İSTATİSTİK: JOURNAL OF THE TURKISH STATISTICAL ASSOCIATION
İSTATİSTİK
Vol. 10, No. 1, January 2017, pp. 14–23
ISSN 1300-4077 | 17 | 1 | 23 | 23

PARTIALLY MOMENT RELATIONS FOR THE PRODUCT MOMENTS OF ORDER STATISTICS FROM THE STANDARD TWO-SIDED POWER DISTRIBUTION

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Abstract: In this work, moment relations for the product moments of order statistics from the standard two-sided power (STSP) distribution are obtained. Since the probability density function (pdf) of the STSP distribution is a piecewise function, we consider the pieces separately and develop certain moment relations based on these pieces. Also, the usefulness of these moment relations in evaluating the product moments of order statistics from the STSP distribution is discussed.

Key words: Order statistics; Product moments; Moment relations; The standard two-sided power distribution
History: Submitted: 10 June 2016; Revised: 20 December 2016; Accepted: 29 December 2016

1. Introduction

The moment relations and identities are important for computational efficiency since they reduce the amount of calculations required for the evaluations of the moments. From the view of order statistics, there are many works about moment recurrence relations and identities for the single and product moments of order statistics. General recurrence relations for moments of order statistics can be found in Khan et al.(1983), Arnold et al.(1992), David and Nagaraja(2003). For specific distributions a lot of works on this subject can be found in the literature. For example, many classical distributions such as normal, Cauchy and logistic distributions (Balakrishnan et al.,1988), beta distribution (Thomas and Samuel,2008) and Topp-Leone distribution (Genç,2012).

On the other hand, the standard two-sided power distribution (STSP) was introduced by van Dorp and Kotz (2002) as a peaked alternative to the beta distribution.It also gives a better alternative way of modelling excess kurtosis of data to the Laplace distribution in case of a bounded domain. The STSP distribution has also applications in risk analysis such as PERT (Program Evaluation and Review Technique) (Kotz and Van Dorp, 2004). It has the following probability density function (pdf)

$$f(x) \equiv f(x|\alpha,\beta) = \begin{cases} f_1(x) & ,0 < x \leq \beta \\ f_2(x) & ,\beta \leq x < 1 \end{cases} \tag{1.1}$$

where $f_1(x) = \alpha(\frac{x}{\beta})^{\alpha-1}$ and $f_2(x) = \alpha(\frac{1-x}{1-\beta})^{\alpha-1}$ and cdf is

$$F(x) \equiv F(x|\alpha,\beta) = \begin{cases} F_1(x) & ,0 < x \leq \beta \\ F_2(x) & ,\beta \leq x < 1 \end{cases} \tag{1.2}$$

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where $F_1(x) = \beta(\frac{x}{\beta})^\alpha$ and $F_2(x) = 1-(1-\beta)(\frac{1-x}{1-\beta})^\alpha$. It is easy to observe that

$$f(x|\alpha,\beta) = \begin{cases} \frac{\alpha F_1(x)}{x} & ,0 < x \leq \beta \\ \frac{\alpha(1-F_2(x))}{1-x} & ,\beta \leq x < 1 \end{cases} \tag{1.3}$$

where $\alpha > 0$

The STSP distribution has extensive flexibility according to its parameters which determines the shape (α) and reflection (β) of the distribution. For example; for $\beta = 0.5$ it is in a symmetrical structure, for $\alpha = 2$ it corresponds to a triangular distribution, for $\beta = 0.5$ and $\alpha = 2$ it returns to a symmetrical triangular distribution that was recently studied by Nagaraja (2013) and it also simplifies to uniform distribution on $[0,1]$ for $\alpha = 1$ and finally it corresponds to a power function distribution for $\beta = 1$.

Since the STSP distribution both generalizes the power function and triangular distributions from which order statistics are studied in the literature, it is natural to study the order statistics of the STSP distribution which is more flexible than these simple distributions. Çetinkaya and Genç (2016) studied the order statistics from the STSP distribution and derived single moment recurrence relations. Except for this reference, as far as we know, the STSP distribution has not been considered in order statistics point of view in the literature. The product moments of order statistics are also important in deriving variance-covariance matrices of order statistics and correlation computations. The aim of this paper is to present certain moment relations for the product moments of order statistics from the STSP distribution which are useful in checking the computations for product moments of order statistics and reducing the amount of calculations required for the evaluations of these moments.These relations may especially be useful in deriving the best linear unbiased estimators (BLUE) of the parameters.

Let $X_{1:n} \leq X_{2:n} \leq X_{3:n} \leq \cdots \leq X_{n:n}$ be the order statistics of a random sample size n drawn from the STSP distribution. The joint pdf of $X_{r:n}$ and $X_{s:n}$ is

$$f_{r,s}(x,y) = \begin{cases} C_{r,s:n}[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}f(x)f(y) & ,x < y \\ 0 & ,otherwise \end{cases}$$

and

$$E[X_{r:n}^k X_{s:n}^l] = C_{r,s:n}\int_0^1\int_0^y x^k y^l [F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}f(x)f(y)dxdy$$

where $\quad C_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}$ since

$$f(x,y) = \begin{cases} f_1(x)f_1(y) & ,0 < x < \beta, 0 < y < \beta \\ f_1(x)f_2(y) & ,0 < x < \beta, \beta < y < 1 \\ f_2(x)f_1(y) & ,\beta < x < 1, 0 < y < \beta \\ f_2(x)f_2(y) & ,\beta < x < 1, \beta < y < 1 \end{cases}$$

then

$$E[X_{r:n}^k X_{s:n}^l] = C_{r,s:n}\int_0^1\int_0^y x^k y^l [F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}f(x,y)dxdy \tag{1.4}$$

where

$$f(x,y)=\begin{cases} f_1(x)f_1(y) & ,0<x<y<\beta \\ f_1(x)f_2(y) & ,0<x<\beta<y<1 \\ f_2(x)f_2(y) & ,\beta<x<y<1 \end{cases}$$

and (1.4) becomes

$$\begin{aligned} E[X_{r:n}^k X_{s:n}^l] = & \\ & C_{r,s:n}\int_0^\beta\int_0^y x^k y^l F_1(x)^{r-1}[F_1(y)-F_1(x)]^{s-r-1}[1-F_1(y)]^{n-s} f_1(x)f_1(y)dxdy \\ & + C_{r,s:n}\int_\beta^1\int_0^\beta x^k y^l F_1(x)^{r-1}[F_2(y)-F_1(x)]^{s-r-1}[1-F_2(y)]^{n-s} f_1(x)f_2(y)dxdy \\ & + C_{r,s:n}\int_\beta^1\int_\beta^y x^k y^l F_2(x)^{r-1}[F_2(y)-F_2(x)]^{s-r-1}[1-F_2(y)]^{n-s} f_2(x)f_2(y)dxdy \\ & \equiv {}_L\mu_{r,s:n}^{(k,l)} + {}_M\mu_{r,s:n}^{(k,l)} + {}_U\mu_{r,s:n}^{(k,l)} \end{aligned} \tag{1.5}$$

Because of $f_1(x)$,$f_1(y)$, $f_2(x)$ and $f_2(y)$ are not proper pdf's, each partial integral in (1.5) do not give the anticipated product moment experessions. So, these integrals are named as *lower* , *middle* and *upper* product moments and ${}_L\mu_{r,s:n}^{(k,l)}$, ${}_M\mu_{r,s:n}^{(k,l)}$ and ${}_U\mu_{r,s:n}^{(k,l)}$ are the notations of these product moments, respectively. The moment relations for each part are obtained in the following sections.

2. Moment relations for lower partial product moments

For the case of $0<X_{r:n}<X_{s:n}<\beta$, we entitled the first part of the integral in (1.5) as lower partial product moment and used ${}_L\mu_{r,s:n}^{(k,l)}$ for its notation. The moment relations obtained for ${}_L\mu_{r,s:n}^{(k,l)}$ are presented in the following theorems.

THEOREM 1. *If* $1\le r<s-1<n$ *, then*

$${}_L\mu_{r,s,:n}^{(k,l)} = \frac{\alpha r}{k+\alpha r}\ {}_L\mu_{r+1,s,:n}^{(k,l)}$$

PROOF.

$$\begin{aligned} {}_L\mu_{r,s:n}^{(k,l)} &= C_{r,s:n}\int_0^\beta\int_0^y x^k y^l [F_1(x)]^{r-1}[F_1(y)-F_1(x)]^{s-r-1}[1-F_1(y)]^{n-s} f_1(x)f_1(y)dxdy \\ &= C_{r,s:n}\int_0^\beta y^l[1-F_1(y)]^{n-s}f_1(y)dy\left[\int_0^y x^k[F_1(x)]^{r-1}[F_1(y)-F_1(x)]^{s-r-1}f_1(x)dx\right]dy \end{aligned}$$

Firstly, transformation (1.3) is used for $f_1(x)$ in the integral which is based on x and partial integration with the substitution $u=[F_1(x)]^r[F_1(y)-F_1(x)]^{s-r-1}$ and $dv=x^{k-1}dx$ used. Then, we get

$$\begin{aligned} {}_L\mu_{r,s:n}^{(k,l)} = & \frac{\alpha C_{r,s:n}(s-r-1)}{k}\int_0^\beta\int_0^y x^k y^l [F_1(x)]^r[F_1(y)-F_1(x)]^{s-r-2}[1-F_1(y)]^{n-s} \\ & \qquad * f_1(x)f_1(y)dxdy \\ & -\frac{\alpha r C_{r,s:n}}{k}\int_0^\beta\int_0^y x^k y^l [F_1(x)]^{r-1}[F_1(y)-F_1(x)]^{s-r-1}[1-F_1(y)]^{n-s} f_1(x)f_1(y)dxdy \end{aligned}$$

Thus Theorem 1 is derived by simplifying the resulting expression.
This recurrence relation shows that if ${}_L\mu_{r,s:n}^{(k,l)}$ can be obtained for an arbitrary r value, results for other rth order statistic can be evaluated.

THEOREM 2. *If $2 \le r < s \le n$ and $\alpha \in \mathbb{Z}^+$, then*

$$_L\mu_{r,s:n}^{(k,l)} = \frac{n}{\beta^{(\alpha-1)}(r-1)} \ {_L\mu_{r-1,s-1:n-1}^{(k+\alpha,l)}}$$

PROOF.

$$\begin{aligned}
_L\mu_{r,s:n}^{(k,l)} &= C_{r,s:n}\int_0^\beta\int_0^y x^k y^l [F_1(x)]^{r-1}[F_1(y)-F_1(x)]^{s-r-1}[1-F_1(y)]^{n-s} f_1(x)f_1(y)dxdy \\
&= C_{r,s:n}\int_0^\beta\int_0^y x^k y^l [F_1(x)]^{r-2}F_1(x)[F_1(y)-F_1(x)]^{s-r-1}[1-F_1(y)]^{n-s} f_1(x)f_1(y)dxdy \\
&= C_{r,s:n}\int_0^\beta\int_0^y x^k y^l [F_1(x)]^{r-2}\beta(\frac{x}{\beta})^\alpha[F_1(y)-F_1(x)]^{s-r-1}[1-F_1(y)]^{n-s} f_1(x)f_1(y)dxdy \\
&= \frac{C_{r,s:n}}{\beta^{(\alpha-1)}}\int_0^\beta\int_0^y x^{k+\alpha} y^l [F_1(x)]^{r-2}F_1(x)[F_1(y)-F_1(x)]^{s-r-1}[1-F_1(y)]^{n-s} f_1(x)f_1(y)dxdy
\end{aligned}$$

Then Theorem 2 is derived by simplifying the obtained expression.

THEOREM 3. *If $1 \le r < s \le n-1$ and $\alpha \in \mathbb{Z}^+$, then*

$$_L\mu_{r,s:n}^{(k,l)} = \frac{n}{n-s}\left({_L\mu_{r,s:n-1}^{(k,l)}} - \frac{_L\mu_{r,s:n-1}^{(k,l+\alpha)}}{\beta^{(\alpha-1)}} \right)$$

PROOF.

$$_L\mu_{r,s:n}^{(k,l)} = C_{r,s:n}\int_0^\beta\int_0^y x^k y^l [F_1(x)]^{r-1}[F_1(y)-F_1(x)]^{s-r-1}[1-F_1(y)]^{n-s} f_1(x)f_1(y)dxdy$$

Similar to previous theorem; by extracting $[1-F_1(y)]^{n-s}$ as $(1-F_1(y))^{n-s-1}(1-\beta(\frac{y}{\beta})^\alpha)$ we get

$$\begin{aligned}
_L\mu_{r,s:n}^{(k,l)} &= C_{r,s:n}\int_0^\beta\int_0^y x^k y^l [F_1(x)]^{r-1}[F_1(y)-F_1(x)]^{s-r-1}[1-F_1(y)]^{n-s-1} f_1(x)f_1(y)dxdy \\
&-\frac{C_{r,s:n}}{\beta^{\alpha-1}}\int_0^\beta\int_0^y x^k y^{l+\alpha} [F_1(x)]^{r-1}[F_1(y)-F_1(x)]^{s-r-1}[1-F_1(y)]^{n-s-1} f_1(x)f_1(y)dxdy
\end{aligned}$$

Theorem 3 follows by simplifying the resulting expression.

It is clearly seen that when the lower partial product moment is available for *n-1* sample size, it can be obtained for n sample size easily by using this relation.

THEOREM 4. *If $3 \le r < s-1 \le n-1$ and $2\alpha \in \mathbb{Z}^+$, then*

$$_L\mu_{r,s:n}^{(k,l)} = \frac{\alpha n(n-1)}{\beta^{2(\alpha-1)}(k+2\alpha)(r-1)}\left({_L\mu_{r-1,s-2:n-2}^{(k+2\alpha,l)}} - \ {_L\mu_{r-2,s-2:n-2}^{(k+2\alpha,l)}} \right)$$

PROOF.

$$\begin{aligned}
_L\mu_{r,s:n}^{(k,l)} &= C_{r,s:n}\int_0^\beta\int_0^y x^k y^l [F_1(x)]^{r-1}[F_1(y)-F_1(x)]^{s-r-1}[1-F_1(y)]^{n-s} f_1(x)f_1(y)dxdy \\
&= C_{r,s:n}\int_0^\beta\int_0^y x^k y^l [F_1(x)]^{r-2}F_1(x)[F_1(y)-F_1(x)]^{s-r-1}[1-F_1(y)]^{n-s} f_1(x)f_1(y)dxdy \\
&= C_{r,s:n}\int_0^\beta\int_0^y x^k y^l [F_1(x)]^{r-2}\alpha\frac{x^{2\alpha-1}}{\beta^{2(\alpha-1)}}[F_1(y)-F_1(x)]^{s-r-1}[1-F_1(y)]^{n-s} f_1(y)dxdy \\
&= \frac{\alpha C_{r,s:n}}{\beta^{2(\alpha-1)}}\int_0^\beta y^l[1-F_1(y)]^{n-s}f_1(y)\left[\int_0^y x^{k+2\alpha-1}[F_1(x)]^{r-2}[F_1(y)-F_1(x)]^{s-r-1}dx\right]dy
\end{aligned}$$

By using partial integral substutions as $u=[F_1(x)]^{r-2}[F_1(y)-F_1(x)]^{s-r-1}$ and $x^{k+2\alpha-1}dx=dv$ we get
${}_L\mu_{r,s:n}^{(k,l)}=$

$$\frac{\alpha C_{r,s:n}}{\beta^{2(\alpha-1)}}\frac{s-r-1}{k+2\alpha}\int_0^\beta\int_0^y x^{k+2\alpha}y^l[F_1(x)]^{r-2}[F_1(y)-F_1(x)]^{s-r-2}[1-F_1(y)]^{n-s} *f_1(x)f_1(y)dxdy$$
$$-\frac{\alpha C_{r,s:n}}{\beta^{2(\alpha-1)}}\frac{r-2}{k+2\alpha}\int_0^\beta\int_0^y x^{k+2\alpha}y^l[F_1(x)]^{r-3}[F_1(y)-F_1(x)]^{s-r-1}[1-F_1(y)]^{n-s} *f_1(x)f_1(y)dxdy$$

By simlifying this expression, Theorem 4 is obtained.

Theorem 5. *If $1\le r<s\le n$ and $\alpha j\in\mathbb{Z}^+$, then*

$$ {}_L\mu_{r,s:n}^{(k,l)}=\binom{n}{s}\sum_{j=0}^{n-s}\binom{n-s}{j}(-1)^j\frac{{}_L\mu_{r,s:s}^{(k,l+\alpha j)}}{\beta^{j(\alpha-1)}}$$

Proof.

$${}_L\mu_{r,s:n}^{(k,l)}=C_{r,s:n}\int_0^\beta\int_0^y x^ky^l[F_1(x)]^{r-1}[F_1(y)-F_1(x)]^{s-r-1}[1-F_1(y)]^{n-s}f_1(x)f_1(y)dxdy$$

In this integral, by using binomial expansion for $[1-F_1(y)]^{n-s}=\left[1-\beta(\frac{y}{\beta})^\alpha\right]^{n-s}$ we get

$${}_L\mu_{r,s:n}^{(k,l)}=C_{r,s:n}\sum_{j=o}^{n-s}\binom{n-s}{j}(-1)^j\frac{1}{\beta^{j(\alpha-1)}}\int_0^\beta\int_0^y x^ky^{l+\alpha j}[F_1(x)]^{r-1} [F_1(y)-F_1(x)]^{s-r-1}f_1(x)f_1(y)dxdy$$

Theorem 5 is obtained by simplifying the resulting expression.

Theorem 6. *If $1\le r<s\le n-1$, $n\ge 2$, $c=r+j$ and $\alpha c\in\mathbb{Z}^+$, then*

$${}_L\mu_{r,s:n}^{(k,l)}=\frac{\alpha n!}{(r-1)!(s-r-1)!}\sum_{j=o}^{s-r-1}\binom{s-r-1}{j}(-1)^j \times\frac{\beta^{c(1-\alpha)}(s-c-1)!}{(n-c)!(\alpha c+1)!}\ {}_l\mu_{s-c:n-c}^{(\alpha c+2)}$$

where ${}_l\mu_{s-c:n-c}^{(\alpha c+2)}$ is named as lower single moment of the STSP distribution which is given by Çetinkaya and Genç (2016).

Proof.

$${}_L\mu_{r,s:n}^{(k,l)}=C_{r,s:n}\int_0^\beta\int_0^y x^ky^l[F_1(x)]^{r-1}[F_1(y)-F_1(x)]^{s-r-1}[1-F_1(y)]^{n-s}f_1(x)f_1(y)dxdy$$

Firstly, by using binomial expansion for $[F_1(y)-F_1(x)]^{s-r-1}$, the following expression is obtained.

$${}_L\mu_{r,s:n}^{(k,l)}=C_{r,s:n}\sum_{j=0}^{s-r-1}\binom{s-r-1}{j}(-1)^j\int_0^\beta\left[\int_0^y x[F_1(x)]^{r+j-1}f_1(x)dx\right] \times\ y[F_1(y)]^{s-r-j-1}[1-F_1(y)]^{n-s}f_1(y)dy$$

$$
{}_L\mu_{r,s:n}^{(k,l)} = C_{r,s:n}\alpha \sum_{j=0}^{s-r-1} \binom{s-r-1}{j} (-1)^j \frac{\beta^{c(1-\alpha)}}{\alpha c+1} \int_0^\beta y^{\alpha c+2}[F_1(y)]^{s-c-1} * [1-F_1(y)]^{n-s} f_1(y)dy
$$

where $c = r+j$. Then, by simplifying the result expression, theorem 6 is obtained.

In the theorem above the lower partial single moment of the STSP distribution which is given by Çetinkaya and Genç (2016) is used. It is clearly seen that the lower partial product moment can be evaluated with lower partial single moment by using this relation.

3. Moment relations for middle partial product moments

Similar to lower partial product moment; for the case of $0 < X_{r:n} < \beta < X_{s:n} < 1$, we entitled the second part of the integral in (1.5) as middle partial product moment and used ${}_M\mu_{r,s:n}^{(k,l)}$ for its notation. The moment relations obtained for ${}_M\mu_{r,s:n}^{(k,l)}$ are given in the following.

THEOREM 7. *If $1 \leq r < s \leq n-1$ and $\alpha \in \mathbb{Z}^+$, then*

$$
{}_M\mu_{r,s:n}^{(k,l)} = \frac{n}{(n-s)(1-\beta)^{\alpha-1}} \sum_{j=0}^{\alpha} \binom{\alpha}{j} (-1)^j {}_M\mu_{r,s:n-1}^{(k,l+j)}
$$

PROOF.

$$
{}_M\mu_{r,s:n}^{(k,l)} = C_{r,s:n} \int_\beta^1 \int_0^\beta x^k y^l [F_1(x)]^{r-1}[F_2(y)-F_1(x)]^{s-r-1}[1-F_2(y)]^{n-s} f_1(x) f_2(y) dxdy
$$

By extracting $[1-F_2(y)]^{n-s}$ as $[1-F_2(y)]^{n-s-1}[1-F_2(y)] = [1-F_2(y)]^{n-s-1}\left[(1-\beta)(\frac{1-y}{1-\beta})^\alpha\right]$ and using binomial expansion for $(1-y)^\alpha$ we get

$$
{}_M\mu_{r,s:n}^{(k,l)} = \frac{C_{r,s:n}}{(1-\beta)^{\alpha-1}} \sum_{j=0}^{\alpha} \binom{\alpha}{j} (-1)^j \int_\beta^1 \int_0^\beta x^k y^{l+j} [F_1(x)]^{r-1} * [F_2(y)-F_1(x)]^{s-r-1}[1-F_2(y)]^{n-s-1} f_1(x) f_2(y) dxdy
$$

The theorem is obtained by simplifying the result expression.

It is clearly seen that when the middle partial product moment is available for $n-1$ sample size, it can be obtained for n sample size easily by using this relation.

THEOREM 8. *If $2 \leq r < s \leq n-1$ and $\alpha \in \mathbb{Z}^+$, then*

$$
{}_M\mu_{r,s:n}^{(k,l)} = \frac{n(n-1)}{(n-s)(r-1)[\beta(1-\beta)]^{\alpha-1}} \sum_{j=0}^{\alpha} \binom{\alpha}{j} (-1)^j {}_M\mu_{r-1,s-1:n-2}^{(k+\alpha,l+j)}
$$

PROOF.

$$
{}_M\mu_{r,s:n}^{(k,l)} = C_{r,s:n} \int_\beta^1 \int_0^\beta x^k y^l [F_1(x)]^{r-1}[F_2(y)-F_1(x)]^{s-r-1}[1-F_2(y)]^{n-s} f_1(x) f_2(y) dxdy
$$

By extracting both $[F_1(x)]^{r-1}$ as $[F_1(x)]^{r-2}F_1(x) = [F_1(x)]^{r-2}\beta(\frac{x}{\beta})^\alpha$ and $[1-F_2(y)]^{n-s}$ as $[1-F_2(y)]^{n-s-1}[1-F_2(y)] = [1-F_2(y)]^{n-s-1}\left[(1-\beta)(\frac{1-y}{1-\beta})^\alpha\right]$ we get

$$
{}_M\mu_{r,s:n}^{(k,l)} = \frac{C_{r,s:n}}{[\beta(1-\beta)]^{\alpha-1}} \sum_{j=0}^{\alpha} \binom{\alpha}{j} (-1)^j \int_\beta^1 \int_0^\beta x^{k+\alpha} y^{l+j} [F_1(x)]^{r-2} * [F_2(y)-F_1(x)]^{s-r-1}[1-F_2(y)]^{n-s-1} f_1(x) f_2(y) dxdy
$$

by simplifying the resulting expression, the theorem is derived.

THEOREM 9. *If* $2 \le r < s \le n$ *and* $\alpha \in \mathbb{Z}^+$ *, then*

$$ {}_M\mu_{r,s:n}^{(k,l)} = \frac{n}{(r-1)\beta^{\alpha-1}} \; {}_M\mu_{r-1,s-1:n-1}^{(k+\alpha,l)} $$

PROOF.

$$ {}_M\mu_{r,s:n}^{(k,l)} = C_{r,s:n} \int_\beta^1 \int_0^\beta x^k y^l [F_1(x)]^{r-1} [F_2(y) - F_1(x)]^{s-r-1} [1 - F_2(y)]^{n-s} f_1(x) f_2(y) dx dy $$

By extracting $[F_1(x)]^{r-1}$ as $[F_1(x)]^{r-2}[F_1(x)] = [F_1(x)]^{r-2}\beta(\frac{x}{\beta})^\alpha$ we get

$$ {}_M\mu_{r,s:n}^{(k,l)} = \frac{C_{r,s:n}}{\beta^{\alpha-1}} \int_\beta^1 \int_0^\beta x^{k+\alpha} y^l [F_1(x)]^{r-2} [F_2(y) - F_1(x)]^{s-r-1} $$
$$ *[1 - F_2(y)]^{n-s-1} f_1(x) f_2(y) dx dy $$

Then, theorem 9 is derived by simplifying the resulting expression.
This theorem is also obtained for the lower partial product moment,too. Thus it can be useful for both the first two parts of the product moment.

THEOREM 10. *If* $1 \le r < s \le n$ *and* $\alpha(n-s) \in \mathbb{Z}^+$ *, then*

$$ {}_M\mu_{r,s:n}^{(k,l)} = \frac{n!}{s!(n-s)!(1-\beta)^{(\alpha-1)(n-s)}} \sum_{j=0}^{\alpha(n-s)} \binom{\alpha(n-s)}{j} (-1)^j {}_M\mu_{r,s:s}^{(k,l+j)} $$

PROOF.

$$ {}_M\mu_{r,s:n}^{(k,l)} = C_{r,s:n} \int_\beta^1 \int_0^\beta x^k y^l [F_1(x)]^{r-1} [F_2(y) - F_1(x)]^{s-r-1} [1 - F_2(y)]^{n-s} f_1(x) f_2(y) dx dy $$

$[1 - F_2(y)]^{n-s}$ equals $\frac{(1-y)^{\alpha(n-s)}}{(1-\beta)^{(\alpha-1)(n-s)}}$. By using binomial expansion for $(1-y)^{\alpha(n-s)}$ we get

$$ {}_M\mu_{r,s:n}^{(k,l)} = \frac{C_{r,s:n}}{(1-\beta)^{(\alpha-1)(n-s)}} \sum_{j=0}^{\alpha(n-s)} \binom{\alpha(n-s)}{j} (-1)^j \int_\beta^1 \int_0^\beta x^k y^{l+j} [F_1(x)]^{r-1} $$
$$ *[F_2(y) - F_1(x)]^{s-r-1} f_1(x) f_2(y) dx dy $$

Thus, theorem is derived by simplifying the resulting expression.

THEOREM 11. *If* $1 \le r < s \le n$ *and* $\alpha(r-1) \in \mathbb{Z}^+$ *, then*

$$ {}_M\mu_{r,s:n}^{(k,l)} = \frac{n!}{(n-r+1)!(r-1)!\beta^{(\alpha-1)(r-1)}} \; {}_M\mu_{1,s-r+1:n-r+1}^{(k+\alpha(r-1),l)} $$

PROOF.

$$ {}_M\mu_{r,s:n}^{(k,l)} = C_{r,s:n} \int_\beta^1 \int_0^\beta x^k y^l [F_1(x)]^{r-1} [F_2(y) - F_1(x)]^{s-r-1} [1 - F_2(y)]^{n-s} f_1(x) f_2(y) dx dy $$

We know that $[F_1(x)]^{r-1} = \left[\frac{x^\alpha}{\beta^{\alpha-1}}\right]^{r-1}$. By using this identity we get

$$ {}_M\mu_{r,s:n}^{(k,l)} = \frac{C_{r,s:n}}{\beta^{(\alpha-1)(r-1)}} \int_\beta^1 \int_0^\beta x^{k+\alpha(r-1)} y^l [F_2(y) - F_1(x)]^{s-r-1} $$
$$ *[1 - F_2(y)]^{n-s} f_1(x) f_2(y) dx dy $$

By simplifying the resulting expression, theorem is derived.

4. Moment relations for upper partial product moments

The last part of the integral in (1.5) is named as upper partial product moment for the case of $\beta < X_{r:n} < X_{s:n} < 1$ and ${}_U\mu_{r,s:n}^{(k,l)}$ is used for its notation.The obtained relations are given in the following.

THEOREM 12. *If $1 \le r < s \le n-1$ and $\alpha \in \mathbb{Z}^+$, then*

$$ {}_U\mu_{r,s:n}^{(k,l)} = \frac{n}{(n-s)(1-\beta)^{(\alpha-1)}} \sum_{j=0}^{\alpha} \binom{\alpha}{j} (-1)^j_U \mu_{r,s:n-1}^{(k,l+j)} $$

PROOF.

$$ {}_U\mu_{r,s:n}^{(k,l)} = C_{r,s:n} \int_\beta^1 \int_\beta^y x^k y^l [F_2(x)]^{r-1} [F_2(y) - F_2(x)]^{s-r-1} [1 - F_2(y)]^{n-s} f_2(x) f_2(y) dx dy $$

By extracting $[1 - F_2(y)]^{n-s}$ as $[1 - F_2(y)]^{n-s-1}[1 - F_2(y)] = [1 - F_2(y)]^{n-s-1}(1-\beta)\left(\frac{1-y}{1-\beta}\right)^\alpha$ and using binomial expansion for $(1-y)^\alpha$ we get

$$ \begin{aligned} {}_U\mu_{r,s:n}^{(k,l)} = \frac{C_{r,s:n}}{(1-\beta)(\alpha-1)} \sum_{j=0}^{\alpha} \binom{\alpha}{j} (-1)^j \int_\beta^1 \int_\beta^y x^k y^{l+j} [F_2(x)]^{r-1} \\ *[F_2(y) - F_2(x)]^{s-r-1} [1 - F_2(y)]^{n-s-1} f_2(x) f_2(y) dx dy \end{aligned} $$

By simplifying the resulting expression, the relation above is derived. It is clearly seen that when the upper partial product moment is available for $n-1$ sample size, it can be obtained for n sample size easily by using this relation.

THEOREM 13. *If $2 \le r < s \le n$ and $\alpha \in \mathbb{Z}^+$, then*

$$ {}_U\mu_{r,s:n}^{(k,l)} = \frac{n}{r-1} \left[{}_U\mu_{r-1,s-1:n-1}^{(k,l)} - \frac{\sum_{j=0}^{\alpha} \binom{\alpha}{j} (-1)^j \ {}_U\mu_{r-1,s-1:n-1}^{(k+j,l)}}{(1-\beta)^{(\alpha-1)}} \right] $$

PROOF.

$$ {}_U\mu_{r,s:n}^{(k,l)} = C_{r,s:n} \int_\beta^1 \int_\beta^y x^k y^l [F_2(x)]^{r-1} [F_2(y) - F_2(x)]^{s-r-1} [1 - F_2(y)]^{n-s} f_2(x) f_2(y) dx dy $$

After extracting $[F_2(x)]^{r-1}$ as $[F_2(x)]^{r-2} F_2(x) = [F_2(x)]^{r-2} \left[1 - (1-\beta\left(\frac{1-x}{1-\beta}\right)^\alpha\right]$, and using the binomial expansion for $(1-x)^\alpha$ we get

$$ \begin{aligned} {}_U\mu_{r,s:n}^{(k,l)} = C_{r,s:n} \int_\beta^1 \int_\beta^y x^k y^l [F_2(x)]^{r-2} [F_2(y) - F_2(x)]^{s-r-1} [1 - F_2(y)]^{n-s} f_2(x) f_2(y) dx dy \\ -(1-\beta)^{1-\alpha} C_{r,s:n} \sum_{j=0}^{\alpha} \binom{\alpha}{j} (-1)^j \int_\beta^1 \int_\beta^y x^{k+j} y^l [F_2(x)]^{r-2} [F_2(y) - F_2(x)]^{s-r-1} \\ *[1 - F_2(y)]^{n-s} f_2(x) f_2(y) dx dy \end{aligned} $$

Then, the moment relation in theorem 13 is derived by simplifying the resulting expression.

THEOREM 14. *If $1 \le r < s \le n$ and $\alpha(n-s) \in \mathbb{Z}^+$, then*

$$ {}_U\mu_{r,s:n}^{(k,l)} = \frac{n!}{s!(n-s)!(1-\beta)^{(\alpha-1)(n-s)}} \sum_{j=0}^{\alpha(n-s)} \binom{\alpha(n-s)}{j} (-1)^j \ {}_U\mu_{r,s:s}^{(k,l+j)} $$

PROOF.

$$_U\mu_{r,s:n}^{(k,l)} = C_{r,s:n}\int_\beta^1\int_\beta^y x^k y^l [F_2(x)]^{r-1}[F_2(y)-F_2(x)]^{s-r-1}[1-F_2(y)]^{n-s} f_2(x)f_2(y)dxdy$$

It is known that $[1-F_2(y)]^{n-s}$ equals $\frac{(1-y)^{\alpha(n-s)}}{(1-\beta)^{(\alpha-1)(n-s)}}$. By using binomial expansion for $(1-y)^{\alpha(n-s)}$ we get

$$_U\mu_{r,s:n}^{(k,l)} = \frac{C_{r,s:n}}{(1-\beta)^{(\alpha-1)(n-s)}}\sum_{j=0}^{\alpha(n-s)}\binom{\alpha(n-s)}{j}(-1)^j\int_\beta^1\int_\beta^y x^k y^{l+j} \\ *[F_2(x)]^{r-1}[F_2(y)-F_2(x)]^{s-r-1} f_2(x)f_2(y)dxdy$$

Thorem is derived by simplifying the resulting expression.
This theorem is obtained for the upper partial product moment, too. So these theorems can be used for both two part of the product moment.

THEOREM 15. *If $2 \le r < s \le n-1$ and $\alpha \in \mathbb{Z}^+$, then*

$$_U\mu_{r,s:n}^{(k,l)} = \frac{n!}{(r-1)!(n-r+1)!}\sum_{j=0}^{r-1}\binom{r-1}{j}(-1)^j\frac{1}{(1-\beta)^{j(\alpha-1)}} \\ \times\sum_{i=0}^{\alpha j}\binom{\alpha j}{i}(-1)^i \ {_U\mu_{1,s-r+1:n-r+1}^{(k+i,l)}}$$

PROOF.

$$_U\mu_{r,s:n}^{(k,l)} = C_{r,s:n}\int_\beta^1\int_\beta^y x^k y^l [F_2(x)]^{r-1}[F_2(y)-F_2(x)]^{s-r-1}[1-F_2(y)]^{n-s} f_2(x)f_2(y)dxdy \qquad (4.1)$$

It is known that $[F_2(x)]^{r-1} = \left[1-\frac{(1-x)^\alpha}{(1-\beta)^{\alpha-1}}\right]^{r-1}$. By using binomial expansion for this identity we get

$$\left[1-\frac{(1-x)^\alpha}{(1-\beta)^{\alpha-1}}\right]^{r-1} = \sum_{j=0}^{r-1}\binom{r-1}{j}(-1)^j\frac{(1-x)^{\alpha j}}{(1-\beta)^{j(\alpha-1)}}$$

Then, by using binomial expansion for $(1-x)^{\alpha j}$ in the resulting expression above, $[F_2(x)]^{r-1}$ is obtained as

$$[F_2(x)]^{r-1} = \sum_{j=0}^{r-1}\binom{r-1}{j}(-1)^j\frac{1}{(1-\beta)^{j(\alpha-1)}}\sum_{i=0}^{\alpha j}\binom{\alpha j}{i}(-1)^i x^i$$

replacing $[F_2(x)]^{r-1}$ with the resulting expression above in (4.1) we get

$$_U\mu_{r,s:n}^{(k,l)} = C_{r,s:n}\sum_{j=0}^{r-1}\binom{r-1}{j}(-1)^j\frac{1}{(1-\beta)^{j(\alpha-1)}}\sum_{i=0}^{\alpha j}\binom{\alpha j}{i}(-1)^i \\ *\int_\beta^1\int_\beta^y x^{k+i}y^l[F_2(y)-F_2(x)]^{s-r-1}[1-F_2(y)]^{n-s}f_2(x)f_2(y)dxdy$$

Finally, by simplifying the resulting expression, theorem 15 is derived.

Similar to Theorem 11, when we obtain the result for the minimum rth order statistics we can obtain the results for the upper partial product moments of the other rth order statistic.

5. Conclusion

In this paper, we derived some moment relations about the product moments of order statistics from the STSP distribution. We observed from the results that these relations are not much computationally friendly due to the piecewise definition of the pdf of the STSP distribution.

Acknowledgments

We thank the anonymous referee for the valuable comments that enhanced the presentation of the paper.

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