Available online: January 10, 2019

Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.
Volume 68, Number 1, Pages 1163–1174 (2019)
DOI: 10.31801/cfsuasmas.509926
ISSN 1303–5991 E-ISSN 2618-6470

http://communications.science.ankara.edu.tr/index.php?series=A1

ACTIONS OF SOFT GROUPS

GULAY OGUZ, ILHAN ICEN, AND M. HABIL GURSOY

Abstract. The soft set theory proposed by Molodtsov is a recent mathematical approach for modeling uncertainty and vagueness. The main aim of this study is to introduce the concept of soft action by combining soft set theory with the action which is an important concept in dynamical systems theory. Moreover, different types of soft action are presented and some important characterizations are given. Finally, we define the concept of soft symmetric group and present the relation between the soft action and soft symmetric group, as a similar result to the classical Cayley's Theorem.

1. Introduction

There are many complex problems in the real world. It is not always possible to find complete and precise solutions to these problems with traditional methods. For this reason, scientists have developed some theories to solve such problems. One of these theories modeled uncertainty is the soft set theory introduced by Molodtsov in 1999 [2]. Although this theory has not had a long history, it has created a wide range of applications in many disciplines. Especially, this theory has been studied by mathematicians in different ways such as algebraic, topological and categorical [1, 3, 6, 8, 10, 12].

On the other hand, there are some concepts that are interdisciplinary. The action is one of the such concepts and it has a great important in mathematics and physics [5, 13]. It is an integral part of the theory of group and dynamical systems which represent a mathematical equation class that defines time-based systems with specific properties [4, 11]. Furthermore, the theory of dynamical systems, a mathematical theory such as soft set theory, is directly related to the developments in the understanding of complex and nonlinear systems of physics and mathematics [7, 9].

In this study, soft action is defined and studied as a new concept. Examples of this concept are given and some important properties are presented. Some concepts

Received by the editors: October 11, 2017; Accepted: June 29, 2018.
2010 *Mathematics Subject Classification.* Primary 20A05; Secondary 03E99, 58E40.
Key words and phrases. Soft set, soft group, soft action, soft symmetric group.

related to the action such as stabilizers, centralizers and normalizers are defined in the soft approach. Finally, the concept of soft symmetric group is introduced and the some relations between the soft symmetric group and the soft action are investigated.

Briefly, it can be said that this study is the construction of a new bridge between the soft set theory and the group theory, and presents a new and different algebraic viewpoint for soft set theory .

2. Preliminaries

In this section, we review some main concepts and properties of soft sets and soft groups for the sake of completeness. For more details, we refer to $[1-3,6,8,12]$.

Molodtsov described the soft set in the following way [2]. Let X be an initial universe set and let E be a set of parameters. Let $P(X)$ denotes the power set of X and $A \subset E$.

Definition 2.1. [2] *A pair (F,A) is called a soft set over X, where F is a mapping defined by*

$$F : A \longrightarrow P(X)$$

In the light of this definition, it is said that a soft set over X can be considered as a parametrized family of subsets of the universe X. We will sometimes use the notation (X,F,A) instead of soft set (F,A) over X.

Definition 2.2. [8] *Let (F,A) and (G,B) be two soft sets over the common universe X. Then, (F,A) is called a soft subset of (G,B) if*
i) *$A \subset B$, and*
ii) *$\forall a \in A$, $f(a)$ and $G(a)$ are identical approximations.*
We denote it as $(F,A)\widetilde{\subset}(G,B)$.

Definition 2.3. [8] *A soft set (F,A) over X is called a null soft set denoted by Φ, if $F(a)=\emptyset$ for all $a \in A$.*

Definition 2.4. [8] *A soft set (F,A) over X is called an absolute soft set denoted by $\tilde{A}$, if $F(a)=X$ for all $a \in A$.*

Definition 2.5. [8] *Let (F,A) and (G,B) be two soft sets over the common universe X. Their intersection is a soft set (H,C) such that $C = A \cap B$ and $H(a) = F(a) \cap G(a)$, $\forall a \in C$.*
We write it as $(F,A)\widetilde{\cap}(G,B)=(H,C)$.

Definition 2.6. [8] *Let (F,A) and (G,B) are both soft sets over the common universe X. Their union is a soft set (H,C), where $C = A \cup B$, and $\forall a \in C$,*

$$H(e) = \begin{cases} F(a), & \text{if } a \in A-B \\ G(a), & \text{if } a \in B-A \\ F(a) \cup G(a), & \text{if } a \in A \cap B \end{cases}$$

We write it as $(F,A)\widetilde{\cup}(G,B)=(H,C)$.

Let us recall the concept of soft group. For properties of soft group, we refer the reader to [29]. Throughout this section, G is a group and A is a nonempty set.

Definition 2.7. [3] *For the soft set (F, A) over G, it is said that (F, A) is a soft group over G if and only if $F(a) < G$ for all $a \in A$.*

From the definition, it is easy to see that the soft group (F, A) is a parameterized family of subgroups of the group G.

In what follows, the triplet (G, F, A) stands for the soft group (F, A) over G.

Example 2.8. [3] *Suppose that $G = A = S_3 = \{e, (12), (13), (23), (123), (132)\}$, and $F(e) = \{e\}$, $F(12) = \{e, (12)\}$, $F(13) = \{e, (13)\}$, $F(23) = \{e, (23)\}$, $F(123) = F(132) = \{e, (123), (132)\}$. It is easy to verify that $F(a)$ is a subgroup of G for all $a \in A$. Hence, (F, A) is a soft group over G.*

Proposition 2.9. [3] *Let (F, A) and (H, B) be two soft groups over G. Then, their intersection $(F, A)\widetilde{\cap}(H, B)$ is also a soft group over G.*

Proposition 2.10. [3] *Let (F, A) and (H, B) be two soft groups over G. If $A \cap B = \emptyset$, then $(F, A)\widetilde{\cup}(H, B)$ is also a soft group over G.*

Definition 2.11. [3] *Let (F, A) be a soft set over G. Then, for all $a \in A$*
i) *(F, A) is called an identity soft group over G if $F(a) = \{e\}$, where e is the identity element of G.*
ii) *(F, A) is called an absolute soft group over G if $F(a) = G$.*

Definition 2.12. [3] *Let (F, A) and (H, B) are both soft groups over G. Then, (H, B) is said to be a soft subgroup of (F, A), denoted by $(H, K)\tilde{<}(F, A)$, if*
i) *$B \subset A$,*
ii) *$H(a) < F(a)$ for all $a \in B$.*

Definition 2.13. [3] *Let (F, A) be a soft group over G and let (H, B) be a soft subgroup of (F, A). Then, we say that (H, B) is a normal soft subgroup of (F, A), written $(H, B)\tilde{\lhd}(F, A)$, if $H(a)$ is a normal subgroup of $F(a)$, i.e., $H(a) \lhd F(a)$ for all $a \in B$.*

Definition 2.14. [3] *Let (F, A) and (H, B) be two soft groups over G and K, respectively, and let $f : G \longrightarrow K$ and $g : A \longrightarrow B$ be two functions. Then, (f, g) is called a soft homomorphism if*
i) *f is a homomorphism from G onto K,*
ii) *g is a mapping from A onto B, and*
iii) *$f(F(a)) = H(g(a))$ for all $a \in A$.*

Thus, (F, A) is called soft homomorphic to (H, B) and this situation denoted by $(F, A) \sim (H, B)$.

3. Actions of Soft Groups

In this section, we will describe the concept of soft action and exemplify it. In addition, we will establish some characterizations about it.

Definition 3.1. *Let $G = (G, F, A)$ be a soft group and let $X = (X, F', A)$ be a soft set. Then, a left soft (group) action of G on X is a binary operation*

$$\Pi_a : F(a) \times F'(a) \longrightarrow F'(a)$$

satisfying the following two axioms for all $a \in A$:

i) $\Pi_a(e, x) = x$ *for all $x \in F'(a)$*

ii) $\Pi_a(g, \Pi_a(h, x)) = \Pi_a(gh, x)$ *for all $x \in F'(a)$ and $g, h \in F(a)$.*

Namely, each Π_a is a left (group) action mapping for all $a \in A$. Then, the soft set X is called a $G-$soft set.

Example 3.2. *Suppose that $A = G = S_3$, the group of permutations on the $S = \{1, 2, 3\}$. The soft group $G = (G, F, A)$ is defined by*

$$\begin{aligned} F(e) &= \{e\}, \\ F(12) &= \{e, (12)\}, \\ F(13) &= \{e, (13)\}, \\ F(23) &= \{e, (23)\}, \\ F(123) = F(132) &= \{e, (123), (132)\}, \end{aligned}$$

The soft set $X = (X, F, A)$ with $X = \{1, 2, 3\}$ is defined by

$$\begin{aligned} F'(e) &= \{1\}, \\ F'(12) &= \{1, 2\}, \\ F'(13) &= \{1, 3\}, \\ F'(23) &= \{2, 3\}, \\ F(123) = F(132) &= \{1, 2, 3\}, \end{aligned}$$

For all $a \in A$, the mapping

$$\begin{array}{rcl} \Pi_a : F(a) \times F'(a) & \longrightarrow & F'(a) \\ (g, x) & \mapsto & \Pi_a(g, x) = \sigma(x) \end{array}$$

is a left (group) action. Note that $e \in G$ is the identity element, and $\Pi_a(e, x) = x$ as desired. For axiom (2), note that if $g, h \in F(a)$, then $\Pi_a(g, \Pi_a(h, x)) = \Pi_a(gh, x)$ as desired.

Remark 3.3. *In the above definition, if the direction of the left (group) action mapping Π_a is reversed for all $a \in A$, then a right soft (group) action of G on X is obtained.*

As it can be seen clearly, right soft (group) actions are not very different from the left soft (group) actions. Only difference between them is direction of the action. For this reason, the dual of the examples of left soft (group) action can be given as examples of right soft (group) actions. Here, we exemplify the concept of right soft (group) action as follows:

Example 3.4. *Let G be a soft group and $H \subset G$ soft subgroup. Suppose that*

$$X = \{Hg : \ g \in F(a)\}$$

is a right soft coset of H. Then, for all $g_1 \in G$, the each mapping

$$\begin{array}{rcl} \Pi_a : F(a)' \times F(a) & \longrightarrow & F'(a) \\ (Hg, g_1) & \mapsto & \Pi_a(Hg, g_1) = H(gg_1) \end{array}$$

is a group action, namely,

i) $\Pi_a(Hg, e) = H(ge) = Hg$

ii) $$\begin{aligned} \Pi_a(\Pi_a(Hg, g_1), g_2) &= \Pi_a(H(gg_1), g_2) \\ &= H((gg_1)g_2) \\ &= H(g(g_1g_2)) \\ &= Hg(g_1g_2) \\ &= \Pi_a(Hg, g_1g_2) \end{aligned}$$

for all $g_1, g_2 \in F(a)$.

Thus, the soft group G acts on the set of right soft cosets of H on the right. Similarly, we can show that the soft group G acts on the set of left soft cosets of H on the left.

From now on, we will use the left soft (group) actions unless otherwise stated.

Proposition 3.5. *Let X be a $G-$soft set. For $x, y \in F'(a)$, let $x \sim y$ if and only if there exists $g \in F(a)$ such that $y = \Pi_a(g, x)$. Then, the relation $\sim$ is an equivalence relation on X.*

Proof. Straightforward. □

It is also easy to see that

Remark 3.6. *Denote $Orb_G(x)$ the orbit of $\sim$ containing x. Clearly*

$$Orb_G(x) = \{\Pi_a(g, x) : \ g \in F(a)\}$$

is a G-soft set. Morever, the pair (Orb_G, X) is a soft set over X with the mapping

$$Orb_G : X \longrightarrow P(X)$$

In the light of this information, the following proposition is given.

Proposition 3.7. *If $x \in F'(a)$, then $x \in Orb_G(x)$*

Proof. $x = \Pi_a(e, x) \in Orb_G(x)$. □

In addition, two proposition that exist in the group theory are expressed by the soft approach as follows:

Proposition 3.8. *Let X be a $G-$soft set. Then, the (distinct) orbits of G partition X.*

Proof. We need to prove the following two conditions.
i) Every element of X is in some orbit.
ii) If $Orb_G(x) \cap Orb_G(x') \neq \emptyset$, then $Orb_G(x) = Orb_G(x')$.

Because $x \in Orb_G(x)$ and Proposition 3.7, the first condition is clear. For condition (ii), let $y \in Orb_G(x) \cap Orb_G(x')$. Then, there exist elements $g, h \in F(a)$ such that $y = \Pi_a(g, x) = \Pi_a(h, x')$. Therefore,

$$x = \Pi_a(e, x) = \Pi_a(gg^{-1}, x) = \Pi_a(g^{-1}, \Pi_a(g, x)) = \Pi_a(g^{-1}, \Pi_a(h, x')) = \Pi_a(g^{-1}h, x')$$

and so $x \in Orb_G(x') = \{\Pi_a(g_1, x') : g_1 \in F(a)\}$. Thus,

$$\begin{aligned} Orb_G(x) = \{\Pi_a(g', x) : g' \in F(a)\} &\subseteq \{\Pi_a(g', g_1 x') : g', g_1 \in F(a)\} \\ &\subseteq \{\Pi_a(g', x') : g' \in F(a)\} = Orb_G(x') \end{aligned}$$

where the last condition holds since we can take $g_1 = e$. On the other hand, from the symmetry property, we also get $Orb_G(x') \subseteq Orb_G(x)$. Hence, we have $Orb_G(x) = Orb_G(x')$.

Thus, These two conditions prove that X is covered by disjoint orbits. □

Proposition 3.9. *Suppose that $G = (G, F, A)$ acts on a soft set $X = (X, F', A)$. If $x \in F'(a)$, $g \in F(a)$, and $y = \Pi_a(g, x)$, then $x = \Pi_a(g^{-1}, y)$. Furthermore, if $x \neq x'$ then $\Pi_a(g, x) \neq \Pi_a(g, x')$.*

Proof. Because of $y = \Pi_a(g, x)$, we obtain

$$\Pi_a(g^{-1}, y) = \Pi_a(g^{-1}, \Pi_a(g, x)) = \Pi_a(g^{-1}g, x) = \Pi_a(e, x) = x$$

We suppose that if $x \neq x'$ then $\Pi_a(g, x) = \Pi_a(g, x')$. Applying g^{-1} to both sides, we have

$$\Pi_a(g^{-1}, \Pi_a(g, x)) = \Pi_a(g^{-1}, \Pi_a(g, x'))$$
$$\Pi_a(g^{-1}g, x) = \Pi_a(g^{-1}g, x')$$
$$\Pi_a(e, x) = \Pi_a(e, x')$$
$$x = x'$$

This contradicts by the fact that $x \neq x'$. Thus, the proof is completed. □

Here, we will present several trivial examples of the soft action.

Example 3.10. *Every soft group G acts on itself as follows. Take $X = G$. Then for all $a \in A$, the mapping*

$$\begin{aligned} \Pi_a : F(a) \times F(a) &\longrightarrow F(a) \\ (g, h) &\mapsto \Pi_a(g, h) = h \end{aligned}$$

is group action. Thus, this action is called trivial soft action of G on itself.

Example 3.11. *Every soft group G acts on itself by multiplication as follows. Let us take the soft group G as soft set X. Then for all $a \in A$, the mapping*

$$\begin{array}{rcl} \Pi_a : F(a) \times F(a) & \longrightarrow & F(a) \\ (g,h) & \mapsto & \Pi_a(g,h) = gh \end{array}$$

is a group action.

Example 3.12. *Every soft group G acts on itself by conjugation as follows. Take $X = G$. Then for all $a \in A$, the mapping*

$$\begin{array}{rcl} \Pi_a : F(a) \times F(a) & \longrightarrow & F(a) \\ (g,h) & \mapsto & \Pi_a(g,h) = ghg^{-1} \end{array}$$

is a group action. It is clear that axioms of the soft action are verified.

Remark 3.13. *Note that if soft group G is abelian, then the soft action of G on itself by conjugation is trivial.*

In addition to these examples, two specific examples can be given as follows:

Example 3.14. *Let $Y = (Y, F'', A)$ be any soft set, and let $G = (G, F, A)$ be a soft group acting on a soft set $X = (X, F', A)$ as shown below.*

$$\begin{array}{rcl} \Pi_a : F(a) \times F'(a) & \longrightarrow & F'(a) \\ (g,x) & \mapsto & \Pi_a(g,x) \end{array}$$

Let $f : X \longrightarrow Y$ be an injective function. Then, we can construct a soft action of G on Y by $F'' = f(F')$ as follows:

$$\begin{array}{rcl} \Pi'_a : F(a) \times f(F'(a)) & \longrightarrow & f(F'(a)) \\ (g, f(x)) & \mapsto & \Pi'_a(g, f(x)) = f(\Pi_a(g,x)) \end{array}$$

It is easy to see that Π'_a provides the conditions in Definition 3.1 for all $a \in A$.

Example 3.15. *Let $G = (G, F, A)$ be a soft group with $A = G = S_n$ symmetric groups and $X = (X, F', A)$ be a soft set with $X = f(P_1, P_2, \ldots, P_n)$, being the polynomials of the set $\{P_1, P_2, \ldots, P_n\}$. The soft action of G on $f(P_1, P_2, \ldots, P_n)$ can be defined as*

$$\begin{array}{rcl} \Pi_a : F(a) \times F'(a) & \longrightarrow & F'(a) \\ (\sigma, f(P_1, P_2, \ldots, P_n)) & \mapsto & \Pi_a(\sigma, f(P_1, P_2, \ldots, P_n)) = f(P_{\sigma(1)}, P_{\sigma(2)}, \ldots, P_{\sigma(n)}) \end{array}$$

such that the following two conditions of action hold:

i) For $\sigma = e$, $\Pi_a(e, f(P_1, P_2, \ldots, P_n)) = f(P_{e(1)}, P_{e(2)}, \ldots, P_{e(n)}) = f(P_1, P_2, \ldots, P_n)$

ii) For all $\sigma, \sigma' \in S_n$

$$\begin{array}{rcl} \Pi_a(\sigma, \Pi_a(\sigma', f(P_1, P_2, \ldots, P_n))) & = & \Pi_a(\sigma, f(P_{\sigma(1)}, P_{\sigma(2)}, \ldots, P_{\sigma(n)})) \\ & = & f(P_{\sigma(\sigma'(1))}, P_{\sigma(\sigma'(2))}, \ldots, P_{\sigma(\sigma'(n))}) \\ & = & \Pi_a(\sigma\sigma', f(P_1, P_2, \ldots, P_n))) \end{array}$$

It is generally agreed that some concepts that exist in one theory can be redefined by appropriately transferring to another theory. Here, our purpose is to examine some action types that exist in group theory by the soft approach.

Definition 3.16. *A soft action of the soft group* (G, F, A) *on the soft set* (X, F', A) *is called transitive if for each pair* x, y *in* $F'(a)$ *there exists an element* g *in* $F(a)$ *such that* $\Pi_a(g, x) = x$.

Definition 3.17. *A soft action of the soft group* (G, F, A) *on the soft set* (X, F', A) *is called effective (or faithful) if for every two distinct elements* g, h *in* $F(a)$ *there is an element* x *in* $F'(a)$ *such that* $\Pi_a(g, x) \neq \Pi_a(h, x)$.

This is equivalent to saying that different elements of $F(a)$ *act on* $F'(a)$ *in different ways.*

Definition 3.18. *A soft action of the soft group* (G, F, A) *on the soft set* (X, F', A) *is called free if given* g, h *in* $F(a)$*, the existence of an element* x *in* $F'(a)$ *with* $\Pi_a(g, x) = \Pi_a(h, x)$ *implies* $g = h$.

In other words, if g *is a soft group element and there exists an element* x *in* $F'(a)$ *with* $\Pi_a(g, x) = x$*, then* g *is the identity.*

Proposition 3.19. *Every free soft action is faithful.*

Proof. From the Definition 3.17 and 3.18, the proof is clear. □

Definition 3.20. *A soft action of the soft group* (G, F, A) *on the soft set* (X, F', A) *is called regular if it is both transitive and free. Equivalently, for every elements* x, y *in* $F'(a)$ *there exists only one element* g *in* $F(a)$ *such that* $\Pi_a(g, x) = y$.

Example 3.21. *The soft action of any soft group* G *on itself by multiplication is both regular and faithful.*

4. Stabilizers, Centralizers and Normalizers for Soft Actions

In this last section, we have discussed concepts such as stabilizer, centralizer and normalizer related to the soft action, and examined the relationship between them. By introducing the concept of soft symmetric group which is in the focus of this study, we complete this study with an important result which gives the relation between the soft action and soft symmetric group.

Definition 4.1. *Let* $X = (X, F', A)$ *be a* $G-$*soft set. Then, for all* $a \in A$ *and* $x \in F'(a)$*, the stabilizer of* x *is the set*

$$Stab_G(x) = \{g \in F(a) : \ \Pi_a(g, x) = x\}$$

More generally, for any soft subset $Y \widetilde{\subseteq} X$ *we can consider elements of the subgroup* $F(a)$ *which fix* Y

$$Fix_G Y = \{g \in F(a) : \ \Pi_a(g, x) = x, \ x \in F'(a) \cap Y\}$$

Proposition 4.2. *The sets $Stab_G(x)$ and $Fix_G Y$ defined as above are soft groups over G.*

Proof. We define the map

$$\begin{array}{rcl} Stab_G : X & \longrightarrow & P(G) \\ x & \mapsto & Stab_G(x). \end{array}$$

For all $x \in X$, $Stab_G(x)$ is a subgroup of G. Thus, the triplet $(G, Stab_G, X)$ is a soft group. Similarly, we can define the map

$$\begin{array}{rcl} Fix_G : Y & \longrightarrow & P(G) \\ x & \mapsto & Fix_G(x). \end{array}$$

For all $x \in Y$, $Fix_G(x)$ is a subgroup of G. This proves that the pair (Fix_G, Y) is a soft group over G. □

Remark 4.3. *We note that $Stab_G(x)$ and $Fix_G Y$ are not soft subgroups of the soft group G as different from the classical theory.*

Proposition 4.4. *Let $Stab_G(x)$ and $Fix_G Y$ be defined as above. Then*

$$Fix_G Y = \cap_{x \in F'(a) \cap Y} Stab_G(x)$$

Proof. Let $g \in Fix_G Y$. Then $\Pi_a(g, x) = x$ for all $x \in F'(a) \cap Y$, which proves that $g \in Stab_G(x)$. This means that $g \in \cap_{x \in F'(a) \cap Y} Stab_G(x)$ and so $Fix_G Y \subseteq \cap_{x \in F'(a) \cup Y} Stab_G(x)$ can be written. Conversely, let $g \in \cap_{x \in F'(a) \cap Y} Stab_G(x)$. Then $g \in Stab_G(x)$ for all $x \in F'(a) \cap Y$ and so $\Pi_a(g, x) = x$. Therefore, $g \in Fix_G Y$ is obtained, which completes the proof. □

Definition 4.5. *Let $G = (G, F, A)$ acts on itself by conjugation soft action . For all $a \in A$ and $h \in F(a)$, the set*

$$C_G(h) = \{g \in F(a) : \ \Pi_a(g, h) = \Pi_a(h, g)\}$$

of the orbits of h is called the centralizer of h in $F(a)$.

Furthermore, the intersection of all the centralizers of elements of $F(a)$ is denoted as follows

$$Z(G) = \{g \in F(a) : \ \Pi_a(g, h) = \Pi_a(h, g), \forall h \in F(a)\}$$

and is called the center of the soft group G.

Proposition 4.6. *The centralizer $C_G(h)$ of h defined above is a soft group over G.*

Proof. For all $h \in F(a)$, we can define

$$\begin{array}{rcl} C_G : G & \longrightarrow & P(G) \\ h & \mapsto & C_G(h). \end{array}$$

It is clear that $C_G(h)$ is a subgroup of G. Thus, $C_G(h)$ is a soft group over G. □

Definition 4.7. *Let $G = (G, F, A)$ act on itself by conjugation soft action and let H be a soft subset of G. The isotropy soft subgroup, which is the soft subset of H for all $a \in A$, is denoted*

$$N_G(H) = \{g \in F(a) : \ \Pi_a(g, h) = h, \ h \in F(a) \cap H\}$$

and is called the normalizer of H in the soft group G.

Note that if H is a soft subgroup of the soft group G, then H is a soft subgroup of $N_G(H)$. Moreover, if H is a normal soft subgroup of the soft group G, $N_G(H)$ is the largest soft subgroup of G.

Let us now describe the concept of the soft symmetric group.

Definition 4.8. *Let X be a set and A be a set of parameters. The group of permutations of X is denoted by $Sym(X)$. Then the triplet $(Sym(X), F, A)$ is called soft symmetric group if $F(a)$ is a subgroup of $Sym(X)$ for all $a \in A$.*

In general, it can be said that there is a relation between the soft symmetric group and soft action. This relation can be constructed as follows, similar to the relation between the symmetric group and group action existing in the classical Cayley Theorem.

Theorem 4.9. *Let $G = (G, F, A)$ be soft group acting on a soft set $X = (X, F', A)$. Then, there is a soft homomorphism from G to $Sym(X)$.*

Proof. Suppose that $G = (G, F, A)$ is a soft action on a soft set $X = (X, F', A)$. Hence, for all $a \in A$

$$\begin{array}{rcl} \Pi_a : F(a) \times F'(a) & \longrightarrow & F'(a) \\ (g, x) & \mapsto & \Pi_a(g, x) = \sigma(x) \end{array}$$

is an action. Also, for each $g \in F(a)$

$$\begin{array}{rcl} \sigma_g : F'(a) & \longrightarrow & F'(a) \\ x & \mapsto & \sigma_g(x) = \Pi_a(g, x) \end{array}$$

is a permutation of $F'(a)$. Morever, for all $a \in A$

$$\begin{array}{rcl} F(a) & \longrightarrow & Sym(F'(a)) \\ g & \mapsto & \sigma_g \end{array}$$

is a homomorphism. Thus, we obtain a soft group homomorphism from G to $Sym(X)$. □

Theorem 4.10. *Each finite soft group can be embedded in a soft symmetric group.*

Proof. Suppose that a finite soft group $G = (G, F, A)$ acts on itself by multiplication, namely for all $a \in A$, we have

$$\begin{array}{rcl} \Pi_a : F(a) \times F(a) & \longrightarrow & F(a) \\ (g, h) & \mapsto & \Pi_a(g, h) = gh \end{array}$$

Also, for each $g \in F(a)$

$$\begin{array}{rcl} \ell_g : F(a) & \longrightarrow & F(a) \\ h & \mapsto & \ell_g(h) = \Pi_a(g,h) = gh \end{array}$$

is a permutation of $F(a)$. Morever, the mapping $F(a) \longrightarrow Sym(F(a))$ defined by $g \mapsto \ell_g$ is a homomorphism for all $a \in A$. Since this homomorphism is one-to-one, the sending $g \mapsto \ell_g$ is an embedding of $F(a)$ as a subgroup of $Sym(F(a))$. Thus, the finite soft group G can be embedded in $Sym(G)$. □

After these theorems, one can easily obtain the following corollary.

Corollary 4.11. *Soft action of $G = (G, F, A)$ on the soft set $X = (X, F', A)$ are the same as soft homomorphism from G to $Sym(X)$.*

References

[1] Atagun, A. O. and Sezgin, A., Soft substructures of rings, fields and modules, *Computers and Math.with Appl.*, 61 (2011), 592-601.

[2] Molodtsov, D. A., Soft set theory-First results, *Comput. Math. Appl.*, 37(4-5) (1999), 19-31.

[3] Aktas, H. and Cagman, N., Soft sets and soft groups, *Inform. Sci.*, 77(13) (2007), 2726-2735.

[4] Rotman, J. J., An Introduction to the Theory of Groups. 4th ed. Springer, New York, 1995.

[5] Veress, L. A., Group actions on sets and automata theory, *Applied Mathematics and Computation*, 113(2-3) (2000), 289-304.

[6] Shabir, M. and Naz, M., On soft topological spaces, *Comput. Math. Appl.*, 61(7) (2011), 1786-1799.

[7] Hirsch, M.W., Smale, S. and Devaney, R.L., Differential Equations, Dynamical Systems and an Introduction to Chaos, Elsevier, San Diego, CA, 2004.

[8] Maji, P. K., Biswas, R. and Roy, A. R., Soft set theory, *Comput. Math. Appl.*, 45(4-5) (2003), 555-562.

[9] Smale, S., Differentiable Dynamical Systems, *Bull. Amer. Math. Soc.*, 73 (1967), 747-817.

[10] Sardar, S. K. and Gupta, S., Soft category theory-an introduction, *Journal of Hyperstructures*, 2 (2013),118-135.

[11] Wagner, S., Free Group Actions from the Viewpoint of Dynamical Systems, *Muenster J. of Math.*, 5(2012), 73-98.

[12] Shah, T. and Shaheen, S., Soft topological groups and rings, *Ann. Fuzzy Math. Inform.*, 7(5)(2014), 725-743.

[13] Santos, W. F. and Rittatore, A., Actions and Invariants of Algebraic Groups, CRC press, 2005.

Current address: GULAY OGUZ: Department of Mathematics, Inonu University, Malatya, TURKEY.

E-mail address: gulay.oguz@inonu.edu.tr

ORCID Address: http://orcid.org/0000-0003-4302-8401

Current address: ILHAN ICEN: Department of Mathematics, Inonu University, Malatya, TURKEY

E-mail address: ilhan.icen@inonu.edu.tr

ORCID Address: http://orcid.org/0000-0003-3576-0731

Current address: M. HABIL GURSOY:Department of Mathematics, Inonu University, Malatya, TURKEY

E-mail address: mhgursoy@gmail.com

ORCID Address: http://orcid.org/0000-0001-9151-2631