Available online: January 15, 2019

Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.
Volume 68, Number 1, Pages 1197–1205 (2019)
DOI: 10.31801/cfsuasmas.513114
ISSN 1303–5991 E-ISSN 2618-6470

http://communications.science.ankara.edu.tr/index.php?series=A1

ANKARA UNIVERSITY FACULTY OF SCIENCES 1948 COMMUNICATIONS SERIES A1

QUASI-SUBORDINATION AND COEFFICIENT BOUNDS FOR CERTAIN CLASSES OF MEROMORPHIC FUNCTIONS OF COMPLEX ORDER

H. M. ZAYED, SERAP BULUT, AND A. O. MOSTAFA

Abstract. In this paper, we obtain Fekete-Szegö functional $\left|a_1 - \mu a_0^2\right|$ for functions of the classes $\Sigma_q^*(\varphi)$ and $\Sigma_{q,\lambda,b}^*(g,\varphi)$ using quasi-subordination. Sharp bounds for the Fekete-Szegö functional $\left|a_1 - \mu a_0^2\right|$ are obtained. Also, applications of the main results for subclasses of functions defined by Bessel function are also considered.

1. Introduction

Let Σ denote the class of meromorphic functions of the form:

$$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open punctured unit disc $\mathbb{U}^* = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\} = \mathbb{U} \backslash \{0\}$. Let $g(z) \in \Sigma$, be given by

$$g(z) = \frac{1}{z} + \sum_{k=0}^{\infty} g_k z^k, \tag{1.2}$$

then the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is given by

$$(f * g)(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k g_k z^k = (g * f)(z).$$

A function $f \in \Sigma$ is meromorphic starlike of order α, denoted by $\Sigma^*(\alpha)$, if

$$-\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \ (0 \leq \alpha < 1;\ z \in \mathbb{U}).$$

Received by the editors: September 12, 2017; Accepted: May 16, 2018.

2010 *Mathematics Subject Classification.* 30C45, 30C50.

Key words and phrases. Analytic, meromorphic, starlike function, Fekete-Szegö problem, convolution, subordination, quasi-subordination, Bessel function.

The class $\Sigma^*(\alpha)$ was introduced and studied by Pommerenke [13] (see also Miller [8]).

For two functions $f(z)$ and $g(z)$, analytic in $\mathbb{U}$, we say that $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$ and written $f(z) \prec g(z)$, if there exists a Schwarz function $w(z)$, analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$ $(z \in \mathbb{U})$. Furthermore, if $g(z)$ is univalent in $\mathbb{U}$, then (see [9]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

Let $\varphi(z)$ be an analytic function with positive real part on $\mathbb{U}$ satisfies $\varphi(0) = 1$ and $\varphi'(0) > 0$ which maps $\mathbb{U}$ onto a region starlike with respect to 1 and symmetric with respect to the real axis. Let $\Sigma^*(\varphi)$ be the class of functions $f \in \Sigma$ for which

$$-\frac{zf'(z)}{f(z)} \prec \varphi(z) \ (z \in \mathbb{U}).$$

The class $\Sigma^*(\varphi)$ was introduced and studied by Silverman et al. [15] (see also [2]). The class $\Sigma^*(\alpha)$ is a special case of the class $\Sigma^*(\varphi)$ when $\varphi(z) = \dfrac{1+(1-2\alpha)z}{1-z}$ $(0 \leq \alpha < 1)$.

Robertson [14] introduced the concept of quasi-subordination. For two functions $f(z)$ and $g(z)$, analytic in $\mathbb{U}$, we say that the function $f(z)$ is quasi-subordinate to $g(z)$ in $\mathbb{U}$ and write $f(z) \prec_q g(z)$, if there exists analytic functions $\phi(z)$ and $w(z)$, with $|\phi(z)| < 1$, $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = \phi(z)g(w(z))$ $(z \in \mathbb{U})$. When $\phi(z) = 1$, then $f(z) = g(w(z))$, so that $f(z) \prec g(z)$ in $\mathbb{U}$. Also, if $w(z) = z$, then $f(z) = \phi(z)g(z)$ and it is said that $f(z)$ is majorized by $g(z)$ and written $f(z) \ll g(z)$ in $\mathbb{U}$ (see Goyal and Goswami [6]). Hence it is obvious that quasi-subordination is a generalization of subordination as well as majorization.

Definition 1. *Let $\Sigma_q^*(\varphi)$ be the class of functions $f(z) \in \Sigma$ satisfying the quasi-subordination*

$$-\frac{zf'(z)}{f(z)} - 1 \prec_q \varphi(z) - 1 \ (z \in \mathbb{U}).$$

The above-mentioned class $\Sigma_q^*(\varphi)$ is the meromorphic analogue of the class $S_q^*(\varphi)$, introduced and studied by Mohd and Darus [10]*, which consists of functions $f(z)$ of the form $z + \sum_{k=2}^{\infty} a_k z^k$ for which*

$$\frac{zf'(z)}{f(z)} - 1 \prec_q \varphi(z) - 1 \ (z \in \mathbb{U}).$$

Definition 2. *For $b \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$ and $\lambda \in \mathbb{C}\backslash(0,1]$, $\Re(\lambda) \geq 0$, let $\Sigma_{q,\lambda,b}^*(g,\varphi)$ be the subclass of Σ consisting of functions $f(z)$ of the form (1.1), the functions $g(z)$ of the form (1.2) with $g_k > 0$ and satisfying the analytic criterion:*

$$\frac{1}{b}\left[\frac{-(1-2\lambda)z\,(f*g)'(z) + \lambda z^2\,(f*g)''(z)}{(1-\lambda)(f*g)(z) - \lambda z\,(f*g)'(z)} - 1\right] \prec_q \varphi(z) - 1.$$

In this paper, we obtain the Fekete-Szegö inequality for meromorphic functions in the classes $\Sigma_q^*(\varphi)$ and $\Sigma_{q,\lambda,b}^*(g,\varphi)$. Also, we investigate an applications for subclasses of functions defined by Bessel function.

2. Fekete-Szegö problem

Let Ω be the class of functions of the form

$$w(z) = w_1 z + w_2 z^2 + w_3 z^3 + ...,$$

satisfying $|w(z)| < 1$ for $z \in \mathbb{U}$.

To prove our results, we need the following lemma.

Lemma 1. [7]. *If $w \in \Omega$, then for any complex number t,*

$$\left|w_2 - t w_1^2\right| \leq \max\{1; |t|\}.$$

The result is sharp for the functions given by

$$w(z) = z \text{ or } w(z) = z^2.$$

Theorem 1. *Let $\varphi(z) = 1 + B_1 z + B_2 z^2 + ..., B_1 > 0$ and $\phi(z) = c_0 + c_1 z + c_2 z^2 + ...$ If $f(z)$ given by (1.1) belongs to the class $\Sigma_q^*(\varphi)$ and μ is a complex number, then*

$$\left|a_1 - \mu a_0^2\right| \leq \frac{B_1}{2}\left[1 + \max\left\{1, \left|\frac{B_2}{B_1}\right| + B_1 |1 - 2\mu|\right\}\right]. \tag{2.1}$$

The result is sharp.

Proof. If $f(z) \in \Sigma_q^*(\varphi)$, then there exist analytic functions $\phi(z)$ and $w(z)$, with $|\phi(z)| < 1,\ w(0) = 0$ and $|w(z)| < 1$ such that

$$-\frac{zf'(z)}{f(z)} - 1 = \phi(z)\left[\varphi(w(z)) - 1\right].$$

Since

$$-\frac{zf'(z)}{f(z)} = 1 - a_0 z + (a_0^2 - 2a_1)z^2 + ...,$$

$$\varphi(w(z)) = 1 + w_1 B_1 z + (w_1^2 B_2 + w_2 B_1)z^2 + (w_3 B_1 + 2w_1 w_2 B_2 + w_1^3 B_3)z^3 + ...,$$

and

$$\phi(z)\left[\varphi(w(z)) - 1\right] = c_0 w_1 B_1 z + \left(c_0 w_1^2 B_2 + c_0 w_2 B_1 + c_1 w_1 B_1\right) z^2 + ..., \tag{2.2}$$

then

$$\begin{aligned} a_0 &= -c_0 w_1 B_1, \\ a_1 &= -\frac{B_1 c_0}{2}\left[w_2 + w_1 \frac{c_1}{c_0} + w_1^2\left(\frac{B_2}{B_1} - B_1 c_0\right)\right]. \end{aligned}$$

Thus

$$a_1 - \mu a_0^2 = -\frac{B_1 c_0}{2}\left[w_2 + w_1\frac{c_1}{c_0} + w_1^2\left(\frac{B_2}{B_1} - B_1 c_0 + 2\mu B_1 c_0\right)\right],$$

and

$$\left|a_1-\mu a_0^2\right| \leq \frac{B_1\left|c_0\right|}{2}\left[\left|w_1\frac{c_1}{c_0}\right|+\left|w_2+w_1^2\left(\frac{B_2}{B_1}-B_1c_0+2\mu B_1c_0\right)\right|\right].$$

Since $\phi(z)$ is analytic and bounded in $\mathbb{U}$, we have (see [12])

$$\left|c_n\right| \leq 1-\left|c_0\right|^2 \leq 1 \ (n>0).$$

By using this fact and the well-known inequality, $|w_1| \leq 1$, we get

$$\left|a_1-\mu a_0^2\right| \leq \frac{B_1}{2}\left[1+\left|w_2+w_1^2\left(\frac{B_2}{B_1}-B_1c_0+2\mu B_1c_0\right)\right|\right].$$

The result (2.1) follows by an application of Lemma 1 and the result is sharp for the functions

$$-\frac{zf'(z)}{f(z)}-1=\phi(z)\left[\varphi(2z^2)-1\right],$$

and

$$-\frac{zf'(z)}{f(z)}-1=\phi(z)\left[\varphi(z)-1\right].$$

This completes the proof of Theorem 1. □

Remark 1. *Putting $\phi(z)=1$ in Theorem 1, we obtain the result obtained by Silverman et al.* [15, Theorem 2.1].

Theorem 2. *If $f(z) \in \Sigma$ satisfies*

$$-\frac{zf'(z)}{f(z)}-1 \ll \varphi(z)-1 \ (z \in \mathbb{U}),$$

then for any complex number μ,

$$\left|a_1-\mu a_0^2\right| \leq \frac{B_1}{2}\left[1+\left|\frac{B_2}{B_1}\right|+B_1\left|1-2\mu\right|\right].$$

Proof. The result follows by taking $w(z)=z$ in the proof of Theorem 1. □

Theorem 3. *Let $\varphi(z)=1+B_1z+B_2z^2+...$, $B_1>0$ and $\phi(z)=c_0+c_1z+c_2z^2+...$. If $f(z)$ given by (1.1) belongs to the class $\Sigma^*_{q,\lambda,b}(g,\varphi)$ $(\lambda \in \mathbb{C}\backslash(0,1],\ \Re(\lambda)\geq 0)$ and μ is a complex number, then*

$$\left|a_1-\mu a_0^2\right| \leq \frac{B_1}{2g_1}\left|\frac{b}{1-2\lambda}\right|\left[1+\max\left\{1,\left|\frac{B_2}{B_1}\right|+B_1\left|b\left[1-2\mu\frac{(1-2\lambda)g_1}{(1-\lambda)^2g_0^2}\right]\right|\right\}\right]. \tag{2.3}$$

The result is sharp.

Proof. If $f(z) \in \Sigma^{*}_{q,\lambda,b}(g,\varphi)$, then there exist analytic functions $\phi(z)$ and $w(z)$, with $|\phi(z)| < 1,\ w(0) = 0$ and $|w(z)| < 1$ such that

$$\frac{1}{b}\left[\frac{-(1-2\lambda)z(f*g)'(z)+\lambda z^2(f*g)''(z)}{(1-\lambda)(f*g)(z)-\lambda z(f*g)'(z)}-1\right]=\phi(z)\left[\varphi(w(z))-1\right].$$

Since

$$\frac{-(1-2\lambda)z(f*g)'(z)+\lambda z^2(f*g)''(z)}{(1-\lambda)(f*g)(z)-\lambda z(f*g)'(z)}=$$
$$1-(1-\lambda)a_0g_0z+\left[(1-\lambda)^2a_0^2g_0^2-2(1-2\lambda)a_1g_1\right]z^2+...,$$

and from (2.2), we get

$$\begin{aligned} a_0 &= -\frac{B_1c_0bw_1}{(1-\lambda)g_0}, \\ a_1 &= -\frac{B_1c_0b}{2(1-2\lambda)g_1}\left[w_2+w_1\frac{c_1}{c_0}+w_1^2\left(\frac{B_2}{B_1}-B_1c_0b\right)\right]. \end{aligned}$$

Thus

$$a_1-\mu a_0^2=-\frac{B_1c_0b}{2(1-2\lambda)g_1}\left[w_2+w_1\frac{c_1}{c_0}+w_1^2\left(\frac{B_2}{B_1}-B_1c_0b+2\mu\frac{(1-2\lambda)B_1c_0bg_1}{(1-\lambda)^2g_0^2}\right)\right],$$

and

$$\left|a_1-\mu a_0^2\right|\le\frac{B_1}{2g_1}\left|\frac{c_0b}{1-2\lambda}\right|\left[\left|w_1\frac{c_1}{c_0}\right|+\left|w_2+w_1^2\left(\frac{B_2}{B_1}-B_1c_0b+2\mu\frac{(1-2\lambda)B_1c_0bg_1}{(1-\lambda)^2g_0^2}\right)\right|\right].$$

Since $|c_0| \le 1,\ |c_1| \le 1$ and $|w_1| \le 1$ as in Theorem 1, we deduce that

$$\left|a_1-\mu a_0^2\right|\le\frac{B_1}{2g_1}\left|\frac{c_0b}{1-2\lambda}\right|\left[1+\left|w_2+w_1^2\left(\frac{B_2}{B_1}-B_1c_0b+2\mu\frac{(1-2\lambda)B_1c_0bg_1}{(1-\lambda)^2g_0^2}\right)\right|\right].$$

The result (2.3) follows by an application of Lemma 1. The result is sharp for the functions

$$\frac{1}{b}\left[\frac{-(1-2\lambda)z(f*g)'(z)+\lambda z^2(f*g)''(z)}{(1-\lambda)(f*g)(z)-\lambda z(f*g)'(z)}-1\right]=\phi(z)\left[\varphi(2z^2)-1\right],$$

and

$$\frac{1}{b}\left[\frac{-(1-2\lambda)z(f*g)'(z)+\lambda z^2(f*g)''(z)}{(1-\lambda)(f*g)(z)-\lambda z(f*g)'(z)}-1\right]=\phi(z)\left[\varphi(z)-1\right].$$

This completes the proof of Theorem 3. □

Remark 2. *Putting $\phi(z) = 1$ and $b = 1$ in Theorem 3, we obtain the result obtained by Silverman et al.* [15, Theorem 2.2].

Theorem 4. *If $f(z) \in \Sigma$ satisfies*

$$\frac{1}{b}\left[\frac{-(1-2\lambda)z(f*g)'(z)+\lambda z^2(f*g)''(z)}{(1-\lambda)(f*g)(z)-\lambda z(f*g)'(z)}-1\right]\ll\varphi(z)\ \ (z\in\mathbb{U}),$$

then for any complex number μ,

$$\left|a_1-\mu a_0^2\right| \le \frac{B_1}{2g_1}\left|\frac{b}{1-2\lambda}\right|\left[1+\left|\frac{B_2}{B_1}\right|+B_1\left|b\left[1-2\mu\frac{(1-2\lambda)g_1}{(1-\lambda)^2 g_0^2}\right]\right|\right].$$

Proof. The result follows by taking $w(z)=z$ in the proof of Theorem 3. □

3. Applications to functions defined by Bessel function

In this section, let us consider the second order linear homogenous differential equation (see, Baricz [3, p. 7]):

$$z^2 w''(z)+\alpha z w'(z)+\left[\beta z^2-\upsilon^2+(1-\alpha)\right]w(z)=0. \tag{3.1}$$

The function $w_{\upsilon,\alpha,\beta}(z)$, which is called the generalized Bessel function of the first kind of order υ, is defined a particular solution of (3.1). The function $w_{\upsilon,\alpha,\beta}(z)$ has the representation

$$w_{\upsilon,\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{(-\beta)^k}{\Gamma(k+1)\Gamma\left(k+\upsilon+\frac{\alpha+1}{2}\right)}\left(\frac{z}{2}\right)^{2k+\nu}.$$

Let us define

$$\begin{aligned}\mathcal{L}_{\upsilon,\alpha,\beta}(z) &= \frac{2^{\upsilon}\Gamma\left(\upsilon+\frac{\alpha+1}{2}\right)}{z^{\upsilon/2+1}}w_{\upsilon,\alpha,\beta}(z^{1/2})\\ &= \frac{1}{z}+\sum_{k=0}^{\infty}\frac{(-\beta)^{k+1}\Gamma\left(\upsilon+\frac{\alpha+1}{2}\right)}{4^{k+1}\ \Gamma(k+2)\Gamma\left(k+\upsilon+1+\frac{\alpha+1}{2}\right)}z^k,\end{aligned}$$

where υ,α,β are non-zero real positive numbers. The operator $\mathcal{L}_{\upsilon,\alpha,\beta}$ is a modification of the operator introduced by Deniz [5] (see also Baricz et al. [4]) for analytic functions.

By using the convolution, we define the operator $\mathcal{L}_{\upsilon,\alpha,\beta}$ as follows:

$$\begin{aligned}(\mathcal{L}_{\upsilon,\alpha,\beta}f)(z) &= \mathcal{L}_{\upsilon,\alpha,\beta}(z)*f(z)\\ &= \frac{1}{z}+\sum_{k=0}^{\infty}\frac{(-\beta)^{k+1}\Gamma\left(\upsilon+\frac{\alpha+1}{2}\right)}{4^{k+1}\ \Gamma(k+2)\Gamma\left(k+\upsilon+1+\frac{\alpha+1}{2}\right)}a_k z^k.\end{aligned}$$

The operator $\mathcal{L}_{\upsilon,\alpha,\beta}$ was introduced and studied by Mostafa et al. [11] (see also Aouf et al. [2]).

Definition 3. *Let* $\Sigma^{*q}_{\upsilon,\alpha,\beta}(\varphi)$ *be the class of functions* $f(z)\in\Sigma$ *satisfying the quasi-subordination*

$$-\frac{z(\mathcal{L}_{\upsilon,\alpha,\beta}f)'(z)}{(\mathcal{L}_{\upsilon,\alpha,\beta}f)(z)}-1 \prec_q \varphi(z)-1\ (z\in\mathbb{U}).$$

Definition 4. *For $b \in \mathbb{C}^*$, $\lambda \in \mathbb{C}\backslash(0,1]$, $\Re(\lambda) \geq 0$ and υ, α, β are non-zero real positive numbers, let $\Sigma^*_{q,\lambda,b}(\upsilon, \alpha, \beta; g, \varphi)$ be the subclass of Σ consisting of functions $f(z)$ of the form (1.1) and satisfying the analytic criterion:*

$$\frac{1}{b}\left[\frac{-(1-2\lambda)z(\mathcal{L}_{\upsilon,\alpha,\beta}f)'(z)+\lambda z^2(\mathcal{L}_{\upsilon,\alpha,\beta}f)''(z)}{(1-\lambda)(\mathcal{L}_{\upsilon,\alpha,\beta}f)(z)-\lambda z(\mathcal{L}_{\upsilon,\alpha,\beta}f)'(z)}-1\right] \prec_q \varphi(z)-1.$$

Using similar arguments to the proof of the previous theorems, we obtain the following theorems.

Theorem 5. *Let $\varphi(z) = 1+B_1z+B_2z^2+\ldots$, $B_1 > 0$ and $\phi(z) = c_0+c_1z+c_2z^2+\ldots$. If $f(z)$ given by (1.1) belongs to the class $\Sigma^{*q}_{\upsilon,\alpha,\beta}(\varphi)$ and μ is a complex number, then*

$$\begin{aligned}\left|a_1-\mu a_0^2\right| \quad &\leq \quad \frac{4^2\left(\upsilon+\frac{\alpha+1}{2}\right)\left(\upsilon+1+\frac{\alpha+1}{2}\right)B_1}{\beta^2} \\ &\quad \times\left[1+\max\left\{1,\left|\frac{B_2}{B_1}\right|+B_1\left|1-\mu\left(\frac{\upsilon+\frac{\alpha+1}{2}}{\upsilon+1+\frac{\alpha+1}{2}}\right)\right|\right\}\right].\end{aligned}$$

The result is sharp.

Theorem 6. *If $f(z) \in \Sigma$ satisfies*

$$-\frac{z(\mathcal{L}_{\upsilon,\alpha,\beta}f)'(z)}{(\mathcal{L}_{\upsilon,\alpha,\beta}f)(z)}-1 \ll \varphi(z)-1 \ (z \in \mathbb{U}),$$

then for any complex number μ,

$$\left|a_1-\mu a_0^2\right| \leq \frac{4^2\left(\upsilon+\frac{\alpha+1}{2}\right)\left(\upsilon+1+\frac{\alpha+1}{2}\right)B_1}{\beta^2}\left[1+\left|\frac{B_2}{B_1}\right|+B_1\left|1-\mu\left(\frac{\upsilon+\frac{\alpha+1}{2}}{\upsilon+1+\frac{\alpha+1}{2}}\right)\right|\right].$$

Theorem 7. *Let $\varphi(z) = 1+B_1z+B_2z^2+\ldots, B_1 > 0$ and $\phi(z) = c_0+c_1z+c_2z^2+\ldots$. If $f(z)$ given by (1.1) belongs to the class $\Sigma^*_{q,\lambda,b}(\upsilon, \alpha, \beta; g, \varphi)$ and μ is a complex number, then*

$$\begin{aligned}\left|a_1-\mu a_0^2\right| \quad &\leq \quad \frac{4^2\left(\upsilon+\frac{\alpha+1}{2}\right)\left(\upsilon+1+\frac{\alpha+1}{2}\right)B_1}{\beta^2}\left|\tfrac{b}{1-2\lambda}\right| \\ &\quad \times\left[1+\max\left\{1,\left|\frac{B_2}{B_1}\right|+B_1\left|b\left[1-\mu\frac{\left(\upsilon+\frac{\alpha+1}{2}\right)(1-2\lambda)}{\left(\upsilon+1+\frac{\alpha+1}{2}\right)(1-\lambda)^2}\right]\right|\right\}\right].\end{aligned}$$

The result is sharp.

Theorem 8. *If $f(z) \in \Sigma$ satisfies*

$$\frac{1}{b}\left[\frac{-(1-2\lambda)z(\mathcal{L}_{\upsilon,\alpha,\beta}f)'(z)+\lambda z^2(\mathcal{L}_{\upsilon,\alpha,\beta}f)''(z)}{(1-\lambda)(\mathcal{L}_{\upsilon,\alpha,\beta}f)'(z)-\lambda z(\mathcal{L}_{\upsilon,\alpha,\beta}f)'(z)}-1\right] \ll \varphi(z) \ (z \in \mathbb{U}),$$

then for any complex number μ,

$$\begin{aligned}\left|a_1-\mu a_0^2\right| \quad \leq \quad & \frac{4^2\left(\upsilon+\frac{\alpha+1}{2}\right)\left(\upsilon+1+\frac{\alpha+1}{2}\right)B_1}{\beta^2}\left|\frac{b}{1-2\lambda}\right| \\ & \times\left[1+\left|\frac{B_2}{B_1}\right|+B_1\left|b\left[1-\mu\frac{\left(\upsilon+\frac{\alpha+1}{2}\right)(1-2\lambda)}{\left(\upsilon+1+\frac{\alpha+1}{2}\right)(1-\lambda)^2}\right]\right|\right].\end{aligned}$$

References

[1] Aouf, M. K., El-Ashwah R. M. and Zayed, H. M., Fekete–Szego inequalities for certain class of meromorphic functions, *J. Egyptian Math. Soc.*, 21 (2013), 197–200.
[2] Aouf, M. K., Mostafa, A. O. and Zayed, H. M., Convolution properties for some subclasses of meromorphic functions of complex order, *Abstr. Appl. Anal.*, 2015 (2015), 1-6.
[3] Baricz, A., Generalized Bessel Functions of the First Kind, Lecture Notes in Math., Vol. 1994, Springer-Verlag, Berlin, 2010.
[4] Baricz, A., Deniz, E., Caglar, M. and Orhan, H., Differential subordinations involving the generalized Bessel functions, *Bull. Malays. Math. Sci. Soc.*, 38 (2015), no. 3, 1255-1280.
[5] Deniz, E., Differential subordination and superordination results for an operator associated with the generalized Bessel function, Preprint.
[6] Goyal, S. P. and Goswami, P., Majorization for certain classes of meromorphic functions defined by integral operator, *Ann. Univ. Mariae Curie Sklodowska Lublin-Polonia*, 2 (2012), 57–62.
[7] Keogh, F. R. and Merkes, E. P., A coefficient inequality for certain classes of analytic functions, *Proc. Amer. Math. Soc.*, 20 (1969), 8-12.
[8] Miller, J. E., Convex meromrphic mapping and related functions, *Proc. Amer. Math. Soc.*, 25 (1970), 220-228.
[9] Miller, S. S. and Mocanu, P. T., Differential Subordinations: Theory and Applications, Series on Monographs and Textbooks in Pure and Appl. Math., vol. 255, Marcel Dekker, Inc., New York, 2000.
[10] Mohd, M. H. and Darus, M., Fekete-Szego problems for quasi-subordination classes, *Abstr. Appl. Anal.*, (2012), Art. ID 192956, 1-14.
[11] Mostafa, A. O., Aouf, M. K. and Zayed, H. M., Convolution properties for some subclasses of meromorphic bounded functions of complex order, *Int. J. Open Problems Complex Analysis,* 8 (2016), no. 3 , 12-19.
[12] Nehari, Z., Conformal Mapping, McGraw-Hill, New York, 1952.
[13] Pommerenke, Ch., On meromrphic starlike functions, *Pacific J. Math.,* 13 (1963), 221-235.
[14] Robertson, M. S., Quasi-subordination and coefficient conjectures, *Bull. Amer. Math. Soc.,* 76 (1970), 1-9.
[15] Silverman, H., Suchithra, K., Stephen, B. A. and Gangadharan, A., Cofficient bounds for certain classes of meromorphic functions, *J. Inequal. Pure Appl. Math.*, (2008), 1-9.

Current address: H. M. Zayed: Department of Mathematics, Faculty of Science, Menofia University, Shebin Elkom 32511, Egypt

E-mail address: hanaa_zayed42@yahoo.com

ORCID Address: http://orcid.org/0000-0001-8814-8162

Current address: Serap Bulut: Kocaeli University, Civil Aviation College, Arslanbey Campus, TR-41285 Izmit-Kocaeli, Turkey

E-mail address: serap.bulut@kocaeli.edu.tr

ORCID Address: http://orcid.org/0000-0002-6506-4588

Current address: A. O. Mostafa: Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt

E-mail address: adelaeg254@yahoo.com