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A NEW PERSPECTIVE OF TRANSMUTED DISTRIBUTION

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Abstract. Regarding the concept of quadratic rank transmutation, a new distribution with convex combinations of the life distributions of two-component systems (series and parallel systems) whose component lifetimes are not identical is obtained. This proposed distribution has extra parameters compared to the known transmuted distribution. It can also be represented by two different baseline distributions. So, it is very flexible in modeling. A description of the various structural properties of the subject distribution along with its reliability behavior is provided. Finally, a real data analysis is performed for this distribution and it is found that this class is more flexible.

1. Introduction

In the recent years, there has been a growing interest in different types of mixture distributions and several of them have been studied. Adding parameters to a well-established distribution is a time-honored device for providing more flexible distributions. An interesting method of adding a new parameter to known distributions was pioneered by [21]. This method has been named as transmuted distribution. In literature, there are some transmuted distributions that have been studied. Some of them are listed below.

Shaw [22] defined the theory of transmuted distribution. Through this work, many studies have been done. Aryal and Tsokos [6] defined the transmuted extreme value distribution. Also, Aryal and Tsokos [7] presented a new generalization of Weibull distribution called the transmuted Weibull distribution. Recently, Aryal [5] proposed and studied the various properties of the transmuted log-logistic distribution. Subsequently, Elbatal and Aryal [10] presented on the transmuted additive Weibull distribution. Merovci [15] studied transmuted Lindley distribution and applied it to bladder cancer data. In another article Merovci [16] introduced

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transmuted exponentiated exponential distribution and transmuted rayleigh distribution is introduced by [17]. Moreover, Merovci and Elbatal [18] studied transmuted Lindley-geometric distribution and Merovci and Puka [19] defined transmuted Pareto distribution and discussed theirs various properties. Transmuted exponentiated Lomax distribution is studied by [8], Eltehiwy and Ashour [12] obtained transmuted exponentiated modified Weibull distribution, transmuted exponentiated gamma distribution is studied by [13], transmuted exponentiated Fréchet distribution is studied by [11] , exponentiated transmuted Weibull distribution is studied by [1].

In this study, we introduce a new distribution by using convex combination of two exponential distribution with different parameters. It is important to note that the reparametrization is done on convex combination parameter, then the transmuted parameter can be achieved. The main aim of this article is to discuss about the principal idea of transmutation method.

Second part of this article contains some main definitions, and the analytical shapes of the probability density, survival, cumulative hazard rate and hazard rate functions are presented of the model under study. Statistical properties including moments, generating function and skewness and kurtosis, random number generation, Rényi entropy of proposed distribution are discussed in subsections of Section 2. In part 3, two exponential distributions with different parameters for the baseline distributions to investigate a special case of the new distribution introduced in part 2 are considered. In addition to the same mathematical properties of the new distribution studied in section 2, mean residual function, maximum likelihood estimates and order statistics are also studied. Finally, three real data applications are presented to illustrate the proposed distribution.

2. The New Family

Two-component systems will be given in this section. Firstly, let's consider the series system. Failure of the serial system is due to failure of either of the two parts. T_1 and T_2 represent the component lifetimes which are independent random variables having the distribution functions $F(t)$ and $G(t)$ respectively and $T_{min} = min\{T_1, T_2\}$ stands for the series system lifetime. Hence, the probability of failure of this system is given by

$$H_{T_{min}}(t) = P(T_{min} \leq t) = P(min\{T_1, T_2\} \leq t) = F(t) + G(t) - F(t)G(t)$$

In the same way, parallel system loses its functioning, if the two components of the system are not functioning. Accordingly, $T_{max} = max\{T_1, T_2\}$ stands for the parallel system lifetime that the probability of failure of this system can be written as

$$H_{T_{max}}(t) = P(T_{max} \leq t) = P(max\{T_1, T_2\} \leq t) = F(t)G(t)$$

Component lifetimes T_1 and T_2 can be stochastically ordered as $T_{min} \leq T_i \leq T_{max}$, $i = 1, 2$. Namely, we have $H_{T_{max}}(t) \leq F(t) \leq H_{T_{min}}(t)$. Then, for $\lambda \in [0, 1]$, the convex combination can be written as follows

$$\begin{aligned} H(t) &= \lambda H_{T_{min}}(t) + (1-\lambda) H_{T_{max}}(t) \\ &= \lambda \ (F(t) + G(t) - F(t)\ G(t)) + (1-\lambda) F(t) G(t) \\ &= \lambda \ (F(t) + G(t)) + (1-2\lambda) F(t) G(t). \end{aligned} \tag{1}$$

If the transformation is done by applying $\lambda = \frac{1+\delta}{2}$ in (1), we can write

$$\begin{aligned} H(t) &= \frac{1+\delta}{2}(F(t)+G(t)) - \delta F(t)G(t), \qquad (2) \\ &= \frac{1+\delta}{2} H_{T_{min}}(t) + \frac{1-\delta}{2} H_{T_{max}}(t) \end{aligned}$$

where $|\delta| \leq 1$. Thus, by this method we have obtained a new univariate distribution which looks like to the transmuted distribution extremely. So, if the baseline distributions are taken identical, namely, $F(t) = G(t)$ in the latter equation, we can obtain the transmuted distribution which has become very popular in the recent years. Here parameter space for proposed distribution is defined as $P = \{\Theta_f \cup \Theta_g \cup \{\delta\}\}$. When a comparison of the parameter space of the known transmuted distributions is made, this new distribution has an extra parameter set Θ_g. By using cumulative distribution function in equation (1), the corresponding probability density function (p.d.f.) is given by

$$\begin{aligned} h(t) &= \lambda(f(t)+g(t)) + (1-2\lambda)(f(t)G(t) + F(t)g(t)), \qquad (3) \\ &= \lambda(f(t)+g(t) - f(t)G(t) - F(t)g(t)) + (1-\lambda)(f(t)G(t) + F(t)g(t)) \\ &= \lambda h_{T_{min}}(t) + (1-\lambda) h_{T_{max}}(t) \end{aligned}$$

2.1. **Survival Function of Proposed Distribution.** According to (2), the corresponding survival function of proposed distribution is given as follows,

$$\begin{aligned} \bar{H}(t) &= 1 - \lambda(F(t)+G(t)) - (1-2\lambda)F(t)G(t) \\ &= (1-\lambda)(\bar{F}(t) + \bar{G}(t)) - (1-2\lambda)\bar{F}(t)\bar{G}(t) \\ &= (1-\lambda)(\bar{F}(t) + \bar{G}(t) - \bar{F}(t)\bar{G}(t)) + \lambda \bar{F}(t)\bar{G}(t). \end{aligned}$$

2.2. **Cumulative Hazard Rate and Hazard Rate Function of Proposed Distribution.** The cumulative hazard rate and hazard rate functions of this distribution are defined by

$$R(t) = -\log \bar{H}(t) = -\log\left(1 - \lambda\ (F(t)+G(t)) - (1-2\lambda)F(t)G(t)\right),$$

$$r(t) = \frac{h(t)}{\bar{H}(t)} = \frac{\lambda\ (f(t)+g(t)) + (1-2\lambda)(f(t)G(t)+F(t)g(t))}{(1-\lambda)(\bar{F}(t)+\bar{G}(t)-\bar{F}(t)\bar{G}(t)) + \lambda\bar{F}(t)\bar{G}(t)}.$$

Note that, the hazard rate function of the proposed distribution can also be obtained as a function of the hazard rate of the baseline distributions as follows:

$$r(t) = r_f(t) + r_g(t) - \frac{(1-\lambda)(r_f(t)\bar{G}(t) + r_g(t)\bar{F}(t))}{(1-\lambda)(\bar{F}(t)+\bar{G}(t)-\bar{F}(t)\bar{G}(t)) + \lambda\bar{F}(t)\bar{G}(t)}$$

In special case, if we get $\lambda = \frac{1}{2}$, distribution and probability density functions of this special case are $H^*(t) = \frac{F(t)+G(t)}{2}$ and $h^*(t) = \frac{f(t)+g(t)}{2}$ respectively. That is, H^* represents average of the baseline distributions. So, the hazard rate funcion of H^* is defined by

$$\begin{aligned} r_{h^*}(t) &= \frac{\frac{f(t)+g(t)}{2}}{1-\frac{F(t)+G(t)}{2}} = \frac{f(t)+g(t)}{\bar{F}(t)+\bar{G}(t)} \\ &= \frac{r_f(t)\bar{F}(t)+r_g(t)\bar{G}(t)}{\bar{F}(t)+\bar{G}(t)} = w(t)r_f(t)+(1-w(t))r_g(t), \end{aligned}$$

where $w(t) = \frac{\bar{F}(t)}{\bar{F}(t)+\bar{G}(t)}$. It is clear that the hazard rate function of this case can be written as a weighted expression of the hazard rate functions of the two baseline distributions.

2.3. **Moment Generating Function and Moments of Proposed Distribution.** The moment generating function of random variable T is obtained as

$$\begin{aligned} M_T(k) &= \int_0^\infty e^{kt}(\lambda\ (f(t)+g(t))+(1-2\lambda)\ (f(t)G(t)+F(t)g(t)))\ dt \\ &= \lambda\int_0^\infty e^{kt}(f(t)+g(t))\ dt+(1-2\lambda)\int_0^\infty e^{kt}h_{T_{max}}(t)\ dt \\ &= \lambda(M_{T_f}(k)+M_{T_g}(k))+(1-2\lambda)M_{T_{max}}(k) \end{aligned}$$

Moment of T random variable is defined as

$$E(T^k) = \int_0^\infty t^k\ h(t)dt = \lambda(E(T_f^k)+E(T_g^k))+(1-2\lambda)E(T_{max}^k), \quad (4)$$

Then the first four moments can be obtained by taking $k = 1, 2, 3, 4$ in equation (4).

2.4. **Skewness and Kurtosis of Proposed Distribution.** Based on the first four moments of the random variable under consideration, the skewness and kurtosis measures are given by

$$\gamma_1 = E\left[\left(\frac{T-E(T)}{\sqrt{Var(T)}}\right)^3\right] = \frac{1}{(E(T^2)-E(T)^2)^{\frac{3}{2}}} \times \begin{pmatrix} \lambda(E(T_f^3)+E(T_g^3))+(1-2\lambda)E(T_{max}^3) \\ -3(\lambda(E(T_f)+E(T_g))+(1-2\lambda)E(T_{max})) \\ (\lambda(E(T_f^2)+E(T_g^2))+(1-2\lambda)E(T_{max}^2)) \\ +2(\lambda(E(T_f)+E(T_g))+(1-2\lambda)E(T_{max}))^3 \end{pmatrix}$$

$$\gamma_2 = E\left[\left(\frac{T-E(T)}{\sqrt{Var(T)}}\right)^4\right] = \frac{1}{(E(T^2)-E(T)^2)^2}$$

$$\times \left(\begin{array}{l} \lambda(E(T_f^4) + E(T_g^4)) + (1-2\lambda)E(T_{max}^4) \\ -4(\lambda(E(T_f) + E(T_g)) + (1-2\lambda)E(T_{max})) \\ (\lambda(E(T_f^3) + E(T_g^3)) + (1-2\lambda)E(T_{max}^3)) \\ +6(\lambda(E(T_f) + E(T_g)) + (1-2\lambda)E(T_{max}))^2 \\ (\lambda(E(T_f^2) + E(T_g^2)) + (1-2\lambda)E(T_{max}^2)) \\ -3(\lambda(E(T_f) + E(T_g)) + (1-2\lambda)E(T_{max}))^4 \end{array} \right)$$

2.5. **Random Number Generation from Proposed Distribution.** In this section, the mixture method will be used to generate the random number from suggested distribution. Distribution function that is defined in (2) can be written as a mixture of T_{min} and T_{max} distribution functions as the latter equality in (2). So, the random number is generated from this nested mixture by the following steps:

Step 1: Generate $t_1 \sim F$ and $t_2 \sim G$ independently,

Step 2. Generate $u \sim U(0,1)$,

Step 3. If $u \leq \lambda$ then $t = min\{t_1, t_2\}$, otherwise $t = max\{t_1, t_2\}$.

2.6. **Rényi Entropy of Proposed Distribution.** The entropy of a random variable is a measure of variation of the uncertainty, see [20]. Then the Rényi entropy function of the random variable T with p.d.f. (3) is defined by

$$I_R(\rho) = \frac{1}{1-\rho} \log \int_0^\infty (h(t))^\rho dt, \tag{5}$$

where $\rho > 0$, $\rho \neq 1$. We have the following series representation of $(h(t))^\rho$ by applying the generalized Binomial theorem to obtain Rényi entropy for proposed distribution.

$$\begin{aligned} (h(t))^\rho &= (\lambda h_{T_{min}}(t) + (1-\lambda) h_{T_{max}}(t))^\rho \\ &= \sum_{j=0}^{\infty} \binom{\rho}{j} (\lambda(f(t) + g(t) - (f(t)G(t) + F(t)g(t))))^{\rho-j} \\ &\qquad \times((1-\lambda)(f(t)G(t) + F(t)g(t))^j \end{aligned}$$

Then, an equality for Rényi entropy can be written as follow

$$\begin{aligned} I_R(\rho) = \frac{1}{1-\rho} \log [&\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{l=0}^{\infty} (-1)^k \binom{\rho}{j}\binom{\rho-j}{k}\binom{j}{m}\binom{\rho-j-k}{n}\binom{k}{l} \\ &\times \lambda^{\rho-j}(1-\lambda)^j \int_0^\infty ((f(t))^{\rho-n-l-m}(g(t))^{n+l+m}(F(t))^{l+m}(G(t))^{k+j-l-m})dt] \end{aligned}$$

3. The Application of the Proposed Method

We assume random variables T_1 and T_2 have exponential distribution with parameters β_1 and β_2, respectively. The baseline distributions in equation (2) are considered as two different exponential distributions. Then we have

$$H(t) = 1 - (1-\lambda)\left(e^{-\beta_1 t} + e^{-\beta_2 t}\right) + (1-2\lambda)\, e^{-(\beta_1+\beta_2)t}, \tag{6}$$

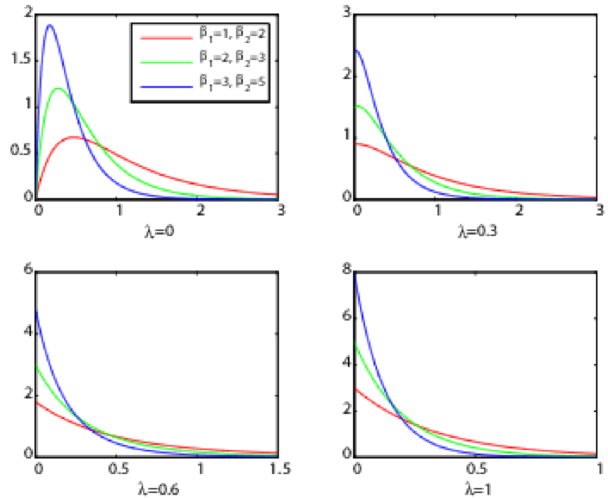

FIGURE 1. P.d.f. of the CCE-E distribution for some parameter values

where $\lambda \in [0, 1]$. So, the new univariate distribution function is obtained. This distribution is called the convex combination of two exponential distributions with different parameters (CCE-E). Then, the p.d.f. can be obtained from the cumulative distribution function that defined in (6) as

$$h(t) = (1-\lambda)\left(\beta_1 e^{-\beta_1 t} + \beta_2 e^{-\beta_2 t}\right) - (1-2\lambda)(\beta_1+\beta_2)\, e^{-(\beta_1+\beta_2)t},$$

where $t \geq 0, \beta_1, \beta_2 \geq 0$. Now, the shapes of the p.d.f. of the CCE-E distribution can be analyzed as follows

$$h'(t) = (1-\lambda)\left(-\beta_1^2 e^{-\beta_1 t} - \beta_2^2 e^{-\beta_2 t}\right) + (1-2\lambda)(\beta_1+\beta_2)^2 e^{-(\beta_1+\beta_2)t}$$

by examining this derivation, it is clear that when $\lambda \geq \frac{1}{2}$, $h^{'}(t) < 0$ is obtained and we can say p.d.f. is decreasing. Also, in order for p.d.f. to be unimodal, it must be $\lambda < \frac{1}{2}$. Figure 1 illustrates some possible shapes of the p.d.f. of the CCE-E for different values of the parameters β_1, β_2 and λ.

It is clear from Figure 1 that the p.d.f. of proposed distribution can take different shapes. The p.d.f. of the CCE-E distribution is compared with their baseline distributions, that is, p.d.f.s of two exponential distributions with parameters $\beta_1 = 2$ and $\beta_2 = 3$ in Figure 2.

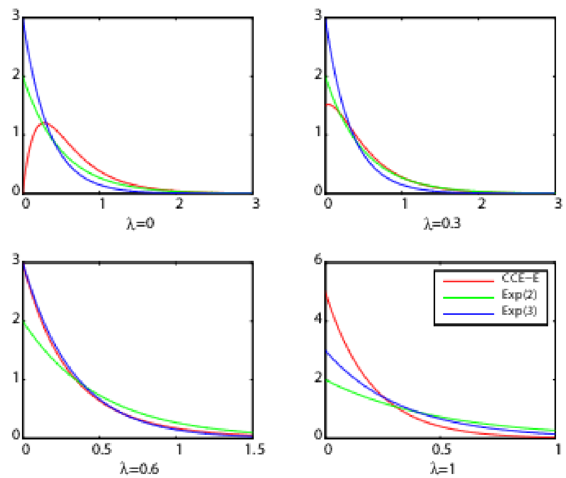

FIGURE 2. Compare plots of p.d.f. of the CCE-E with baseline distribution for some parameter values

3.1. **Survival Function of the CCE-E Distribution.** The survival function of the CCE-E distribution is given as follows

$$\bar{H}(t) = (1-\lambda)\ \left(e^{-\beta_1 t} + e^{-\beta_2 t}\right) - (1-2\lambda)\, e^{-(\beta_1+\beta_2)t} \tag{7}$$

3.2. **Cumulative Hazard Rate and Hazard Rate Function of the CCE-E Distribution.** The cumulative hazard rate $R(t)$ and hazard rate $r(t)$ functions can be found as follows

$$R(t) = -\log \bar{H}(t) = -\log\left((1-\lambda)\left(e^{-\beta_1 t} + e^{-\beta_2 t}\right) - (1-2\lambda)\, e^{-(\beta_1+\beta_2)t}\right)$$

$$\begin{aligned} r(t) &= \frac{h(t)}{\bar{H}(t)} = \frac{(1-\lambda)\left(\beta_1 e^{-\beta_1 t} + \beta_2 e^{-\beta_2 t}\right) - (1-2\lambda)(\beta_1+\beta_2)\, e^{-(\beta_1+\beta_2)t}}{(1-\lambda)\left(e^{-\beta_1 t} + e^{-\beta_2 t}\right) - (1-2\lambda)\, e^{-(\beta_1+\beta_2)t}} \\ &= (\beta_1+\beta_2) - \frac{(1-\lambda)\left(\beta_2 e^{-\beta_1 t} + \beta_1 e^{-\beta_2 t}\right)}{(1-\lambda)\left(e^{-\beta_1 t} + e^{-\beta_2 t}\right) - (1-2\lambda)\, e^{-(\beta_1+\beta_2)t}}, \end{aligned}$$

Let's investigate the hazard rate function.

$$\begin{aligned} r'(t) = &\frac{-(1-\lambda)^2(\beta_1+\beta_2)^2 e^{-(\beta_1+\beta_2)t}}{\left((1-\lambda)\left(e^{-\beta_1 t} + e^{-\beta_2 t}\right) - (1-2\lambda)\, e^{-(\beta_1+\beta_2)t}\right)^2} \\ &+ \frac{(1-\lambda)(1-2\lambda)\left(\beta_2^2 e^{-(2\beta_1+\beta_2)t} + \beta_1^2 e^{-(\beta_1+2\beta_2)t}\right)}{\left((1-\lambda)\left(e^{-\beta_1 t} + e^{-\beta_2 t}\right) - (1-2\lambda)\, e^{-(\beta_1+\beta_2)t}\right)^2} \end{aligned}$$

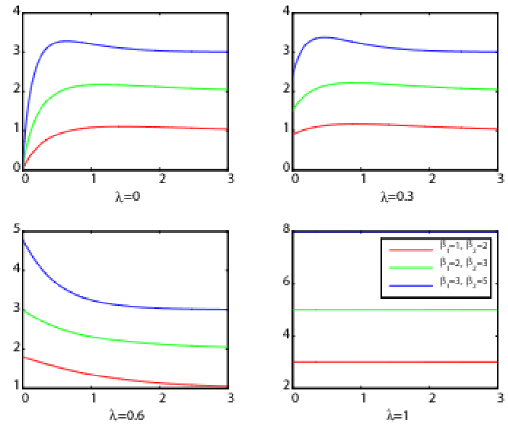

FIGURE 3. Hazard rate function of the CCE-E distribution for some parameter values

It is clear from above derivation, $(1-2\lambda)$ must be negative so that the derivative sign can be negative. That is, it is enough that $\lambda \geq \frac{1}{2}$. So, the hazard rate function is decreasing. Also, in order for hazard rate function to be unimodal, that is, in order for $r'(t) = 0$ to have a single solution, it must be $\lambda < \frac{1}{2}$. We are primarily interested in extreme points of $r(t)$ and we have

$$r(0) = \lambda(\beta_1 + \beta_2)$$

$$\lim_{t\to\infty} r(t) = \begin{cases} \beta_1 & \text{if } \beta_1 < \beta_2; \\ \beta_2 & \text{if } \beta_1 > \beta_2. \end{cases}$$

Thus, the hazard rate function of this distribution is changed from $\lambda(\beta_1+\beta_2)$ to $min\{\beta_1, \beta_2\}$. Now, we find the values of the hazard rate function for extreme points of combination parameters.

$$r(t) = \begin{cases} (\beta_1+\beta_2) - \frac{\beta_2 e^{-\beta_1 t} + \beta_1 e^{-\beta_2 t}}{e^{-\beta_1 t} + e^{-\beta_2 t} - e^{-(\beta_1+\beta_2)t}} & \text{if } \lambda = 0; \\ (\beta_1+\beta_2) - \frac{\beta_2 e^{-\beta_1 t} + \beta_1 e^{-\beta_2 t}}{e^{-\beta_1 t} + e^{-\beta_2 t}} & \text{if } \lambda = \frac{1}{2}. \\ \beta_1 + \beta_2 & \text{if } \lambda = 1 \end{cases}$$

while $\lambda = 1$, hazard rate function is constant. Now, we show some possible shapes of the hazard rate function for selected parameter values in following figures.

Figure 3 shows the hazard rate function that defined in (7) with different choices of parameters. This distribution has an decreasing hazard rate function for $\lambda \geq \frac{1}{2}$.

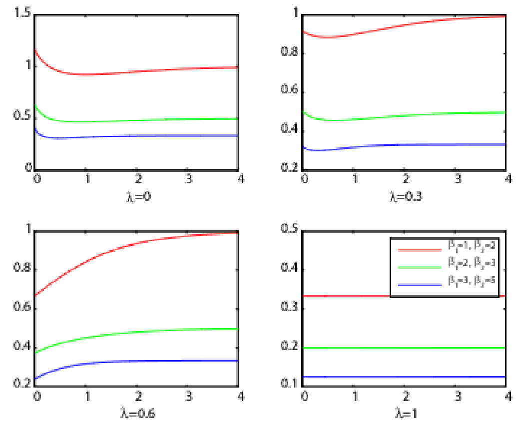

FIGURE 4. Mean residual life function of the CCE-E distribution for some parameter values

If $\lambda < \frac{1}{2}$, the hazard rate function is inverse bathtub curve function, and for $\lambda = 1$, it is constant.

3.3. **Mean Residual Life Function of the CCE-E Distribution.** Now, we will examine the mean residual life function of the CCE-E distribution which is another important characteristic of a random variable.

$$\begin{aligned} m(t) &= E(T-t\,|T>t) = \int_0^{\infty}(k-t)\,dP(T\le k\,|T>t) = \frac{\int_t^{\infty}\bar{H}(k)\,dk}{\bar{H}(t)} \\ &= \frac{(1-\lambda)\left(\frac{1}{\beta_1}e^{-\beta_1 t}+\frac{1}{\beta_2}e^{-\beta_2 t}\right)-(1-2\lambda)\left(\frac{1}{\beta_1+\beta_2}\right)e^{-(\beta_1+\beta_2)t}}{(1-\lambda)\left(e^{-\beta_1 t}+e^{-\beta_2 t}\right)-(1-2\lambda)\,e^{-(\beta_1+\beta_2)t}} \\ &= \left(\frac{1}{\beta_1+\beta_2}\right)-\frac{(1-\lambda)\left(\frac{1}{\beta_2}e^{-\beta_1 t}+\frac{1}{\beta_1}e^{-\beta_2 t}\right)}{(1-\lambda)\left(e^{-\beta_1 t}+e^{-\beta_2 t}\right)-(1-2\lambda)\,e^{-(\beta_1+\beta_2)t}}, \end{aligned} \tag{8}$$

The mean residual life function obtained in (8) is examined as follows:

$$m(0) = \frac{1-\lambda}{\beta_1}+\frac{1-\lambda}{\beta_2}-\frac{1-2\lambda}{\beta_1+\beta_2} = \frac{(1-\lambda)\left(\beta_1^2+\beta_2^2\right)+\beta_1\beta_2}{\beta_1\beta_2\left(\beta_1+\beta_2\right)}$$

$$\lim_{t\to\infty} m(t) = \begin{cases} \frac{1}{\beta_1} & \text{if } \beta_1<\beta_2; \\ \frac{1}{\beta_2} & \text{if } \beta_1>\beta_2. \end{cases}$$

and some possible shapes of the mean residual life function for selected parameter values are given in following figures.

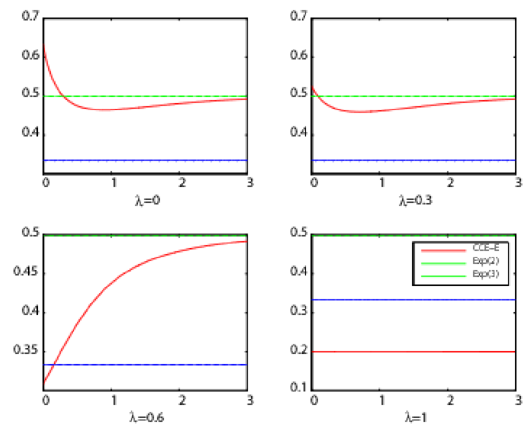

FIGURE 5. Compare plots of The mean residual life function of the CCE-E with baseline distribution for some parameter values

The mean residual life function, defined above, will be compared with the mean remaining life function of the baseline distributions for the values of the parameters $\beta_1 = 2,\ \beta_2 = 3$ in the following graph.

As you can see above (Figure 5), as T goes to infinity, the mean residual life function of the CCE-E distribution approaches the mean residual life function of exponential distribution with small parameter.

3.4. **Moment Generating Function and Moments of the CCE-E Distribution.** The moment generating function of the CCE-E random variable can be expressed as

$$\begin{aligned} M_T(k) &= (1-\lambda)\left(\frac{\beta_1}{\beta_1 - k} + \frac{\beta_2}{\beta_2 - k}\right) - (1-2\lambda)\left(\frac{\beta_1+\beta_2}{\beta_1+\beta_2-k}\right) \\ &= (1-\lambda)\left(M_{Exp(\beta_1)}(k) + M_{Exp(\beta_2)}(k)\right) - (1-2\lambda)\, M_{Exp(\beta_1+\beta_2)}(k) \end{aligned}$$

where $k < min(\beta_1, \beta_2)$. Thus, we give a weighted expression of the moment generating functions of the three exponential distributions with parameters $\beta_1,\ \beta_2$ and

$\beta_1+\beta_2$. The k^{th} moment of a random variable Tcan be obtained from (3) as

$$\begin{aligned} E(T^k) &= \Gamma(k+1)\left((1-\lambda)\left(\frac{1}{{\beta_1}^k}+\frac{1}{{\beta_2}^k}\right)-\frac{1-2\lambda}{(\beta_1+\beta_2)^k}\right), \qquad (9) \\ &= (1-\lambda)\left(E\left(T^k_{Exp(\beta_1)}\right)+E\left(T^k_{Exp(\beta_2)}\right)\right)-(1-2\lambda)\,E\left(T^k_{Exp(\beta_1+\beta_2)}\right) \end{aligned}$$

The expressions for the expected value is

$$E\left(T\right)=\left(1-\lambda\right)\left(\frac{1}{\beta_1}+\frac{1}{\beta_2}\right)-\frac{1-2\lambda}{\beta_1+\beta_2}$$

Then, the 2^{th}, 3^{th} and 4^{th} moments of a random variable T from equation (9) are given by

$$E\left(T^2\right)=\Gamma\left(3\right)\left(\left(1-\lambda\right)\left(\frac{1}{{\beta_1}^2}+\frac{1}{{\beta_2}^2}\right)-\frac{1-2\lambda}{\left(\beta_1+\beta_2\right)^2}\right)$$

$$E\left(T^3\right)=\Gamma\left(4\right)\left(\left(1-\lambda\right)\left(\frac{1}{{\beta_1}^3}+\frac{1}{{\beta_2}^3}\right)-\frac{1-2\lambda}{\left(\beta_1+\beta_2\right)^3}\right)$$

$$E\left(T^4\right)=\Gamma\left(5\right)\left(\left(1-\lambda\right)\left(\frac{1}{{\beta_1}^4}+\frac{1}{{\beta_2}^4}\right)-\frac{1-2\lambda}{\left(\beta_1+\beta_2\right)^4}\right)$$

3.5. **Skewness and Kurtosis of the CCE-E Distribution.** Based on the first four moments of the CCE-E random variable, the skewness measure of this random variable is given by

$$\gamma_1=\frac{1}{\left(\Gamma\left(3\right)\left(\left(1-\lambda\right)\left(\frac{1}{{\beta_1}^2}+\frac{1}{{\beta_2}^2}\right)-\frac{1-2\lambda}{\left(\beta_1+\beta_2\right)^2}\right)-\left(\left(1-\lambda\right)\left(\frac{1}{\beta_1}+\frac{1}{\beta_2}\right)-\frac{1-2\lambda}{\beta_1+\beta_2}\right)^2\right)^{\frac{3}{2}}}$$

$$\times\left(\begin{array}{l}\Gamma\left(4\right)\left(\left(1-\lambda\right)\left(\frac{1}{{\beta_1}^3}+\frac{1}{{\beta_2}^3}\right)-\frac{1-2\lambda}{\left(\beta_1+\beta_2\right)^3}\right)\\ -\Gamma\left(4\right)\left(\left(1-\lambda\right)\left(\frac{1}{\beta_1}+\frac{1}{\beta_2}\right)-\left(1-2\lambda\right)\left(\frac{1}{\beta_1+\beta_2}\right)\right)\\ \left(\left(1-\lambda\right)\left(\frac{1}{{\beta_1}^2}+\frac{1}{{\beta_2}^2}\right)-\frac{1-2\lambda}{\left(\beta_1+\beta_2\right)^2}\right)\\ +\Gamma\left(3\right)\left(\left(1-\lambda\right)\left(\frac{1}{\beta_1}+\frac{1}{\beta_2}\right)-\left(1-2\lambda\right)\left(\frac{1}{\beta_1+\beta_2}\right)\right)^3\end{array}\right), \qquad (10)$$

The values of the obtained skewness measures in (10) at the extreme values of λ will be examined,

$$\gamma_1=\begin{cases} 2 & \text{if } \lambda=0,1; \\ 2.5134 & \text{if } \lambda=\frac{1}{2}. \end{cases}$$

The skewness measures of the CCE-E distribution will be compared with the skewness measure of the exponential distribution with the following graph.

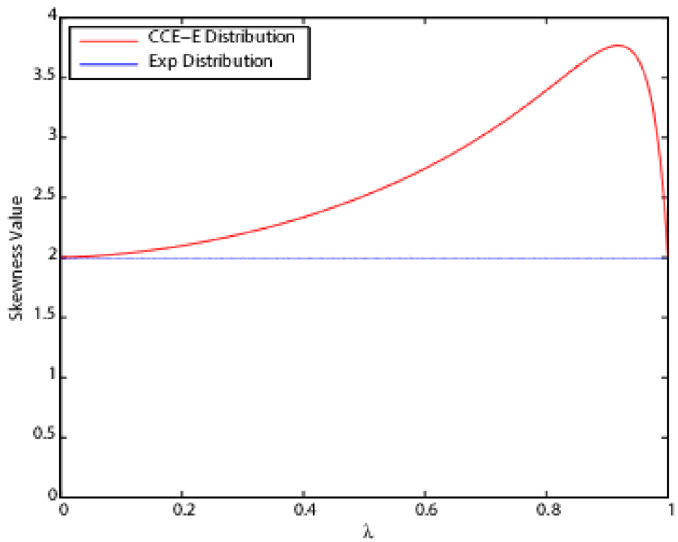

FIGURE 6. Comparison skewness measure of the CCE-E distribution with the skewness measures of the exponential distribution

The skewness measure of the exponential distribution is 2 and we can see that the skewness measure of the CCE-E distribution is positive and larger than 2, that is, the CCE-E distribution is a right skewed distribution. So, the skewness measure of this distribution is larger than the exponential distribution, it is more right skewed distribution than exponential distribution. When $\lambda = 0$ and $\lambda = 1$, the skewness of the CCE-E distribution is equal to the skewness measure of the exponential distribution. Now, the kurtosis measure of the CCE-E distribution will be found as follows

$$\gamma_2 = \frac{1}{\left(\Gamma(3)\left((1-\lambda)\left(\frac{1}{{\beta_1}^2}+\frac{1}{{\beta_2}^2}\right)-\frac{1-2\lambda}{(\beta_1+\beta_2)^2}\right)-\left((1-\lambda)\left(\frac{1}{\beta_1}+\frac{1}{\beta_2}\right)-\frac{1-2\lambda}{\beta_1+\beta_2}\right)^2\right)^2}$$

$$\times\left(\begin{array}{l} \Gamma(5)\left((1-\lambda)\left(\frac{1}{{\beta_1}^4}+\frac{1}{{\beta_2}^4}\right)-(1-2\,\lambda)\left(\frac{1}{(\beta_1+\beta_2)^4}\right)\right) \\ -\Gamma(5)\left((1-\lambda)\left(\frac{1}{\beta_1}+\frac{1}{\beta_2}\right)-(1-2\lambda)\left(\frac{1}{\beta_1+\beta_2}\right)\right) \\ \left((1-\lambda)\left(\frac{1}{{\beta_1}^3}+\frac{1}{{\beta_2}^3}\right)-\frac{1-2\lambda}{(\beta_1+\beta_2)^3}\right) \\ -\Gamma(4)\left((1-\lambda)\left(\frac{1}{\beta_1}+\frac{1}{\beta_2}\right)-(1-2\lambda)\left(\frac{1}{\beta_1+\beta_2}\right)\right)^4 \end{array}\right), \qquad (11)$$

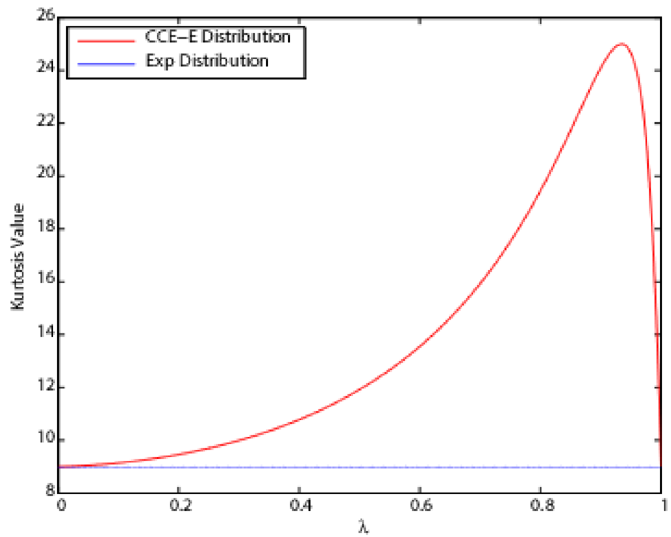

FIGURE 7. Comparison kurtosis measure of the CCE-E distribution with the kurtosis measures of the exponential distribution

The values of the obtained kurtosis measures in (11) at the extreme values of λ will be examined,

$$\gamma_2 = \begin{cases} 9 & \text{if } \lambda = 0, 1; \\ 11.9623 & \text{if } \lambda = \frac{1}{2}. \end{cases}$$

Then the kurtosis measures of the CCE-E distribution will be compared with the kurtosis measure of the exponential distribution with the following graph.

The kurtosis measure of the CCE-E distribution is larger than the kurtosis measure of the exponential distribution for $\lambda \in (0, 1)$. So, the p.d.f. of the CCE-E distribution has a more sharp-pointed shape than the p.d.f. of exponential distribution. For $\lambda = 0$ and $\lambda = 1$ the kurtosis measure of the CCE-E distribution is equal to the kurtosis measure of the exponential distribution.

3.6. **Estimation byMaximum Likelihood and the Information Matrix of the CCE-E Distribution.** Let $t = (t_1, t_2, \cdots, t_n)$ be observed values from this distribution with parameters β_1, β_2 *and* λ. The likelihood function for $\Theta = \{\beta_1, \beta_2, \lambda\}$ is given by

$$L(\Theta; t) = \prod_{i=1}^{n} \left((1-\lambda) \left(\beta_1 e^{-\beta_1 t} + \beta_2 e^{-\beta_2 t} \right) - (1-2\lambda) (\beta_1 + \beta_2) \; e^{-(\beta_1+\beta_2)t} \right).$$

Throughout this subsection, the log-likelihood function is denoted by $l = \log L(\Theta; t)$ for brevity. We differentiate l with respect to β_1, β_2 and λ as follows

$$\frac{\partial l}{\partial \beta_1} = \sum_{\imath=1}^{n} \frac{(1-\lambda)(1-\beta_1 t_i) e^{-\beta_1 t_i} - (1-2\lambda)(1-\beta_1 t_i - \beta_2 t_i) e^{-(\beta_1+\beta_2)t_i}}{h(t_i;\Theta)}, \quad (12)$$

$$\frac{\partial l}{\partial \beta_2} = \sum_{\imath=1}^{n} \frac{(1-\lambda)(1-\beta_2 t_i) e^{-\beta_2 t_i} - (1-2\lambda)(1-\beta_1 t_i - \beta_2 t_i) e^{-(\beta_1+\beta_2)t_i}}{h(t_i;\Theta)}, \quad (13)$$

$$\frac{\partial l}{\partial \lambda} = \sum_{\imath=1}^{n} \frac{-\left(\beta_1 e^{-\beta_1 t_i} + \beta_2 e^{-\beta_2 t_i}\right) + 2(\beta_1+\beta_2) e^{-(\beta_1+\beta_2)t_i}}{h(t_i;\Theta)}, \quad (14)$$

The maximum likelihood estimators as $\hat{\beta}_1$, $\hat{\beta}_2$ and $\hat{\lambda}$ are obtained by equating these three equations (12), (13) and (14) to zero and solving the equations simultaneously. For these three parameters, we will get the second order derivative of logarithms of the likelihood functions for obtaining the elements of the Fisher-Information Matrix.

$$\begin{aligned} I_{\beta_1\beta_1} = \frac{\partial^2 l}{\partial \beta_1^2} &= -\sum_{\imath=1}^{n} \frac{\left((1-\lambda) e^{-\beta_1 t_i} - (1-2\lambda) e^{-(\beta_1+\beta_2)t_i}\right)^2}{(h(t_i;\Theta))^2} \\ &+ \sum_{\imath=1}^{n} \frac{(1-\lambda)^2 (2-\beta_1 t_i) \beta_2 t_i e^{-(\beta_1+\beta_2)t_i}}{(h(t_i;\Theta))^2} \\ &+ \sum_{\imath=1}^{n} \frac{(1-\lambda)(1-2\lambda)(2-\beta_1 t_i - \beta_2 t_i) \beta_2 t_i e^{-(\beta_1+2\beta_2)t_i}}{(h(t_i;\Theta))^2} \end{aligned}$$

$$\begin{aligned} I_{\beta_2\beta_2} = \frac{\partial^2 l}{\partial \beta_2^2} &= -\sum_{\imath=1}^{n} \frac{\left((1-\lambda) e^{-\beta_2 t_i} - (1-2\lambda) e^{-(\beta_1+\beta_2)t_i}\right)^2}{(h(t_i;\Theta))^2} \\ &+ \sum_{\imath=1}^{n} \frac{(1-\lambda)^2 (2-\beta_2 t_i) \beta_1 t_i e^{-(\beta_1+\beta_2)t_i}}{(h(t_i;\Theta))^2} \\ &- \sum_{\imath=1}^{n} \frac{(1-\lambda)(1-2\lambda)(2-\beta_1 t_i - \beta_2 t_i) \beta_1 t_i e^{-(2\beta_1+\beta_2)t_i}}{(h(t_i;\Theta))^2} \end{aligned}$$

$$I_{\lambda\lambda} = \frac{\partial^2 l}{\partial \lambda^2} = -\sum_{\imath=1}^{n} \left(\frac{-\left(\beta_1 e^{-\beta_1 t_i} + \beta_2 e^{-\beta_2 t_i}\right) + 2(\beta_1+\beta_2) e^{-(\beta_1+\beta_2)t_i}}{h(t_i;\Theta)} \right)^2$$

$$\begin{aligned} I_{\beta_1\beta_2} = I_{\beta_2\beta_1} &= \frac{\partial^2 l}{\partial \beta_2 \partial \beta_1} \\ &= -\sum_{\imath=1}^{n} \frac{-(1-\lambda)(1-2\lambda)\left((1-\beta_1 t_i) e^{-(\beta_1+2\beta_2)t_i} + (1-\beta_2 t_i) e^{-(2\beta_1+\beta_2)t_i}\right)}{(h(t_i;\Theta))^2} \end{aligned}$$

$$-\sum_{\imath=1}^{n}\frac{(1-2\lambda)^2\left(e^{-2(\beta_1+\beta_2)t_i}-(1-\beta_1t_i)(1-\beta_2t_i)\,e^{-(\beta_1+\beta_2)t_i}\right)}{(h(t_i;\Theta))^2}$$

$$I_{\beta_1\lambda}=I_{\lambda\beta_1}=\frac{\partial^2 l}{\partial\lambda\partial\beta_1}=-\sum_{\imath=1}^{n}\frac{\beta_2e^{-(2\beta_1+\beta_2)t_i}-(1-\beta_1t_i-\beta_2t_i)\,\beta_2e^{-(\beta_1+2\beta_2)t_i}}{(h(t_i;\Theta))^2}$$

$$I_{\beta_2\lambda}=I_{\lambda\beta_2}=\frac{\partial^2 l}{\partial\lambda\partial\beta_2}=-\sum_{\imath=1}^{n}\frac{\beta_1e^{-(\beta_1+2\beta_2)t_i}-(1-\beta_1t_i-\beta_2t_i)\,\beta_1e^{-(2\beta_1+\beta_2)t_i}}{(h(t_i;\Theta))^2}$$

Thus, Fisher information matrix, $I_n(\Theta)$ of sample size n for Θ is as follows:

$$I_n(\Theta)=-E\begin{pmatrix} I_{\beta_1\beta_1} & I_{\beta_1\beta_2} & I_{\beta_1\lambda}\\ I_{\beta_2\beta_1} & I_{\beta_2\beta_2} & I_{\beta_2\lambda}\\ I_{\lambda\beta_1} & I_{\lambda\beta_2} & I_{\lambda\lambda}\end{pmatrix}$$

Inverse of the Fisher-information matrix of single observation, i.e., $I_1^{-1}(\Theta)$ indicates asymptotic variance-covariance matrix of maximum likelihood estimates of Θ. Hence, joint distribution of maximum likelihood estimator for Θ is asymptotically normal with mean Θ and variance-covariance matrix $I_1^{-1}(\Theta)$. Namely

$$\sqrt{n}\left(\begin{bmatrix}\hat{\beta}_1\\ \hat{\beta}_2\\ \hat{\lambda}\end{bmatrix}-\begin{bmatrix}\beta_1\\ \beta_2\\ \lambda\end{bmatrix}\right)\sim AN\left(\begin{pmatrix}0\\0\\0\end{pmatrix},I_1^{-1}(\Theta)\right). \tag{15}$$

By solving this inverse dispersion matrix these solutions will yield asymptotic variance and covariances of these ML estimators for these parameters. We can approximate $100(1-\gamma)\,\%$ confidence intervals for β_1, β_2 and λ by using (15) are obtained respectively as

$$\hat{\beta}_1\pm z_{1-\frac{\gamma}{2}}\sqrt{I^{-1}_{\beta_1\beta_1}},\ \hat{\beta}_2\pm z_{1-\frac{\gamma}{2}}\sqrt{I^{-1}_{\beta_2\beta_2}},\ \hat{\lambda}\pm z_{1-\frac{\gamma}{2}}\sqrt{I^{-1}_{\lambda\lambda}},$$

where $z_{\frac{\gamma}{2}}$ is the upper 100γ quantile of the standard normal distribution.

3.7. **Rényi Entropy of the CCE-E Distribution.**

By using (3) for $h(t)$ in the equation (5) and applying the generalized Binomial expansion, then we have

$$(h(t))^{\rho}=\sum_{j=0}^{\infty}\binom{\rho}{j}\left((1-\lambda)\left(\beta_1e^{-\beta_1t}+\beta_2e^{-\beta_2t}\right)\right)^{\rho-j}\left(-(1-2\lambda)\ (\beta_1+\beta_2)\ e^{-(\beta_1+\beta_2)t}\right)^j$$

$$=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^j\binom{\rho}{j}\binom{\rho-j}{k}(1-\lambda)^{\rho-j}(1-2\lambda)^j\beta_1^{\rho-j-k}\beta_2^{k}(\beta_1+\beta_2)^je^{-((\rho-k)\beta_1+(k+j)\beta_2)t}$$

Rényi entropy can be obtained as follows

$$I_R(\rho)=\frac{1}{1-\rho}\log\left[\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^j\binom{\rho}{j}\binom{\rho-j}{k}(1-\lambda)^{\rho-j}(1-2\lambda)^j\frac{\beta_1^{\rho-j-k}\beta_2^{k}(\beta_1+\beta_2)^j}{(\rho-k)\,\beta_1+(k+j)\,\beta_2}\right]$$

3.8. **Order Statistics of the CCE-E Distribution.** Let's $T_{(1)}, T_{(2)}, \ldots, T_{(n)}$ denote the order statistics of a random sample $T_1, T_2, \ldots, T_n$ from a continuous population with p.d.f. $h(t)$ and cdf $H(t)$, then the p.d.f. of $T_{(j)}$ is given as follows

$$\begin{aligned} f_{T_{(j)}}(t) &= \frac{n!}{(j-1)!\,(n-j)!} h(t)\,[H(t)]^{j-1}[1-H(t)]^{n-j}, \qquad j = 1, 2, \ldots, n \\ &= \frac{n!}{(j-1)!\,(n-j)!} \left((1-\lambda)\left(\beta_1 e^{-\beta_1 t} + \beta_2 e^{-\beta_2 t}\right) - (1-2\lambda)(\beta_1+\beta_2)\, e^{-(\beta_1+\beta_2)t}\right) \\ &\qquad \times \left[1-(1-\lambda)\left(e^{-\beta_1 t} + e^{-\beta_2 t}\right) + (1-2\lambda)\, e^{-(\beta_1+\beta_2)t}\right]^{j-1} \\ &\qquad \times \left[(1-\lambda)\left(e^{-\beta_1 t} + e^{-\beta_2 t}\right) - (1-2\lambda)\, e^{-(\beta_1+\beta_2)t}\right]^{n-j} \end{aligned}$$

therefore, the p.d.f. of the first order statistics $T_{(1)}$ is given by

$$\begin{aligned} f_{T_{(1)}}(t) &= n\left((1-\lambda)\left(\beta_1 e^{-\beta_1 t} + \beta_2 e^{-\beta_2 t}\right) - (1-2\lambda)(\beta_1+\beta_2)\, e^{-(\beta_1+\beta_2)t}\right) \\ &\qquad \times \left[(1-\lambda)\left(e^{-\beta_1 t} + e^{-\beta_2 t}\right) - (1-2\lambda)\, e^{-(\beta_1+\beta_2)t}\right]^{n-1} \end{aligned}$$

and the p.d.f. of the n. order statistics $T_{(n)}$ is given

$$\begin{aligned} f_{T_{(n)}}(t) &= n\left((1-\lambda)\left(\beta_1\ e^{-\beta_1 t} + \beta_2 e^{-\beta_2 t}\right) - (1-2\lambda)(\beta_1+\beta_2)\, e^{-(\beta_1+\beta_2)t}\right) \\ &\qquad \times \left[1-(1-\lambda)\left(e^{-\beta_1 t} + e^{-\beta_2 t}\right) + (1-2\lambda)\, e^{-(\beta_1+\beta_2)t}\right]^{j-1} \end{aligned}$$

Note that $\lambda = 1$ yields the order statistics of the exponential distribution with parameter $(\beta_1 + \beta_2)$.

4. Numerical Examples

We illustrate the applicability of proposed distribution by considering three different data sets which have been examined by a lot of other researchers. We compare the CCE-E distribution with the different distributions that are defined before this work. In addition to, we consider different p.d.f.s for the second baseline distribution in (3), such as Gamma (CCE-G), Lognormal (CCE-Ln), Rayleigh (CCE-R) and Weibull (CCE-W) distributions. In order to compare distributional models, some criteria as K-S (Kolmogorow-Smirnow), $-2LL$ (-2LogL), AIC (Akaike information criterion) and BIC (Bayesian information criterion) are taken into account for the data sets. When comparing the CCE-E distribution with the other distributions, only distributions with small K-S values are considered.

4.1. **Data Set (Waiting time (in minutes) before customer service in Bank B).** The data set is given as the waiting time (in minutes) before customer service at bank B. This data was analyzed by [4] and was also used by [23]. They fit this data to Lindley (L) and generalized Lindley (GL) distributions and we fit this data to proposed distributions. Thus, parameter estimates are $\hat{\beta}_1 = 0.2082$, $\hat{\beta}_2 = 0.2082$ and $\hat{\lambda} = 0.1721$. According to the model selection criteria (K-S) tabulated in Table 1, it is said that the CCE-E takes the best place in amongst these six models.

Table 1. Model selection criteria for Bank B data

Data set	**K-S**	**-2LL**	**AIC**	**BIC**
L	0.080	338.203	340.203	341.759
GL	0.068	338.026	342.026	341.582
CCE-E	0.063	338.142	344.142	350.425
CCE-G	0.093	339.208	345.208	351.492
CCE-R	0.111	343.450	349.450	355.733
CCE-W	0.126	341.910	347.910	354.193

4.2. **Data Set (Exceedances of Wheaton River flood data).** The data consists of the exceedances of flood peaks (in m3/s) of the Wheaton River near Carcross in Yukon Territory, Canada. The data consists of 72 exceedances for the years 1958–1984, rounded to one decimal place. This data was used by [3] to apply Beta-Pareto (BP) distribution. Merovci and Pukab [19] made a comparison between Pareto (P) and transmuted Pareto (TP) distribution and they showed that better model is the transmuted Pareto distribution (TP). Bourguignon et al. [9] proposed Kumaraswamy (Kw) Pareto distribution (Kw-P). Yilmaz et al. [24] have proposed exponential modified discrete Lindley (EMDL) distribution. We fit this data to the CCE-E distribution and get parameter estimates as $\hat{\beta}_1 = 0.0706, \hat{\beta}_2 = 0.9092$ and $\hat{\lambda} = 0.1536$. According to the model selection criteria (AIC, or BIC) tabulated in Table 2, we can see different form of proposed distribution that defined in (3) is the best than other distribution and it is said that the CCE-E takes second place in amongst ten models.

Table 2. Model selection criteria for river flood data

Model	**K-S**	**-2LL**	**AIC**	**BIC**
TP	0.389	572.401	578.4	580.9
P	0.456	606.128	610.1	610.4
BP	0.175	567.400	573.4	580.3
Kw-P	0.170	542.400	548.4	555.3
EMDL	0.116	503.574	507.6	512.1
CCE-E	0.079	499.164	505.164	511.994
CCE-G	0.063	495.971	501.971	508.801
CCE-Ln	0.144	537.013	543.013	549.843
CCE-R	0.093	497.484	503.484	510.314
CCE-W	0.142	503.855	509.855	516.685

4.3. **Data Set (Bladder cancer).** The data is extracted from [14] represents remission times (in months) of a random sample of 128 bladder cancer patients. Several authors analyzed this data set. Merovci [15], observed that the Lindley (L) and transmuted Lindley (TL) distributions work quite well for this data. Also, this data was fitted to the two parameter Lindley (TPL) and transmuted two parameter Lindley (TTL) distribution with [2] to the subject data. This data is fitted to the CCE-E distribution and the parameter estimates are $\hat{\beta}_1 = 0.1064$, $\hat{\beta}_2 = 0.3519$ and $\hat{\lambda} = 0.09$. K-S and AIC values in Table 3 indicate that the CCE-E fits well among the other distributions considered here.

Table 3. Model selection criteria for bladder cancer

Distributions	K-S	-2LL	AIC	BIC
L	0.0740	839.040	841.040	—
TL	0.2265	830.310	834.310	—
TPLD	0.0846	828.684	832.684	—
TTLD	0.0637	825.884	825.884	—
CCE-E	0.0539	822.074	828.074	836.630
CCE-G	0.1222	839.073	845.073	853.629
CCE-Ln	0.1735	873.574	879.574	888.131
CCE-R	0.0433	820.967	826.967	835.523
CCE-W	0.0841	828.207	834.207	842.763

5. Conclusions

This work focuses on two new ideas. The first one is that a transmuted distribution is actually a convex composition of min-max distributions of 2-sized sample. In the last three years under the name of the transmuted distribution, over 50 studies have been carried out. All of these are innovative contributions in terms of statistical modeling. The second is that the proposed new distribution has two baselines. According to the obtained results for real data sets, we can conclude that this method achieves success in modeling by giving more flexibility to the distribution. Therefore, many more new distributions such as Exp-Weib, Weib-Rayleigh, and Weib-Lindley can be derived for subsequent studies. Obviously, this will bring innovation in addition to the existing works.

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