda x}}{\left(e^{\lambda x}+2b-1\right)^{2}} \quad (15)$$

Then, the pdf of $X_{k:n}$, the kth order statistics, is given by

$$f_{X_{k:n}}(x) = \frac{n!}{(k-1)!(n-k)!} f(x)\left[F(x)\right]^{k-1}\left[1-F(x)\right]^{n-k} \quad (16)$$

Using this formula, we obtain the pdf of the kth order statistics as follows:

$$f_{k:n}(x) = \frac{n!}{(k-1)!(n-k)!}\left[\frac{1-e^{-\lambda x}}{1-e^{-\lambda x}+2be^{-\lambda x}}\right]^{k-1}\left[\frac{2be^{-\lambda x}}{1-e^{-\lambda x}+2be^{-\lambda x}}\right]^{n-k} \quad (17)$$

3.3. Maximum Likelihood Estimates

Many estimation methods have argued in literature but the MLE method provides maximum information about the properties of estimated parameters and mostly used. Moreover, normal approximation of these estimators can frankly be managed systematically and mathematically for large sample theory. Consequently, the MLE has adopted to estimate the unknown parameters of the McPF distribution. The sample likelihood function is given by

$$\prod_{i=1}^{n} f(x_i) = \frac{(2b\lambda)^{n}\prod_{i=1}^{n} e^{2\lambda x_i}}{\prod_{i=1}^{n}\left(e^{\lambda x_i}+2b-1\right)} \quad (18)$$

Then, the log-likelihood function is obtained as

$$\ell = n\log(2b\lambda) + 2\lambda\sum_{i=1}^{n} x_i - \ln\left(e^{\lambda^{n}\sum_{i=1}^{n} x_i} + 2b - 1\right) \quad (19)$$

Therefore, the MLE's of parameters which maximize the above log-likelihood function must satisfy the normal equations. We take the first derivative of the above log-likelihood equation with respect to parameters and equate to zero respectively.

$$\frac{\partial \ell}{\partial b} = n2\lambda\log(2b\lambda) + \frac{2}{\ln\left(e^{\lambda^{n}\sum_{i=1}^{n} x_i} + 2b - 1\right)} \quad (20)$$

$$\frac{\partial \ell}{\partial \lambda} = n2b\log(2b\lambda) + \frac{n\lambda^{n-1}\sum_{i=1}^{n} x_i}{\ln\left(e^{\lambda^{n}\sum_{i=1}^{n} x_i} + 2b - 1\right)} \quad (21)$$

The exact solution of above-derived ML estimator for unknown parameters is not possible. So it is more convenient to use non-linear optimization algorithms such as Newton-Raphson algorithm to numerically maximize the above likelihood function.

4. Simulation

In this section, we examine the performance of the MOHL distribution by conducting various simulations for different example sizes and different parameter values by using Monte Carlo Simulation using R language. Inverse transform method is used to generate random data from MOHL distribution. The simulation study is repeated N=10000 times each with sample size n=50, 100, 300 and the parameter values I: $(b=0.5,\ \lambda=2)$ and II: $(b=1.5, \lambda=3)$

Two quantities are computed in this simulation: Average bias of the MLE $\hat{\theta}$ of the parameters defined as

$$Average\ Bias = \frac{1}{N}\sum_{i=1}^{N}\left(\hat{\theta} - \theta\right) \tag{22}$$

Root mean square error (RMSE) of the MLE $\hat{\theta}$ of the parameters defined as

$$RMSE = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(\hat{\theta} - \theta\right)^2} \tag{23}$$

Average bias and RMSE of the values of the parameters are listed in Table 2. The values in Table 2 indicate that as the sample size increases the biases and RMSEs of the estimates decrease.

Parameter	**n**	***Set-I***		***Set-II***	
		Avg. Bias	**RMSE**	**Avg. Bias**	**RMSE**
λ	50	-0.145	0.262	-1.043	1.045
	150	-0.122	0.224	-0.952	1.007
	250	-0.087	0.169	-0.765	0.865
b	50	3.232	4.301	3.448	4.922
	150	2.583	3.452	3.232	4.854
	250	1.291	1.701	2.336	3.565

Table 2. Monte Carlo simulation results

5. Conclusion

In this paper we obtain a new distribution namely Marshall Olkin half-logistic (MOHL) distribution by extending the half logistic distribution. Some statistical properties of the MOHL distribution are investigated including skewness, kurtosis, survival, hazard rate and quantile functions and order statistics. The parameters of the MOHL distribution are estimated by using the maximum likelihood method. A simulation study is also given the to present the performance of the new distribution. The results show that the proposed distribution is flexible and can be a good alternative to existing distributions.

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