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Fuchsian Groups and Continued Fractions

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Abstract. The suborbital graph is a directed graph arisen from the transitive group action. We investigate suborbital graphs forming by the action of $N_{\mathbb{P}}(\Gamma)$ which is the normalizer of modular group in the Picard group. We give necessary and sufficient conditions for paired and self-paired graphs.

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1. Introduction

Continued fractions were studied by the great mathematicians of the seventeenth and eighteenth centuries and are a subject of active investigation today. They provide much insight into many mathematical problems, particularly into the nature of numbers. Nearly all books on the theory of numbers include a chapter on continued fractions. Most remarkable properties are as follows:

- Rational fractions and irrational numbers can be expanded into continued fractions and infinite continued fractions respectively;
- Continued fractions can be used to give better rational approximations to irrational numbers;
- The continued fraction expansion of every quadratic irrational is periodic. This fact is then used as the key to the solution of Diophantine and Pell's equations [8].

On the other hand, the concept of suborbital graph was introduced by Sims in 1967 for finite permutation groups [7]. Sarma et al. showed that the trees in suborbital graphs of modular group can be defined as a new kind of contined fraction and that any irrrational numbers has a unique subgraph $F_{1,2}$ expansion as an example [6]. This was followed by a similar study where the case of subgraph $F_{1,3}$ and subgraph $F_{1,4}$ were examined [4]. In that year, Nathanson published a work that reveals the relationship between continued fractions and trees produced by linear fractional transformations [5]. Actually, Jones et al. also pointed out same idea in [3]. We conclude that graphs of the objects like as modular group might be worth examining from this point of view. In [9], some properties of the graphs of the normalizer of modular group were studied following from the case of Γ. Nevertheless, connectivity of the graph was

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not yet examined. First step for this, we obtain paired and self paired graphs in this short note which may be thought as a sequel of [9].

2. Suborbital Graphs

Let $PSL(2,\mathbb{R})$ denote the group of all linear fractional transformations

$$T : z \to \frac{az+b}{cz+d}\text{ , where } a,b,c \text{ and } d \text{ are real and } ad-bc=1.$$

In terms of matrix representation, the elements of $PSL(2,\mathbb{R})$ correspond to the matrices

$$\pm\begin{pmatrix} a & b \\ c & d \end{pmatrix}; \quad a,b,c,d\in\mathbb{R} \text{ and } ad-bc=1.$$

This is the automorphism group of the upper half plane $\mathbb{H} := \{z\in\mathbb{C} : \mathrm{Im(z)}>0\}$. Γ, the modular group, is the subgroup of $PSL(2,\mathbb{R})$ such that a,b,c and d are integers. $\mathbb{P} = PSL(2,\mathbb{Z}[i])$, the Picard group, is the subgroup of $PSL(2,\mathbb{C})$ such that a,b,c and d are Gaussian integers. A Fuchsian group is a discrete subgroup of $PSL(2,\mathbb{R})$. It is known that every finitely generated Fuchsian groups has a unique presentation with generators and relations [2]. The presentation of $N_{\mathbb{P}}(\Gamma)$ is

$$N_{\mathbb{P}}(\Gamma) = \langle u = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, y = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, r = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}; u^2 = y^3 = r^2 = (ry)^2 = (ru)^2 = 1\rangle \text{ [9]}.$$

Let (G,Δ) be a transitive permutation group, consisting of a group G acting on a set Δ transitively. An equivalence relation $\approx$ on Δ is called *G-invariant* if, whenever $\alpha,\beta\in\Delta$ satisfy $\alpha\approx\beta$, then $g(\alpha)\approx g(\beta)$ for all $g\in G$.

The equivalence classes are called blocks, and the block containing α is denoted by $[\alpha]$.

We call (G,Δ) *imprimitive* if Δ admits some G-invariant equivalence relation different from

i. the identity relation, $\alpha\approx\beta$ if and only if $\alpha=\beta$;
ii. the universal relation, $\alpha\approx\beta$ for all $\alpha,\beta\in\Delta$.

Otherwise (G,Δ) is called *primitive*. These two relations are supposed to be trivial relations. Clearly, a primitive group must be transitive, for if not the orbits would form a system of blocks. The converse is false, but we have the following useful result.

Lemma 2.1. *[1] Let (G,Δ) be a transitive permutation group. (G,Δ) is primitive if and only if G_α,the stabilizer of $\alpha\in\Delta$, is a maximal subgroup of G for each $\alpha\in\Delta$.*

From the above lemma we see that whenever, for some α, $G_\alpha \nleq H \leq G$, then Ω admits some G-invariant equivalence relation other than the trivial cases. Because of the transitivity, every element of Ω has the form $g(\alpha)$ for some $g\in G$. Thus one of the non-trivial G-invariant equivalence relation on Ω is given as follows:

$$g(\alpha)\approx g'(\alpha) \text{ if and only if } g'\in gH.$$

Lemma 2.2 ([9])**.** *The elements of $N_{\mathbb{P}}(\Gamma)$ consist of the following form :*

$$\begin{pmatrix} ai^k & bi^k \\ ci^k & di^k \end{pmatrix}$$

such that $a,b,c,d\in\mathbb{Z}$ and $k=0,1$.

If we set $G = N_{\mathbb{P}}(\Gamma)$, $\Delta = \hat{\mathbb{Q}}$, $H = \bar{\Gamma}_0(N) = \left\{\begin{pmatrix} ai^k & bi^k \\ ci^k & di^k \end{pmatrix} \in N_{\mathbb{P}}(\Gamma) | \ c\equiv 0 \, mod(N)\right\}$. and $G_\alpha = N_{\mathbb{P}}(\Gamma)_\infty$, then we clearly see that $N_{\mathbb{P}}(\Gamma)_\infty \leq \bar{\Gamma}_0(N) \leq N_{\mathbb{P}}(\Gamma)$.

We define the following $N_{\mathbb{P}}(\Gamma)$ invariant equivalence relation " $\underset{N}{\approx}$ " on $\hat{\mathbb{Q}}$. Since $N_{\mathbb{P}}(\Gamma)$ acts transitively on $\hat{\mathbb{Q}}$, every element of $\hat{\mathbb{Q}}$ has the form $g(\infty)$ for some $g\in N_{\mathbb{P}}(\Gamma)$. So, it is easily seen that,

$$g(\infty)\underset{N}{\approx} g'(\infty) \Longleftrightarrow g'\in gN_{\mathbb{P}}(\Gamma)$$

gives a $N_{\mathbb{P}}(\Gamma)-$invariant imprimitive equivalence relation.

Theorem 2.3 ([9] Block condition). *Let* $v = \frac{r}{s}$, $w = \frac{x}{y} \in \hat{\mathbb{Q}}$. *Then* $v \underset{N}{\approx} w$ *if and only if* $ry - sx \equiv 0(modN)$ *or* $sx - ry \equiv 0(modN)$.

Let (G, Δ) be a transitive permutation group. Then G acts on $\Delta \times \Delta$ by $g(\alpha, \beta) = (g(\alpha), g(\beta))$, $(g \in G, \alpha, \beta \in \Delta)$. The orbits of this action are called *suborbitals* of G.

In this study, G is $N_{\mathbb{P}}(\Gamma)$ and Δ is $\hat{\mathbb{Q}}$. We now consider the suborbital graphs for the action $N_{\mathbb{P}}(\Gamma)$ on $\hat{\mathbb{Q}}$. Since $N_{\mathbb{P}}(\Gamma)$ acts transitively on $\hat{\mathbb{Q}}$, each suborbital contains a pair $(\infty, u/N)$ for some $u/N \in \hat{\mathbb{Q}}$ such that $(u, N) = 1$. We denote this suborbital by $\bar{O}(u, N)$ and corresponding suborbital graph $\bar{G}(u, N)$ by $\bar{G}_{u,N}$.

Theorem 2.4 ([9] Edge condition). *[9]* $r/s \longrightarrow x/y$ *is an edge in* $\bar{G}_{u,N}$ *if and only if*

(i) $x \equiv ur\,(modN)$, $y \equiv us\,(modN)$, $ry - sx = N$ *or*
(ii) $x \equiv -ur\,(modN)$, $y \equiv -us\,(modN)$, $ry - sx = -N$ *or*
(iii) $x \equiv ur\,(modN)$, $y \equiv us\,(modN)$, $ry - sx = -N$ *or*
(iv) $x \equiv -ur\,(modN)$, $y \equiv -us\,(modN)$, $ry - sx = N$.

Corollary 2.5. *If* $uv \equiv \pm 1 \pmod N$, *then the suborbital graph* $\bar{G}_{u,N}$ *is paired with* $\bar{G}_{v,N}$.

Proof. We suppose that $uv \equiv 1 \pmod N$. By using Theorem 2.4, we have

Case 1: It is obtained that $x \equiv ur \pmod N, y \equiv us$ (mod) and $ry - sx = N$. Since $vx \equiv vur \pmod N$ and $vy \equiv vus \pmod N$, we have $r \equiv vx \pmod N$, $s \equiv vy \pmod N$ and $sx - ry = -N$. By Theorem 2.4, $\frac{x}{y} \rightarrow \frac{r}{s}$ is an edge in $\bar{G}_{v,N}$. Therefore $\bar{G}_{u,N}$ is paired with $\bar{G}_{v,N}$.

Case 2: It is obtained that $x \equiv -ur \pmod N, y \equiv -us$ (mod) and $ry - sx = -N$. Since $vx \equiv -vur \pmod N$ and $vy \equiv -vus \pmod N$, we have $r \equiv -vx \pmod N$, $s \equiv -vy \pmod N$ and $sx - ry = N$. By Theorem [?], $\frac{x}{y} \rightarrow \frac{r}{s}$ is an edge in $\bar{G}_{v,N}$. Therefore $\bar{G}_{u,N}$ is paired with $\bar{G}_{v,N}$.

Case 3: It is obtained that $x \equiv ur \pmod N, y \equiv us$ (mod) and $ry - sx = -N$. Since $vx \equiv vur \pmod N$ and $vy \equiv vus \pmod N$, we have $r \equiv vx \pmod N$, $s \equiv vy \pmod N$ and $sx - ry = N$. By Theorem 2.4, $\frac{x}{y} \rightarrow \frac{r}{s}$ is an edge in $\bar{G}_{v,N}$. Therefore $\bar{G}_{u,N}$ is paired with $\bar{G}_{v,N}$.

Case 4: It is obtained that $x \equiv -ur \pmod N, y \equiv -us$ (mod) and $ry - sx = N$. Since $vx \equiv -vur \pmod N$ and $vy \equiv -vus \pmod N$, we have $r \equiv -vx \pmod N$, $s \equiv -vy \pmod N$ and $sx - ry = -N$. By Theorem [?], $\frac{x}{y} \rightarrow \frac{r}{s}$ is an edge in $\bar{G}_{v,N}$. Therefore $\bar{G}_{u,N}$ is paired with $\bar{G}_{v,N}$.

We suppose that $uv \equiv -1 \pmod N$. By using Theorem 2.4, we have

Case 1: It is obtained that $x \equiv ur \pmod N, y \equiv us$ (mod) and $ry - sx = N$. Since $vx \equiv vur \pmod N$ and $vy \equiv vus \pmod N$, we have $r \equiv -vx \pmod N$, $s \equiv -vy \pmod N$ and $sx - ry = -N$. By Theorem 2.4, $\frac{x}{y} \rightarrow \frac{r}{s}$ is an edge in $\bar{G}_{v,N}$. Therefore $\bar{G}_{u,N}$ is paired with $\bar{G}_{v,N}$.

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□

Corollary 2.6. $\bar{G}_{u,N}$ *is self-paired if and only if* $u^2 \equiv \pm 1 \pmod N$.

Proof. Assume that $\bar{G}_{u,N}$ is self-paired. There exists a transformation $T \in N_{\mathbb{P}}(\Gamma)$ such that

$$\left(\infty, \frac{u}{N}\right) \xrightarrow{T} \left(\frac{u}{N}, \infty\right).$$

Thus, T is in the form of $\begin{pmatrix} u & -b \\ N & -u \end{pmatrix}$ or $\begin{pmatrix} ui & -bi \\ Ni & -ui \end{pmatrix}$ for an integer b. Indeed

$$\begin{pmatrix} u & -b \\ N & -u \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} u \\ N \end{pmatrix} \text{ and } \begin{pmatrix} u & -b \\ N & -u \end{pmatrix}\begin{pmatrix} u \\ N \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Since $detT = 1$, we have $u^2 \equiv -1 \pmod N$. Or,

$$\begin{pmatrix} ui & -bi \\ Ni & -ui \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} ui \\ Ni \end{pmatrix} \text{ and } \begin{pmatrix} ui & -bi \\ Ni & -ui \end{pmatrix}\begin{pmatrix} u \\ N \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Since $detT = 1$, we have $u^2 \equiv 1 \pmod N$.

Conversely, let $u^2 \equiv 1 \pmod N$. There exists an integer b such that $-u^2 + bN = -1$. Thus the transformation $\begin{pmatrix} ui & -bi \\ Ni & -ui \end{pmatrix}$ is in $N_{\mathbb{P}}(\Gamma)$, and sends ∞ to $\frac{u}{N}$ and $\frac{u}{N}$ to ∞. This means that $\bar{G}_{u,N}$ is self-paired.

On the other hand, let $u^2 \equiv -1 \pmod N$. There exists an integer b such that $-u^2 + bN = 1$. Thus the transformation $\begin{pmatrix} u & -b \\ N & -u \end{pmatrix}$ is in $N_{\mathbb{P}}(\Gamma)$, and sends ∞ to $\frac{u}{N}$ and $\frac{u}{N}$ to ∞. This means that $\bar{G}_{u,N}$ is self-paired. □

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