Available online: January 11, 2019

Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.
Volume 68, Number 1, Pages 1175–1185 (2019)
DOI: 10.31801/cfsuasmas.000000
ISSN 1303–5991 E-ISSN 2618-6470
http://communications.science.ankara.edu.tr/index.php?series=A1

SMARANDACHE CURVES IN THREE DIMENSIONAL LIE GROUPS

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Abstract. In this paper, we introduce special Smarandache curves and obtain Frenet apparatus of a Smarandache curve in three dimensional Lie groups with a bi-invariant metric. Moreover, we give some relations between a helix or a slant helix curve and its Smarandache curve in three dimensional Lie Groups.

1. Introduction

In the classical differential geometry, curves theory is a most important work area. Special curves and their characterizations have been studied for a long time and are still being studied. The application of special curves is seen in nature, mechanic tools, computer aided design and computer graphics etc.

One of the special curves is Smarandache curve, whose position vector is composed by Frenet frame vectors on an other regular curve. In [1], Ahmad introduced some special Smarandache curves in the Euclidean space. Then, in [2], Smarandache curves were examined according to Darboux frame in Euclidean 3-space. Also, some researchers studied Smarandache curves in Minkowski space, [8, 10]. Furthermore, in [9], Taşköprü and Tosun have investigated Smarandache curves according to Sabban Frame in Euclidean 3-space.

The degenerate semi-Riemannian geometry of Lie group has been studied by Çöken and Çiftçi [6]. In this work, they obtained a naturally reductive homogeneous semi-Riemannian space using the Lie group. Then, Çiftçi [5] defined general helices in three dimensional Lie groups with a bi-invariant metric and obtained a generalization of Lancret's theorem. Also, a relation between the geodesics of the so-called cylinders and general helices is given in the same study.

In [11], Okuyucu et al. defined slant helices in a three dimensional Lie group G with a bi-invariant metric as a curve $\alpha : I \subset \mathbb{R} \longrightarrow G$ whose normal vector field makes a constant angle with a left invariant vector field. Also, they defined Bertrand curves in [12]. Gök et al., in [7], studied Mannheim curves in three dimensional Lie

Received by the editors: May 25, 2018; Accepted: September 28, 2018.
2010 *Mathematics Subject Classification.* Primary 53A04; Secondary 22E15.
Key words and phrases. Smarandache curves, Lie groups, helices.

groups. Bozkurt et al., in [3], investigated the characterizations of the rectifying, normal and osculating curves in a three dimensional compact Lie group with a bi-invariant metric. As a physical application, Okuyucu et al., [13], obtained Spinor Frenet equations of curves and Körpınar, [14, 15], investigated a new version of the energy of curves in three dimensional compact Lie groups.

In this paper, we introduce special Smarandache curves in three dimensional Lie groups with a bi-invariant metric and obtain Frenet apparatus of a Smarandache curve in three dimensional Lie groups.

2. Preliminaries

By G we shall denote a Lie group with a bi-invariant metric $\langle\ ,\ \rangle$ in three dimensional Euclidean space. If $\mathfrak{g}$ is the Lie algebra of G, then we know that $\mathfrak{g}$ is isomorphic to T_eG where e is neutral element of G. Let ∇ be the Levi-Civita connection of Lie group G. If $\langle\ ,\ \rangle$ is a bi-invariant metric on G, we have

$$\langle X, [Y, Z]\rangle = \langle [X, Y], Z\rangle \tag{1}$$

and

$$\nabla_X Y = \frac{1}{2}[X, Y]$$

for all X, Y and $Z \in \mathfrak{g}$.

Let $\alpha : I \subset \mathbb{R} \to G$ be an arc-lenghted regular curve and $\{V_1, V_2, V_3\}$ be an orthonormal basis of $\mathfrak{g}$. In this case, any two vector fields ξ and δ along the curve α can be given as $\xi = \sum_{i=1}^{3} \xi_i V_i$ and $\delta = \sum_{i=1}^{3} \delta_i V_i$ where $\xi_i : I \to \mathbb{R}$ and $\delta_i : I \to \mathbb{R}$ are smooth functions. Furthermore, the Lie bracket of two vector fields ξ and δ is given

$$[\xi, \delta] = \sum_{i,j=1}^{3} \xi_i \delta_j [V_i, V_j]$$

and the covariant derivative of ξ along the curve α with the notation $\nabla_{\alpha'}\xi$ is given as follows

$$\nabla_{\alpha'}\xi = \dot{\xi} + \frac{1}{2}[\mathbb{T}, \xi] \tag{2}$$

where $\mathbb{T} = \alpha'$ and $\dot{\xi} = \sum_{i=1}^{3} \frac{d\xi_i}{dt} V_i$. Note that if ξ is a left-invariant vector field to the curve α then $\dot{\xi} = 0$ (see [4] for details).

Let $\{\mathbb{T}, \mathbb{N}, \mathbb{B}, \kappa, \tau\}$ denote the Frenet apparatus of the curve α, then we have $\kappa = \left\|\dot{\mathbb{T}}\right\|$ in G where $\mathbb{T}$, $\mathbb{N}$ and $\mathbb{B}$ are called, the tangent, the principal normal, the binormal vector fields, respectively. And also, κ is the curvature and τ is the torsion of α.

Definition 1. *Let $\alpha : I \subset \mathbb{R} \to G$ be a parametrized curve with $(\mathbb{T}, \mathbb{N}, \mathbb{B}, \kappa, \tau)$ then*

$$\tau_G = \frac{1}{2}\langle [\mathbb{T}, \mathbb{N}], \mathbb{B}\rangle \tag{3}$$

or

$$\tau_G = \frac{1}{2\kappa^2\tau}\left\langle\left[\ddot{\mathbb{T}},\mathbb{T}\right],\dot{\mathbb{T}}\right\rangle + \frac{1}{4\kappa^2\tau}\left\|\left[\mathbb{T},\dot{\mathbb{T}}\right]\right\|^2$$

(see [5]*).*

Proposition 2. *Let $\alpha : I \subset \mathbb{R} \rightarrow G$ be an arc length parametrized curve with $\{\mathbb{T},\mathbb{N},\mathbb{B}\}$. Then the following equalities hold,*

$$[\mathbb{T},\mathbb{N}] = \langle[\mathbb{T},\mathbb{N}],\mathbb{B}\rangle\,\mathbb{B} = 2\tau_G\mathbb{B},$$
$$[\mathbb{T},\mathbb{B}] = \langle[\mathbb{T},\mathbb{B}],\mathbb{N}\rangle\,\mathbb{N} = -2\tau_G\mathbb{N},$$

(see [11]*).*

Theorem 3. *Let $\alpha : I \subset \mathbb{R} \rightarrow G$ be an arc length parametrized curve with $\{\mathbb{T},\mathbb{N},\mathbb{B},\kappa,\tau\}$. Then the harmonic curvature function of α is defined by*

$$h = \frac{\tau - \tau_G}{\kappa}$$

where $\tau_G = \frac{1}{2}\langle[\mathbb{T},\mathbb{N}],\mathbb{B}\rangle$ (see [11]*).*

Theorem 4. *Let $\alpha : I \subset \mathbb{R} \rightarrow G$ be a parametrized curve with $(\mathbb{T},\mathbb{N},\mathbb{B},\kappa,\tau)$. Then α is a general helix if and only if*

$$\tau = c\kappa + \tau_G$$

where c is a constant (see [5]*). Hence a curve is a general helix if and only if its harmonic curvature function h is a constant function.*

Remark 5. *Let α be a helix with Frenet apparatus $(\mathbb{T},\mathbb{N},\mathbb{B},h)$ in three dimensional Lie group. Then the axis of the curve α can be given as*

$$X = \frac{h}{\sqrt{1+h^2}}\mathbb{T} + \frac{1}{\sqrt{1+h^2}}\mathbb{B} \tag{4}$$

or considering the constant harmonic curvature function $h = \cot\theta$, $\theta =$ constant, can be given

$$X = \cos\theta\mathbb{T} + \sin\theta\mathbb{B} \tag{5}$$

with the help of reference [5]*.*

Theorem 6. *Let $\alpha : I \subset \mathbb{R} \rightarrow G$ be an arc length parametrized curve with $\{\mathbb{T},\mathbb{N},\mathbb{B},\kappa,\tau\}$. Then α is a slant helix if and only if*

$$\sigma_N = \frac{\kappa\left(1+h^2\right)^{\frac{3}{2}}}{h'} = \tan\varphi$$

is a constant, where h is harmonic curvature function of the curve α and $\varphi \neq \frac{\pi}{2}$ is a constant (see [11]*).*

3. Smarandache Curves in a three dimensional Lie group

In this section, we define Smarandache curves and obtain Frenet apparatus of these curves in three dimensional Lie groups with a bi-invariant metric.

Unless otherwise stated, throughout the paper $\alpha : I \subset \mathbb{R} \to G$ is an arc-lenghted regular curve with the Frenet apparatus $\{\mathbb{T}, \mathbb{N}, \mathbb{B}, \kappa, \tau\}$ in three dimensional Lie group G with a bi-invariant metric.

Definition 7. *Let α be a curve in G. $\mathbb{TN}-$Smarandache curve can be defined as*

$$\psi(s_\psi) = \frac{1}{\sqrt{2}} \left(\mathbb{T}(s) + \mathbb{N}(s)\right). \tag{6}$$

Now, we compute Frenet invariants of $\mathbb{TN}-$Smarandache curve. Differentiating Eq. (6) with respect to s, we get

$$\psi' = \frac{d\psi}{ds_\psi}\frac{ds_\psi}{ds} = \frac{1}{\sqrt{2}}[\dot{\mathbb{T}}(s) + \dot{\mathbb{N}}(s)]$$

and

$$\mathbb{T}_\psi \frac{ds_\psi}{ds} = \frac{\kappa}{\sqrt{2}} \left(-\mathbb{T}(s) + \mathbb{N}(s) + h\mathbb{B}(s)\right),$$

where

$$\frac{ds_\psi}{ds} = \frac{\kappa}{\sqrt{2}}\sqrt{2 + h^2}. \tag{7}$$

And so, the tangent vector of ψ can be written as follow,

$$\mathbb{T}_\psi(s_\psi) = \frac{-\mathbb{T}(s) + \mathbb{N}(s) + h\mathbb{B}(s)}{\sqrt{2 + h^2}}. \tag{8}$$

By differentiating Eq. (8), we have

$$\frac{d\mathbb{T}_\psi}{ds_\psi}\frac{ds_\psi}{ds} = \frac{\mathcal{A}_\psi \mathbb{T}(s) + \mathcal{B}_\psi \mathbb{N}(s) + \mathcal{C}_\psi \mathbb{B}(s)}{(2 + h^2)^{\frac{3}{2}}} \tag{9}$$

where

$$\begin{aligned}
\mathcal{A}_\psi &= -\kappa \left(2 + h^2\right) + hh', \\
\mathcal{B}_\psi &= -\kappa \left(1 + h^2\right)\left(2 + h^2\right) - hh', \\
\mathcal{C}_\psi &= \left(\kappa h + h'\right)\left(2 + h^2\right) - h^2 h'.
\end{aligned}$$

Substituting (7) in (9), we get

$$\dot{\mathbb{T}}_\psi = \frac{\sqrt{2}}{\kappa \left(2 + h^2\right)^2} \left(\mathcal{A}_\psi \mathbb{T}(s) + \mathcal{B}_\psi \mathbb{N}(s) + \mathcal{C}_\psi \mathbb{B}(s)\right).$$

Then, the curvature and principal normal vector field of curve ψ are respectively,

$$\kappa_\psi = \left\|\dot{\mathbb{T}}_\psi\right\| = \frac{\sqrt{2}}{\kappa \left(2 + h^2\right)^2}\sqrt{\mathcal{A}_\psi^2 + \mathcal{B}_\psi^2 + \mathcal{C}_\psi^2}$$

and

$$\mathbb{N}_\psi(s_\psi) = \frac{1}{\sqrt{\mathcal{A}_\psi^2 + \mathcal{B}_\psi^2 + \mathcal{C}_\psi^2}} \left(\mathcal{A}_\psi \mathbb{T}(s) + \mathcal{B}_\psi \mathbb{N}(s) + \mathcal{C}_\psi \mathbb{B}(s)\right).$$

So, the binormal vector of curve γ is

$$\begin{aligned}\mathbb{B}_\psi(s_\psi) &= \mathbb{T}_\psi \times \mathbb{N}_\psi \\ &= \frac{1}{\rho q}\left\{\left(\mathcal{C}_\psi - h\mathcal{B}_\psi\right)\mathbb{T}(s) + \left(\mathcal{C}_\psi + h\mathcal{A}_\psi\right)\mathbb{N}(s) - \left(\mathcal{A}_\psi + \mathcal{B}_\psi\right)\mathbb{B}(s)\right\}\end{aligned}$$

where $\rho = \sqrt{2+h^2}$ and $q = \sqrt{\mathcal{A}_\psi^2 + \mathcal{B}_\psi^2 + \mathcal{C}_\psi^2}$.

In order to calculate the torsion of the curve ψ, we differentiate the ψ'

$$\psi'' = \frac{1}{\sqrt{2}}\left\{(-\kappa^2 - \kappa')\mathbb{T}(s) + (\kappa' - \kappa^2(1+h^2))\mathbb{N}(s) + (\kappa^2 h + \kappa' h + \kappa h')\mathbb{B}(s)\right\}$$

and thus

$$\psi''' = \frac{l_\psi \mathbb{T}(s) + m_\psi \mathbb{N}(s) + n_\psi \mathbb{B}(s)}{\sqrt{2}}$$

where

$$\begin{aligned} l_\psi &= -3\kappa\kappa'(1-h) + 2\kappa^2 h', \\ m_\psi &= -\kappa^3(1-h)(1+h^2) + \kappa''(1-h) - 2\kappa' h' - \kappa h'', \\ n_\psi &= 3\kappa\kappa' h(1-h) + \kappa^2 h'(1-3h). \end{aligned}$$

Thus we compute

$$\tau_\psi = \frac{\det\left(\psi', \psi'', \psi'''\right)}{\left\|\psi' \times \psi''\right\|^2},$$

$$\tau_\psi = \frac{\sqrt{2}\left\{\left(\kappa h\rho^2 + h'\right) l_\psi + h' m_\psi + \kappa\rho^2 n_\psi\right\}}{\left(\kappa h\rho^2 + \kappa h'\right)^2 + (\kappa h')^2 + \kappa^4\rho^4}.$$

Corollary 8. *Let α be a curve and ψ be the $\mathbb{TN}-$Smarandache curve of α in G. If the curve α is a helix with the axis X, then the tangent vector of the curve ψ is perpendicular the vector field X.*

Proof. It is obvious using the equations (4) and (8). □

Definition 9. *Let α be a curve in G. $\mathbb{TB}-$Smarandache curve can be defined as*

$$\omega(s_\omega) = \frac{1}{\sqrt{2}}\left(\mathbb{T}(s) + \mathbb{B}(s)\right). \tag{10}$$

Now, we compute Frenet invariants of $\mathbb{TB}-$Smarandache curve. Differentiating Eq. (10) with respect to s, we get

$$\omega' = \frac{d\omega}{ds_\omega}\frac{ds_\omega}{ds} = \frac{1}{\sqrt{2}}[\dot{\mathbb{T}}(s) + \dot{\mathbb{B}}(s)] \tag{11}$$

and

$$\mathbb{T}_\omega \frac{ds_\omega}{ds} = \frac{\kappa}{\sqrt{2}} (1-h) \, \mathbb{N}(s)$$

where

$$\frac{ds_\omega}{ds} = \frac{\kappa}{\sqrt{2}} (1-h) \, . \tag{12}$$

And so, the tangent vector of ω can be written as follow,

$$\mathbb{T}_\omega(s_\omega) = \mathbb{N}(s). \tag{13}$$

By differentiating Eq. (13), we have

$$\frac{d\mathbb{T}_\omega}{ds_\omega}\frac{ds_\omega}{ds} = -\kappa\mathbb{T}(s) + \kappa h \mathbb{B}(s). \tag{14}$$

Substituting (12) in (14), we get

$$\dot{\mathbb{T}}_\omega = \frac{\sqrt{2}}{(1-h)} \left(-\mathbb{T}(s) + h\mathbb{B}(s)\right).$$

Then, the curvature and principal normal vector field of curve ω are respectively,

$$\kappa_\omega = \left\| \dot{\mathbb{T}}_\omega \right\| = \frac{\sqrt{2}}{(1-h)} \sqrt{1+h^2}$$

and

$$\mathbb{N}_\omega(s_\omega) = -\frac{1}{\sqrt{1+h^2}}\mathbb{T}(s) + \frac{h}{\sqrt{1+h^2}}\mathbb{B}(s).$$

So, the binormal vector of curve ω is

$$\mathbb{B}_\omega = \mathbb{T}_\omega \times \mathbb{N}_\omega = \frac{h}{\sqrt{1+h^2}}\mathbb{T}(s) + \frac{1}{\sqrt{1+h^2}}\mathbb{B}(s).$$

In order to calculate the torsion of the curve ω, we differentiate Eq. (11)

$$\omega'' = \frac{1}{\sqrt{2}} \left\{ -\kappa^2(1-h)\mathbb{T}(s) + \left(\kappa'(1-h) - \kappa h'\right) \mathbb{N}(s) + \kappa^2 h(1-h)\mathbb{B}(s) \right\}$$

and thus

$$\omega''' = \frac{l_\omega \mathbb{T}(s) + m_\omega \mathbb{N}(s) + n_\omega \mathbb{B}(s)}{\sqrt{2}}$$

where

$$\begin{aligned} l_\omega &= -3\kappa\kappa'(1-h) + 2\kappa^2 h', \\ m_\omega &= -\kappa^3(1-h)(1+h^2) + \kappa''(1-h) - 2\kappa' h' - \kappa h'', \\ n_\omega &= 3\kappa\kappa' h(1-h) + \kappa^2 h'(1-3h). \end{aligned}$$

Thus we compute

$$\tau_{\omega} = \frac{\det(\omega', \omega'', \omega''')}{\|\omega' \times \omega''\|^2},$$

$$\tau_{\omega} = \frac{\sqrt{2}(n_{\omega} + hl_{\omega})}{\kappa^3(1-h)^2(1+h^2)}.$$

Corollary 10. *Let α be a curve and ω be the $\mathbb{TB}-$Smarandache curve of α in G. If the curve α is a slant helix with the axis X, then the curve ω is a helix. Furthermore, the axis of the curve ω is the same axis of α.*

Proof. One can see by considering the Eq. (13). □

Definition 11. *Let α be a curve in G. $\mathbb{NB}-$Smarandache curve can be defined as*

$$\phi(s_{\phi}) = \frac{1}{\sqrt{2}}\left(\mathbb{N}(s) + \mathbb{B}(s)\right). \tag{15}$$

Now, we compute Frenet invariants of $\mathbb{NB}-$Smarandache curve. Differentiating Eq. (15) with respect to s, we get

$$\phi' = \frac{d\phi}{ds_{\phi}}\frac{ds_{\phi}}{ds} = \frac{1}{\sqrt{2}}[\dot{\mathbb{N}}(s) + \dot{\mathbb{B}}(s)]$$

and

$$\mathbb{T}_{\phi}\frac{ds_{\phi}}{ds} = \frac{\kappa}{\sqrt{2}}[-\mathbb{T}(s) - h\mathbb{N}(s) + h\mathbb{B}(s)],$$

where

$$\frac{ds_{\phi}}{ds} = \frac{\kappa}{\sqrt{2}}\sqrt{1+2h^2}. \tag{16}$$

And so, the tangent vector of ϕ can be written as follow,

$$\mathbb{T}_{\phi} = \frac{-\mathbb{T}(s) - h\mathbb{N}(s) + h\mathbb{B}(s)}{\sqrt{1+2h^2}}. \tag{17}$$

By differentiating Eq. (17), we have

$$\frac{d\mathbb{T}_{\phi}}{ds_{\phi}}\frac{ds_{\phi}}{ds} = \frac{\mathcal{A}_{\phi}\mathbb{T}(s) + \mathcal{B}_{\phi}\mathbb{N}(s) + \mathcal{C}_{\phi}\mathbb{B}(s)}{(1+2h^2)^{\frac{3}{2}}} \tag{18}$$

where

$$\begin{aligned}
\mathcal{A}_{\phi} &= \kappa h\left(1+2h^2\right) + 2hh', \\
\mathcal{B}_{\phi} &= -\kappa\left(1+h^2\right)\left(1+2h^2\right) - h', \\
\mathcal{C}_{\phi} &= -\kappa h^2\left(1+2h^2\right) + h'.
\end{aligned}$$

Substituting (16) in (18), we get

$$\dot{\mathbb{T}}_{\phi} = \frac{\sqrt{2}}{\kappa\left(1+2h^2\right)^2}\left(\mathcal{A}_{\phi}\mathbb{T}(s) + \mathcal{B}_{\phi}\mathbb{N}(s) + \mathcal{C}_{\phi}\mathbb{B}(s)\right).$$

Then, the curvature and principal normal vector field of curve ϕ are respectively,

$$\kappa_\phi = \left\| \dot{\mathbb{T}}_\phi \right\| = \frac{\sqrt{2}}{\kappa\left(1+2h^2\right)^2}\sqrt{\mathcal{A}_\phi^2+\mathcal{B}_\phi^2+\mathcal{C}_\phi^2}$$

and

$$\mathbb{N}_\phi = \frac{1}{\sqrt{\mathcal{A}_\phi^2+\mathcal{B}_\phi^2+\mathcal{C}_\phi^2}}\left(\mathcal{A}_\phi\mathbb{T}(s)+\mathcal{B}_\phi\mathbb{N}(s)+\mathcal{C}_\phi\mathbb{B}(s)\right).$$

So, the binormal vector of curve ϕ is

$$\begin{aligned}\mathbb{B}_\phi(s_\phi) &= \mathbb{T}_\phi(s_\phi)\times\mathbb{N}_\phi(s_\phi)\\ &= \frac{1}{\rho q}\left[-h\left(\mathcal{C}_\phi+\mathcal{B}_\phi\right)\mathbb{T}(s)+\left(\mathcal{C}_\phi+h\mathcal{A}_\phi\right)\mathbb{N}(s)+\left(-\mathcal{B}_\phi+h\mathcal{A}_\phi\right)\mathbb{B}(s)\right]\end{aligned}$$

where $\rho=\sqrt{1+2h^2}$ and $q=\sqrt{\mathcal{A}_\phi^2+\mathcal{B}_\phi^2+\mathcal{C}_\phi^2}$.
In order to calculate the torsion of the curve ϕ, we differentiate the ϕ'

$$\begin{aligned}\phi'' = \frac{1}{\sqrt{2}}\{&(-\kappa'+\kappa^2 h)\mathbb{T}(s)+\left(-\kappa^2\left(1+h^2\right)-\kappa' h-\kappa h'\right)\mathbb{N}(s)\\ &+\left(-\kappa^2h^2+\kappa' h+\kappa h'\right)\mathbb{B}(s)\}\end{aligned}$$

and thus

$$\phi''' = \frac{l_\phi\mathbb{T}(s)+m_\phi\mathbb{N}(s)+n_\phi\mathbb{B}(s)}{\sqrt{2}}$$

where

$$\begin{aligned}l_\phi &= -\kappa''+3\kappa\kappa' h+2\kappa^2h'+\kappa^3(1+h^2),\\ m_\phi &= \kappa^3h(1+h^2)-3\kappa\kappa'-\kappa'' h-2\kappa' h'-\kappa h''-3\kappa h\left(\kappa h\right)',\\ n_\phi &= -\kappa^3h(1+h^2)-3\kappa h\left(\kappa h\right)'+\kappa'' h+2\kappa' h'+\kappa h''.\end{aligned}$$

Thus we compute

$$\tau_\phi = \frac{\det\left(\phi',\phi'',\phi'''\right)}{\left\|\phi'\times\phi''\right\|^2},$$

$$\tau_\phi = \frac{\sqrt{2}\left\{\kappa h\rho^2 l_\phi+h' m_\phi+\left(\kappa\rho^2+h'\right)n_\phi\right\}}{2\kappa^2h'\left(h'+\kappa\rho^2\right)+\kappa^4\rho^4(1+h^2)}.$$

Corollary 12. *Let α be a curve and ϕ be the $\mathbb{NB}-$Smarandache curve of α in G. If the curve α is a helix with the axis X, then the tangent vector of the curve ϕ is perpendicular the vector field X.*

Proof. This is an immediate consequence of the equations (4) and (17). □

Definition 13. *Let α be a curve in G. $\mathbb{TNB}-$Smarandache curve can be defined as*

$$\zeta(s_\zeta) = \frac{1}{\sqrt{3}}\left(\mathbb{T}(s) + \mathbb{N}(s) + \mathbb{B}(s)\right). \tag{19}$$

Now, we compute Frenet invariants of $\mathbb{TNB}-$Smarandache curve. Differentiating Eq. (19) with respect to s, we get

$$\zeta^{'} = \frac{d\zeta}{ds_\zeta}\frac{ds_\zeta}{ds} = \frac{1}{\sqrt{3}}\left(\dot{\mathbb{T}}(s) + \dot{\mathbb{N}}(s) + \dot{\mathbb{B}}(s)\right)$$

and

$$\mathbb{T}_\zeta\frac{ds_\zeta}{ds} = \frac{\kappa}{\sqrt{3}}[-\mathbb{T}(s) + (1-h)\,\mathbb{N}(s) + h\mathbb{B}(s)]$$

where,

$$\frac{ds_\zeta}{ds} = \frac{\sqrt{2}}{\sqrt{3}}\kappa\sqrt{1-h+h^2}. \tag{20}$$

And so, the tangent vector of ζ can be written as follow,

$$\mathbb{T}_\zeta\left(s_\zeta\right) = \frac{-\mathbb{T}(s) + (1-h)\,\mathbb{N}(s) + h\mathbb{B}(s)}{\sqrt{2}\sqrt{1-h+h^2}}. \tag{21}$$

By differentiating Eq. (21), we have

$$\frac{d\mathbb{T}_\zeta}{ds_\zeta}\frac{ds_\zeta}{ds} = \frac{\mathcal{A}_\zeta\mathbb{T}(s) + \mathcal{B}_\zeta\mathbb{N}(s) + \mathcal{C}_\zeta\mathbb{B}(s)}{2\sqrt{2}\left(1-h+h^2\right)^{\frac{3}{2}}} \tag{22}$$

where

$$\begin{aligned}
\mathcal{A}_\zeta &= -2\left(1-h+h^2\right)(1-h) - h^{'}\left(1-2h\right), \\
\mathcal{B}_\zeta &= 2\left(1-h+h^2\right)\left(-\kappa - h^{'} - \kappa h^2\right) + h^{'}\left(1-2h\right)(1-h), \\
\mathcal{C}_\zeta &= 2\left(1-h+h^2\right)\left(\kappa h\left(1-h\right) + h^{'}\right) + hh^{'}\left(1-2h\right).
\end{aligned}$$

Substituting (20) in (22), we get

$$\dot{\mathbb{T}}_\zeta = \frac{\sqrt{3}}{4\kappa\left(1-h+h^2\right)^2}\left(\mathcal{A}_\zeta\mathbb{T}(s) + \mathcal{B}_\zeta\mathbb{N}(s) + \mathcal{C}_\zeta\mathbb{B}(s)\right).$$

Then, the principal curvature and principal normal vector field of curve ζ are respectively,

$$\kappa = \left\|\dot{\mathbb{T}}_\zeta\right\| = \frac{\sqrt{3}}{4\kappa\left(1-h+h^2\right)^2}\sqrt{\mathcal{A}_\zeta^2 + \mathcal{B}_\zeta^2 + \mathcal{C}_\zeta^2}$$

and

$$\mathbb{N}_\zeta = \frac{1}{\sqrt{\mathcal{A}_\zeta^2 + \mathcal{B}_\zeta^2 + \mathcal{C}_\zeta^2}}\left(\mathcal{A}_\zeta\mathbb{T}(s) + \mathcal{B}_\zeta\mathbb{N}(s) + \mathcal{C}_\zeta\mathbb{B}(s)\right).$$

So, the binormal vector of curve ζ is

$$\begin{aligned}\mathbb{B}_\zeta &= \mathbb{T}_\zeta \times \mathbb{N}_\zeta \\ &= \frac{1}{\sqrt{2}\rho q}\left\{\left((1-h)\,\mathcal{C}_\zeta - h\mathcal{B}_\zeta\right)\mathbb{T}(s) + \left(\mathcal{C}_\zeta + h\mathcal{A}_\zeta\right)\mathbb{N}(s) - \left(\mathcal{B}_\zeta + (1-h)\,\mathcal{A}_\zeta\right)\mathbb{B}(s)\right\}\end{aligned}$$

where $\rho = \sqrt{1-h+h^2}$ and $\upsilon = \sqrt{\mathcal{A}_\zeta^2 + \mathcal{B}_\zeta^2 + \mathcal{C}_\zeta^2}$.

In order to calculate the torsion of the curve ζ, we differentiate

$$\begin{aligned}\zeta'' = \frac{1}{\sqrt{3}}&\{\left(-\kappa^2\,(1-h) - \kappa'\right)\mathbb{T}(s) + \left(-\kappa^2\left(1+h^2\right) + \kappa'\,(1-h) - \kappa h'\right)\mathbb{N}(s) \\ &+ \left(\kappa^2 h(1-h) + \kappa' h + \kappa h'\right)\mathbb{B}(s)\}\end{aligned}$$

and thus

$$\zeta''' = \frac{l_\zeta \mathbb{T}(s) + m_\zeta \mathbb{N}(s) + n_\zeta \mathbb{B}(s)}{\sqrt{3}}$$

where

$$\begin{aligned}l_\zeta &= -\kappa'' - 3\kappa\kappa'\,(1-h) + 2\kappa^2 h' + \kappa^3\left(1+h^2\right), \\ m_\zeta &= -\kappa^3\,(1-h)\left(1+h^2\right) - 3\kappa\kappa' + \kappa''\,(1-h) - 2\kappa' h' - \kappa h'' - 3\kappa h\,(\kappa h)', \\ n_\zeta &= -\kappa^3 h\left(1+h^2\right) - 2\kappa^2 h h' + \kappa h\,(1-h)\left(3\kappa' + \kappa\right) + \kappa'' h + 2\kappa' h' + \kappa h''.\end{aligned}$$

Thus we compute

$$\tau_\zeta = \frac{\det\left(\zeta', \zeta'', \zeta'''\right)}{\left\|\zeta' \times \zeta''\right\|^2},$$

$$\tau_\zeta = \frac{\sqrt{3}\left\{\left(2\kappa^2 h\rho^2 + \kappa h'\right) l_\zeta + \kappa h' m_\zeta + \left(2\kappa^2\rho^2 + \kappa h'\right) n_\zeta\right\}}{4\kappa^4\rho^2\left(\kappa h^4\rho^2 + h'(1+h)\right) + 3\kappa^3\left(h'\right)^2}.$$

Corollary 14. *Let α be a curve and ζ be the $\mathbb{TNB}-$Smarandache curve of α in G. If the curve α is a helix with the axis X, then the tangent vector of the curve ζ is perpendicular the vector field X.*

Proof. It is obvious using the equations (4) and (21). □

Acknowledgement. We would like to thank the reviewers for their valuable comments to improve the work.

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