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Green's Functions for Two-Interval Sturm-Liouville Problems in Direct Sum Space

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Abstract. The theme of the development of the theory and applications of Green's Functions is skilfully used to motivate and connect clear accounts of the theory of distributions, Fourier series and transforms, Hilbert spaces, linear integral equations. In this work we analyze the Green's functions of boundary value problems defined on two interval and associated with Schrodinger operators with interaction conditions. We have constructed some special eigensolutions of this problem and presented a formula and the existence condition of Green's function in terms of the general solution of a corresponding homogeneous equation. We have obtained the relation between two Green's functions of two nonhomogeneous problems. It allows us to find Green's function for the same equation but with different additional conditions. These problems include the cases in which the boundary has two, one or none vertices. In each case, the Green's functions, the eigenvalues and the eigenfunctions are given in terms of asymptotic formulas. A preliminary study of two-point regular boundary value problems with additional transmission conditions was developed by the authors of this study under the denomination of two-point transmission boundary value problems. In each case, it is essential to describe the solutions of the Schrodinger equation on the interior nodes of the path. As an consequence of this property, we can characterize those boundary value problems that are regular and then we obtain their corresponding Green's function, as well as the eigenvalues and the eigenfunctions for the regular case.

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1. Introduction

The purpose of this study is to investigate some spectral properties of two-interval Sturm-Liouville problems involving nonclassical terms. Boundary value problems with transmission conditions (but without abstract operator B_1 in the equation) were investigate extensively in the resent years (see, for example [1–6, 8–11]). Let us consider a new type Sturm-Liouville equation containing an abstract linear operator given by

$$-y''(x) + (Ty)(x) = \lambda y, x \in [a,b] \cup [c,d], \tag{1.1}$$

with the boundary conditions

$$b_1 y := y(a) - hy'(a) = 0, \tag{1.2}$$

$$b_2 y := y(d) + Hy'(d) = 0, \tag{1.3}$$

and interaction conditions

$$\ell_1 y := y(b) - y(c) = 0, \tag{1.4}$$

$$\ell_2 y := (\alpha y'(b) - \lambda y(b)) - (\beta y'(c) - \lambda y(c)) = 0, \tag{1.5}$$

where T is an abstract linear operator in the direct sum of Hilbert spaces $L_2[a,b] \oplus L_2[c,d]$; $a < b < c < d$; h, H, α, β-are real numbers.

Remark 1.1. For operator -theoretic realization of considered problem we shall assume that $\alpha > 0$ and $\beta > 0$

2. Operator realization of the problem

For operator realization of the problem we shall introduce a new inner product in the direct sum space $H(\alpha,\beta) := (L_2[a,b] \oplus L_2[c,d]) \oplus C$ by the following equality:

$$< U, V >_{H_{(\alpha,\beta)}} = \alpha \int_a^b u(x)\overline{v(x)}dx + \beta \int_c^d u(x)\overline{v(x)}dx + u_1 v_1 \tag{2.1}$$

for $U = \begin{pmatrix} u(x), \\ u_1 \end{pmatrix}, \quad V = \begin{pmatrix} v(x), \\ v_1 \end{pmatrix} \in H(\alpha,\beta)$

Remark 2.1. It easy to see that the inner product (2.1) is equivalent to the classical inner product of the direct sum space $(L_2[a,b] \oplus L_2[c,d]) \oplus C$ Therefore the space $H(\alpha,\beta)$ can be seen as different realization of the Hilbert space $(L_2[a,b] \oplus L_2[c,d]) \oplus C$.

Now we shall define a linear operator $S : H(\alpha,\beta) \to H(\alpha,\beta)$ with domain of definition

$$D(S) = \left\{ U = \begin{pmatrix} u(x), \\ u_1 \end{pmatrix} \in H(\alpha,\beta) : u \in W_2^2[a,b] \oplus W_2^2[c,d], b_1 u = b_2 u = \ell_1 u = 0, u_1 = u(b) + u(c) \right\} \tag{2.2}$$

and action low

$$SU = \begin{pmatrix} -u'', \\ \alpha u'(b) - \beta u'(c) \end{pmatrix} \tag{2.3}$$

Hence we can rewrite the considered problem (1.1)-(1.5) in the operator form as

$$(S + T)Y = \lambda Y,$$

for

$$Y = \begin{pmatrix} y(x), \\ y_1 \end{pmatrix} \in D(S).$$

Therefore the set of eigenvalues of the problem (1.1)-(1.5) coincide with the set of eigenvalues of the operator $S + T$, and the set of eigenfunctions of the problem (1.1)-(1.5) coincide with the set of the first components of the operator $S + T$.

3. Eigenvalues and eigenfunctions of the pure differential boundary-value problem

Let us consider the pure differential part of the problem, i.e. the differential operator $S : H(\alpha,\beta) \to H(\alpha,\beta)$ which is defined by the equalities (2.2)-(2.3).

Theorem 3.1. *The differential operator S is densely defined and symmetric in the Hilbert space $H(\alpha,\beta)$.*

Proof. By integration by parts, we have the following identity:

$$\begin{aligned} < SU, V >_{H(\alpha,\beta)} - < U, SV >_{H(\alpha,\beta)} &= \alpha W(u,\overline{v};b) - \alpha W(u,\overline{v};a) + \beta W(u,\overline{v};d) - \beta W(u,\overline{v};c) \\ &+ (\alpha u'(b) - \beta u'(c))(\overline{v(b)} - \overline{v(c)}) - (\alpha v'(b) - \beta v'(c))(\overline{u(b)} - \overline{u(c)}) \\ &= -\alpha W(u,\overline{v}a) + \beta W(u,\overline{v};d) \end{aligned}$$

Taking in view that u and v satisfy the boundary conditions 1.2) and (1.3) this implies that

$$W(u,\overline{v}a) = W(u,\overline{v};d) = 0 \tag{3.1}$$

Now, substituting (3.1) in (3.2) we have

$$< SU, V >_{H(\alpha,\beta)} = < U, SV >_{H(\alpha,\beta)} \tag{3.2}$$

for all $U, V \in D(S)$. By applying the same technique as in [4] we can prove that the set $D(S)$ is dense in $H(\alpha,\beta)$, so the operator S is densely defined and symmetric. The proof is complete. □

Corollary 3.2. *The eigenvalues of the differential operators S are real.*

Remark 3.3. Taking in view the Corollary 3.2, we can assume, without loss of generality, that all eigenfunction of the differential operators S are real valued.

Corollary 3.4. *Let $u_1(x)$ and $u_2(x)$ be eigenfunction of the pure differential part of the (1.1)-(1.5) (i.e. in the special case $T = 0$ of the problem (1.1)-(1.5)) corresponding to the distinct eigenvalues λ_1 and λ_2. Then u_1 and u_2 are orthogonal in the sense of the following equality*

$$\alpha\int_a^b u_1(x)u_2(x)dx + \beta\int_c^d u_1(x)u_2(x)dx + u_1(b)u_2(b) + u_1(b)u_2(c) + u_1(c)u_2(b) + u_1(c)u_2(c) = 0. \tag{3.3}$$

4. The characteristic function and the Green's function

By employing the same technique as in [4] we can define the following functions. Let

$$\psi(x,\lambda) = \begin{cases} \psi_\ell(x,\lambda), for x \in [a,b] \\ \psi_r(x,\lambda), for x \in [c,d] \end{cases}$$

be the solution of the equation $-y'' = \lambda y$ satisfying the initial conditions

$$\psi(a,\lambda) = h, \quad \frac{\partial\psi(a,\lambda)}{\partial x} = 1$$

and the interaction conditions

$$\ell_1(\psi(.,\lambda)) = \ell_2(\psi(.,\lambda)) = 0.$$

Similarly, let

$$\chi(x,\lambda) = \begin{cases} \chi_\ell(x,\lambda), for x \in [a,b] \\ \chi_r(x,\lambda), for x \in [c,d] \end{cases}$$

be the solution of the same equation satisfying the initial conditions

$$\chi(d,\lambda) = -H, \quad \frac{\partial\chi(d,\lambda)}{\partial x} = 1$$

and the interaction conditions

$$\ell_1(\chi(.,\lambda)) = \ell_2(\chi(.,\lambda)) = 0.$$

It is clear that the wronskians

$$\nabla_\ell(\lambda) := W(\psi_\ell(.,\lambda),\chi_\ell(.,\lambda))$$

and

$$\nabla_r(\lambda) := W(\psi_r(.,\lambda), \chi_r(.,\lambda))$$

are entire functions of the parameter $\lambda \in C$. Moreover, each of these wronskians is independent on the variable x. We can show easily that

$$\nabla(\lambda) := \nabla_\ell(\lambda) = \delta\nabla_r(\lambda)$$

for some $\delta \neq 0$.

Theorem 4.1. *The eigenvalues of the differential operator $S : H(\alpha,\beta) \to H(\alpha,\beta)$ consist of the zeros of the characteristic function $\nabla(\lambda)$.*

Corollary 4.2. *Let $\lambda = \lambda_0$ be any zero of the characteristic function $\nabla(\lambda)$, i.e. $\nabla(\lambda_0) = 0$. Then each of the functions $\psi(x,\lambda_0)$ and $\chi(x,\lambda_0)$ is the eigenfunction of S, corresponding to the eigenvalue λ_0.*

By using asymptotic representation of the solutions $\psi(x,\lambda)$ and $\chi(x,\lambda)$ we can prove the next theorem.

Theorem 4.3. *The differential operator S has precisely denumerable real eigenvalues λ_n, $n = 1, 2, 3, ...$*

Now, by using the well known results of [7] we can prove the following theorem.

Theorem 4.4. *Let $D(T) \supset D(S)$ and the operator $T(s-\mu I)^{-1}$ is compact for some regular point $\mu \in \rho(S)$ (i.e. T is $S-$compact operator). Then the spectrum of the operator $S+T$(i.e.the spectrum of the problem (1.1)-(1.5))) is discrete.*

Finally, by applying the same technique as in [4] we can construct the Green's function $G(x,y;\lambda)$ which takes the form

$$G(x,y;\lambda) = \begin{cases} \frac{\psi(y,\lambda)\chi(x,\lambda)}{\nabla(\lambda)}, for\ y \le x;\ x,y \neq b,c \\ \frac{\psi(x,\lambda)\chi(y,\lambda)}{\nabla(\lambda)}, for\ y > x;\ x,y \neq b,c \end{cases}$$

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