

UNIMODALITY OF DISTRIBUTION OF
GENERALIZED ORDER STATISTICS

Fazil Aliev

Department of Psychology,
Virginia Commonwealth University,
Richmond, 23219, VA, USA.Department of Actuarial and Risk Management,
Karabuk University, Turkey

Sevgi Yurt Oncel *

Department of Statistics,
Faculty of Arts and Sciences,
KIRIKKALE, Turkey

Abstract: Unimodality of the distribution of Generalized Order Statistics (GOSs) has a substantial role in many parameter and confidence interval problems of statistics, actuarial science and economics. Under some restrictions on model parameters and distributions a number of authors have shown unimodality of distribution of GOSs. In this article, we present some new results on unimodality of distribution of GOSs that extend and generalize recently obtained results. A counter example showing that the conditions of the main Theorem are minimal is also provided.

Key words: Convexity; Generalized Order Statistics; Order Statistics; Record Values; Unimodality

History: Submitted: 23 October 2015; Revised: 10 November 2016; Accepted: 11 January 2017

1. Introduction

A distribution function $F(x)$ is said to be unimodal if there exists $x = a$ such that $F(x)$ is convex for $x < a$ and concave for $x > a$ (see, An [4]), Dharmadhikari and Joag-Dev [10]. Unimodality of the distribution of GOSs has an important role in many parameter and confidence interval problems of statistics and actuarial science. Financial and actuarial stochastic actions such as life insurance and stock sales can be represented as functions of GOSs like first or last order statistics. We also refer the reader for some applications and the unimodality conditions for ordinary order statistics to Alam [1], Huang et al. [11]. It has been shown that the convexity of $1/f$ is sufficient to ensure unimodality of order statistics, where f is a density function, Basak and Basak [5] claimed the same for the same record values. However, Aliev [2] provided a counterexample showing that convexity of $1/f(x)$ is not sufficient for record values to be unimodal. Cramer et al. [9] and Cramer [8] considered special cases of GOSs investigated their unimodality conditions. Huang and Ghosh [11] and Chen [6] investigated strong unimodality conditions for GOSs. Recently, some previously known results for order statistics and record values to the case of GOSs have been generalized by Alimohammadi and Alamatsaz [3]. Nonetheless, the authors have covered not all possible cases and some cases remained open. In this study we provide a more general result that fills the gap in Alimohammadi and Alamatsaz [3]. A counterexample showing that our conditions are minimal is also provided.

2. Generalized order statistics

The idea of generalized order statistics (GOSs) was introduced in Kamps [12] as a unification

* Corresponding author. E-mail address: syoncel@gmail.com

of several models of random variables arranged in ascending order of magnitude with different interpretations and statistical applications. Assuming F to be an absolutely continuous distribution function with density f , and define $\bar{F} = 1 - F$. The random variables $X(r, n, \tilde{m}, k)$, $r = 1, 2, \dots, n$ are called the GOSs based on F if their joint density function is given by

$$f_{X(1,n,\tilde{m},k), \dots, X(n,n,\tilde{m},k)}(x_1, x_2, \dots, x_n) = k \left(\prod_{j=1}^{n-1} \gamma_j \right) \left(\prod_{i=1}^{n-1} [\bar{F}(x_i)]^{m_i} f(x_i) \right) \times [\bar{F}(x_n)]^{k-1} f(x_n),$$

for all $F^{-1}(0) < x_1 \leq x_2 \leq \dots \leq x_n < F^{-1}(1-)$, where $n \in \mathbb{N}$, $k > 0$ and $m_1, \dots, m_{n-1} \in \mathbb{R}$ are such that $\gamma_r = k + n - r + \sum_{j=r}^{n-1} m_j > 0$ for all $r \in \{1, \dots, n-1\}$, and $\tilde{m} = (m_1, \dots, m_{n-1})$, if $n \geq 2$ ($\tilde{m} \in \mathbb{R}$ is arbitrary if $n = 1$).

The sequence $X(r, n, \tilde{m}, k)$, $1 \leq r \leq n$ of g.o.s.'s based on a continuous d.f. F forms a Markov chain. It is known that many ordered variables are special cases of generalized order statistics. By choosing the parameters appropriately, known results for models such as order statistics ($m_1 = \dots = m_{n-1} = 0$, $k = 1$), record values ($m_1 = \dots = m_{n-1} = -1$, $k = 1$), k -th record values ($m_1 = \dots = m_{n-1} = -1$, $k \in \mathbb{N}$), progressive type II censored order statistics ($m_i \in \mathbb{N}$, $k \in \mathbb{N}$), sequential order statistics ($\gamma_i = (n - i + 1) \alpha_i$; $\alpha_1, \alpha_2, \dots, \alpha_n > 0$), Pfeifer's record values ($\gamma_i = \beta_i$, $\beta_1, \beta_2, \dots, \beta_n > 0$) can be obtained as special cases of g.o.s. .

Marginal density functions of GOSs have a form

$$f_{X(r,n,\tilde{m},k)}(x) = c_{r-1} [\bar{F}(x)]^{\gamma_r-1} g_r(F(x)) f(x), \quad x \in \mathbb{R}, \quad (2.1)$$

where $c_{r-1} = \prod_{i=1}^r \gamma_i$, $r = 1, \dots, n$ with $\gamma_n = k$ and $g_1(u) \equiv 1$, $g_r(u) = \int_0^u g_{r-1}(t) [1-t]^{m_{r-1}} dt$, $r > 1$ (see, Cramer and Kamps [7], Kamps and Cramer [13], Alimohammadi and Alamatsaz [3]).

The next Theorem is the most recent Theorem covering the unimodality of distribution of GOS's.

THEOREM 1 (see, Alimohammadi and Alamatsaz [3]). *Let F be an absolutely continuous distribution function with density f . Then, its GOS is unimodal if*

(i) For $r = 1$ and $m_i \in \mathbb{R}$, $i = 1, \dots, n - 1$:

(a) $0 < \gamma_1 < 1$ and f is non-decreasing on its support,

(b) $\gamma_1 = 1$ and f is unimodal,

(c) $1 < \gamma_1$ and either $1/f$ is convex or f is non-increasing on its support,

and

(ii) For $r \geq 2$:

(a) $0 < \gamma_r < 1$, $m_i \in \mathbb{R}$, $i = 1, \dots, n - 1$, and f is non-decreasing on its support,

(b) $1 \leq \gamma_r$, $m = m_1 = \dots = m_{n-1} = -1$, $1/f$ is convex and f is non-decreasing on its support,

(c) $1 \leq \gamma_r$, $0 \leq m_i$, $i = 1, \dots, n - 1$, and $1/f$ is convex.

For the case of negative m_i , $i = 1, \dots, n - 1$ the authors provide an example showing that for different negative values of m_i the GOS may not be unimodal even if $1/f$ is convex. We take into consideration that the case when $r \geq 2$, $\gamma_r \geq 1$ and m_i are negative and equal.

3. Main results

THEOREM 2. *Let F be an absolutely continuous distribution function with density f , $m = m_1 = \dots = m_{n-1} < 0$ ($m \neq -1$), $r \geq 2$ and $\gamma_r \geq 1$. Then, the distribution of GOS is unimodal if f is non-decreasing on its support and $1/f$ is convex.*

REMARK 1. Note that the case $m = m_1 = \dots = m_{n-1} = -1$ is included in Theorem 1 and have not been considered here.

PROOF. We will show that $f'_{X(r,n,\tilde{m},k)}(x)$ changes sign at most once and, if so, only from positive sign to negative sign, which is the sufficient condition GOS to be unimodal. From (2.1) we have

$$f'_{X(r,n,\tilde{m},k)}(x) = c_{r-1} f^2(x) [\bar{F}(x)]^{\gamma_r-1} g_r(F(x)) \times \left[-\frac{\gamma_r-1}{\bar{F}(x)} + \frac{g'_r(F(x))}{g_r(F(x))} + \frac{f'(x)}{f^2(x)} \right]. \quad (3.1)$$

As the expression outside parentheses is positive, sign of the derivative is the same as the sign of the expression in the parentheses. Let's denote

$$Z(u) = \frac{g'_r(u)}{g_r(u)} - \frac{\gamma_r-1}{1-u}, \quad 0 < u < 1.$$

Taking into account $g_{k+1}(u) = \frac{1}{k!(m+1)^k} (1 - (1-u)^{m+1})^k$ and $g'_k(u) = g_{k-1}(u)(1-u)^m$ we get

$$Z(u) = \frac{(r-1)(m+1)(1-u)^m}{1 - (1-u)^{m+1}} - \frac{\gamma_r-1}{1-u},$$

$$Z'(u) = \frac{[-(r-1)(m+1) - (\gamma_r-1)](1-u)^{2(m+1)}}{(1 - (1-u)^{m+1})^2(1-u)^2} + \frac{[2(\gamma_r-1) - (r-1)(m^2+m)](1-u)^{m+1} - (\gamma_r-1)}{(1 - (1-u)^{m+1})^2(1-u)^2}. \quad (3.2)$$

Consider the denominator of $Z'(u)$

$$\begin{aligned} Z_1(u) &= [-(r-1)(m+1) - (\gamma_r-1)](1-u)^{2(m+1)} \\ &\quad + [2(\gamma_r-1) - (r-1)(m^2+m)](1-u)^{m+1} - (\gamma_r-1) \\ &\equiv A(1-u)^{2(m+1)} + B(1-u)^{m+1} + C \\ &\equiv At^2 + Bt + C \end{aligned} \quad (3.3)$$

which is a quadratic expression of $t = (1-u)^{m+1}$ and

$$Z_1(0) = A + B + C = -(r-1)(m+1)^2 < 0. \quad (3.4)$$

Discriminant of $Z_1(u)$ as function of t is

$$\begin{aligned} D &= B^2 - 4AC \\ &= [2(\gamma_r-1) - (r-1)(m^2+m)]^2 + 4[(1-r)(m+1) - (\gamma_r-1)](\gamma_r-1) \\ &= (r-1)(m+1)^2[(r-1)m^2 - 4(\gamma_r-1)] \end{aligned}$$

Rewrite

$$\begin{aligned} Z(u) &= \frac{[(r-1)(m+1) + (\gamma_r-1)](1-u)^{m+1} - (\gamma_r-1)}{(1 - (1-u)^{m+1})(1-u)} \\ &\equiv \frac{-A(1-u)^{m+1} + C}{(1 - (1-u)^{m+1})(1-u)}. \end{aligned} \quad (3.5)$$

From the conditions of Theorem we conclude that the coefficient $C \equiv -(\gamma_r-1) \leq 0$. Case $A \geq 0$: It means $m+1 \leq -(\gamma_r-1)/(r-1) \leq 0$, therefore $(1-u)^{m+1} > 1$ or $1 - (1-u)^{m+1} < 0$. In this case both nominator and denominator of the right hand side of (3.4) are non-positive and $Z(u) \geq 0$. Because

f is non-decreasing that we will be getting $f'/f^2 \geq 0$. Hereby, right side of (3.1) is non-negative that implies the $f_{X(r,n,\tilde{m},k)}(x)$ is non-decreasing and, is unimodal. Case $A < 0$: In this case let's check two subcases when the discriminant of $Z_1(u)$ as function of t is negative and non-negative. Therefore, when discriminant is negative, $Z_1(u)$ as having a negative determinant, does not change the signs and $Z_1(0) < 0$ we conclude that $Z_1(u) < 0$ for all u and by (3.3) we get the $Z'(u) < 0$. Therefore, $Z(u)$ is decreasing function. Additionally, convexity of $1/f$ implies that it's derivative $-f'/f^2$ is non-decreasing, we conclude that $Z(F(x)) + f'/f^2$ is non-increasing, can change sign only once and, if so, only from positive to negative. It follows from (3.1) that $f'_{X(r,n,\tilde{m},k)}(x)$ itself also changes the sign at most once as x moves from $-\infty$ to ∞ which is equivalent to unimodality of $f_{X(r,n,\tilde{m},k)}(x)$. When discriminant is non-negative, intervals where $Z(u)$ increases and decreases depend on the sign of $Z'(u)$ which also depends on the sign of $Z_1(u)$. From (3.4) we conclude that $Z(u)$ is non-increasing around 0. Assuming that u_1, u_2 are two roots of $Z_1(u) = 0$. Then we can calculate values of $Z(u_1)$. By (3.3) we know that the product of roots of quadratic equation $(1 - u_1)^{m+1}(1 - u_2)^{m+1} = C/A$. By taking this into account on (3.5) we get

$$\begin{aligned} Z(u_1) &= \frac{-A(1 - u_1)^{m+1} + C}{(1 - (1 - u_1)^{m+1})(1 - u_1)} \\ &= \frac{C \left(-(1 - u_1)^{m+1} \frac{1}{C/A} + 1 \right)}{(1 - (1 - u_1)^{m+1})(1 - u_1)} \\ &= \frac{C \left(-(1 - u_1)^{m+1} \frac{1}{(1 - u_1)^{m+1}(1 - u_2)^{m+1}} + 1 \right)}{(1 - (1 - u_1)^{m+1})(1 - u_1)} \\ &= \frac{C(-1/(1 - u_2)^{m+1} + 1)}{(1 - (1 - u_1)^{m+1})(1 - u_1)}. \end{aligned}$$

Because $C < 0$, and roots $u_1 \in (0, 1)$ and $u_2 \in (0, 1)$ one can see no matter what is the sign of $m + 1$, $(1 - u_1)^{m+1}$ and $(1 - u_2)^{m+1}$ both are greater than 1 or both are less than 1. In both cases the right hand side of the last equality is positive as C is negative or equally $Z(u_1) > 0$. By the symmetry $Z(u_2) > 0$. It means that $Z(u)$ does not reach 0 on the interval $(0, \max(u_1, u_2))$ and then always decreases on $(\max(u_1, u_2), 1)$. By this way $Z(u)$ can change sign only on the interval $(\max(u_1, u_2), 1)$ and only from positive to negative (for the illustration see Figure 1).

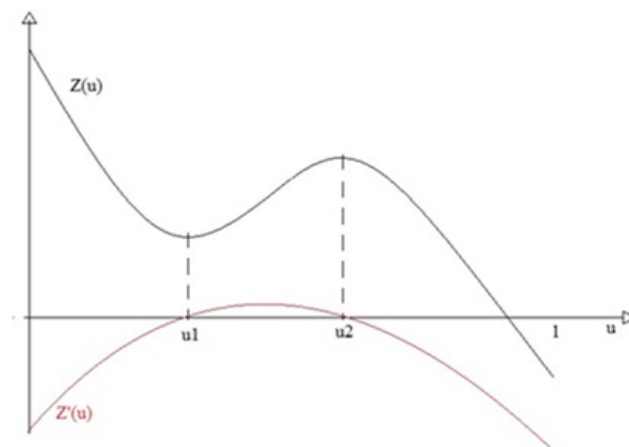


FIGURE 1. Illustration of $Z(u)$

As noted above f'/f^2 is non-increasing and non-negative. So $Z(F(x)) + f'/f^2$ is non-negative on $(0, \max(u_1, u_2))$ as the sum of two non-negative functions and only after decreasing on $(\max(u_1, u_2), 1)$ as the sum of two decreasing and non-increasing functions. As in the case above from (3.1) we conclude that $f'_{X(r,n,\tilde{m},k)}(x)$ changes the sign at most once as x moves from $-\infty$ to ∞ which is equivalent to unimodality of $f_{X(r,n,\tilde{m},k)}(x)$.

References

- [1] Alam, K. (1972). Unimodality of the distribution of an order statistic. *The Annals of Mathematical Statistics*, 43, 2041–2044.
- [2] Aliev, F.A. (2003). A comment on ‘Unimodality of the distribution of record statistics’. *Statistics and Probability Letters*, 64, 39–40.
- [3] Alimohammadi, M. and Alamatsaz, M. H. (2011). Some new results on unimodality of generalized order statistics and their spacings. *Statistics and Probability Letters*, 81(11), 1677–1682.
- [4] An, M.Y. (1998). Logconcavity versus logconvexity: a complete characterization. *Journal of Economic Theory*, 80, 350–369
- [5] Basak, P. and Basak, I. (2002). Unimodality of the distribution of record statistics. *Statistics and Probability Letters*, 56, 395–398.
- [6] Chen, H., Xie, H. and Hu, T. (2009). Log-concavity of generalized order statistics. *Statistics and Probability Letters*, 79, 396–399.
- [7] Cramer, E. and Kamps, U. (2003). Marginal distributions of sequential and generalized order statistics. *Metrika*, 58, 293–310.
- [8] Cramer, E. (2004). Logconcavity and unimodality of progressively censored order statistics. *Statistics and Probability Letters*, 68, 83–90.
- [9] Cramer, E., Kamps, U. and Rychlik, T. (2004). Unimodality of uniform generalized order statistics, with applications to mean bounds. *Annals of the Institute of Statistical Mathematics*, 56, 183–192.
- [10] Dharmadhikari, S. and Joag-Dev, K. (1988). *Unimodality, Convexity, and Applications*. Academic Press, Boston.
- [11] Huang, J.S. and Ghosh, M. (1982). A note on strong unimodality of order statistics. *Journal of the American Statistical Association*, 77, 929–930.
- [12] Kamps, U. (1995). A concept of generalized order statistics. *Journal of Statistical Planning and Inference*, 48, 1–23.
- [13] Kamps, U. and Cramer, E. (2001). On distributions of generalized order statistics. *Statistics*, 35, 269–280.