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THE QUASI XGAMMA-POISSON DISTRIBUTION: PROPERTIES AND APPLICATION

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Abstract: In this work, we introduce a new xgamma-Poisson lifetime model called the quasi xgamma-Poisson distribution. Some of its mathematical properties are derived. The proposed model can be motivated with a physical motivation by compounding the quasi xgamma construction with the truncated Poisson distribution. The quasi xgamma-Poisson model also motivated by the wide use of the xgamma distribution in many applied areas as well as for the fact that the new generalization provides more flexibility to analyze real data. We discuss the maximum likelihood estimation of the quasi xgamma-Poisson model parameters. An application to illustrate that the proposed quasi xgamma-Poisson model provides consistently better fit than the other competitive models.

Key words: Xgamma-Poisson life distributions; zero-truncated poisson distribution; maximum likelihood estimation; order statistics.
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1. Introduction

When the lifetime data present a bathtub shaped hazard rate function, such as human mortality and machine life cycles, practical problems generally require a wider range of possibilities in the medium risk. Researchers in the last years developed various extensions and modified forms of the xgamma distribution to obtain more flexible models with different number of parameters. A state-of-the-art survey on the class of such distributions can be found in Sen et al. (2016) and Sen and Chandra (2017). The xgamma distribution with its delegate structural and distributional properties serves as a potential survival model among the other popular lifetime models in the literature, more details can be seen in Sen et al. (2018). Recently, Sen et al. (2017) have introduced and studied a weighted version of xgamma distribution along with its length biased version for modeling time-to-event data sets. The quasi xgamma distribution, a two-parameter extension or generalization of xgamma distribution, shows superiority over many more life distributions when applied to real life survival and/or reliability data set.

In this present investigation, our aim is to introduce and study a three parameter extension of quasi xgamma distribution for modeling lifetime data. This extension is proposed by mixing quasi xgamma and zero-truncated Poisson distributions similarly Lindley-Poisson distribution (Gui et

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al., 2014). We can interpret the proposed model as follows: A situation where failure of a unit or system (be it mechanical or biological) occurs due to the presence of some unknown number of initial defects of similar kind. If we assume these unknown number of initial defects follow a zero-truncated Poisson distribution and their respective lives follow a quasi xgamma distribution, then the first failure distribution leads to what we call quasi xgamma-Poisson distribution. we aim, in this article, sythesis of the proposed model, its essential properties, method of estimating model parameters and real life application of the model.

The rest of the article is organized as follows. The proposed distribution is synthesized in Section 2. Different properties, such as, survival function, hazard rate function, moments and related measures, distributions of extreme order statistics and stochastic ordering, are discussed and studied in Section 3 and in its deliberate subsections. In Section 4, method of maximum likelihood is proposed for estimating the unknown parameters of the proposed distribution. Algorithm of a simulation is proposed along with a simulation study in Section 5. Section 6 deals with an application of the model with a real data illustration and comparison. Finally, Section 7 concludes.

2. Synthesis

If Y is a random variable (rv) following quasi xgamma (QXG) model with parameters α and θ (Sen and Chandra, 2017), then it has pdf as

$$f(y) = \frac{\theta}{(1+\alpha)} \left(\alpha + \frac{1}{2}\theta^2 y^2 \right) e^{-\theta y}|_{(y>0,\alpha,\theta>0)}. \tag{2.1}$$

Let us denote it by $Y \sim QXG(\alpha, \theta)$, corresponding cdf is given by

$$F(y) = 1 - \frac{\left(1+\alpha+\theta y + \frac{1}{2}\theta^2 y^2\right)}{(1+\alpha)} e^{-\theta y}|_{(y>0,\alpha,\theta>0)}.$$

The new xgamma-Poisson distribution can be synthesized as follows:
Suppose that the life of a unit (be it mechanical or biological) fails due to the presence of M (an unknown number) initial defects for some kind. Let $Y_1, Y_2, \ldots, Y_M$ denote the lives of the initial defects, then the life of the unit, say X, can be expressed as

$$X = Min\{Y_1, Y_2, \ldots, Y_M\}.$$

Suppose that the lives of the initial defects, $Y_1, Y_2, \ldots, Y_M$, follow identically and independently distributed (i.i.d) $QXG(\alpha, \theta)$ and the number of initial defects M follows a zero-truncated Poisson distribution with parameter λ. Then, the probability mass function (pmf) of M is

$$\Pr(M = m|_{\lambda>0, m=1,2,\ldots}) = p(m) = \frac{\lambda^m e^{-\lambda}}{m!\,(1-e^{-\lambda})} = \frac{\lambda^m}{m!\,(-1+e^{\lambda})}.$$

Assuming that the rvs $Y_i (i = 1, 2, \ldots, M)$ and M are independent, the conditional density of X given $M = m$ is

$$f(x|m) = \frac{m\theta\left(\frac{1}{2}\theta^2 x^2 + \alpha\right)\left(\frac{1}{2}\theta^2 x^2 + \theta x + \alpha + 1\right)^{-1+m}}{(1+\alpha)^m e^{m\theta y}}|_{(x>0)}.$$

Then, the marginal pdf of X can be obtained as

$$f(x) = \sum_{m=1}^{\infty} f(x|m) p(m)$$

$$= \sum_{m=1}^{\infty} m\theta \frac{(1+\alpha)^{-m}\left(\frac{1}{2}\theta^2x^2+\alpha\right)e^{-m\theta y}}{\left(\frac{1}{2}\theta^2x^2+\theta x+\alpha+1\right)^{1-m}} \cdot \frac{\lambda^m}{m!\,(-1+e^{\lambda})}$$
$$= \frac{\lambda\theta e^{-\theta x}\left(\frac{1}{2}\theta^2x^2+\alpha\right)}{(-1+e^{\lambda})(1+\alpha)} \sum_{m=1}^{\infty} \lambda^{-1+m} \frac{\left(\frac{1}{2}\theta^2x^2+\theta x+\alpha+1\right)^{-1+m}}{(1+\alpha)^{-1+m}(-1+m)!e^{\theta(-1+m)x}}$$
$$= \frac{\lambda\theta\left(\frac{1}{2}\theta^2x^2+\alpha\right)}{(-1+e^{\lambda})(1+\alpha)} \exp\left[\frac{\lambda\left(\frac{1}{2}\theta^2x^2+\theta x+\alpha+1\right)}{(1+\alpha)e^{\theta x}} - \theta x\right]\Big|_{(x>0,\alpha,\theta,\lambda>0)}.$$

2.1. The quasi xgamma-Poisson distribution

We have the following definition for the new distribution obtained from the above synthesis:

Definition 1. An absolutely continuous rv X will be said to follow quasi xgamma-Poisson (QXGP) distribution with parameters α, θ and λ if its pdf is of the form

$$f(x) = K(\alpha,\theta,\lambda)\left(\frac{1}{2}\theta^2x^2+\alpha\right)e^{\frac{\lambda e^{-\theta x}\left(1+\alpha+\theta x+\frac{1}{2}\theta^2x^2\right)}{(1+\alpha)}-\theta x}, x>0, \alpha,\theta,\lambda>0, \tag{2.2}$$

where $K(\alpha,\theta,\lambda) = \frac{\lambda\theta}{(e^{\lambda}-1)(1+\alpha)}$, a function of α, θ and λ.

We denote it by $X \sim QXGP(\alpha,\theta,\lambda)$.

Special cases:

(i) When $\alpha=\theta$ in (2.2), we obtain a new family of probability distributions, can be termed as xgamma-Poisson (XGP) distribution, with the following pdf:

$$f_1(x) = K_1(\theta,\lambda)\left(1+\frac{\theta}{2}x^2\right)e^{\frac{\lambda e^{-\theta x}\left(1+\theta+\theta x+\frac{1}{2}\theta^2x^2\right)}{(1+\theta)}-\theta x}, x>0, \theta,\lambda>0,$$

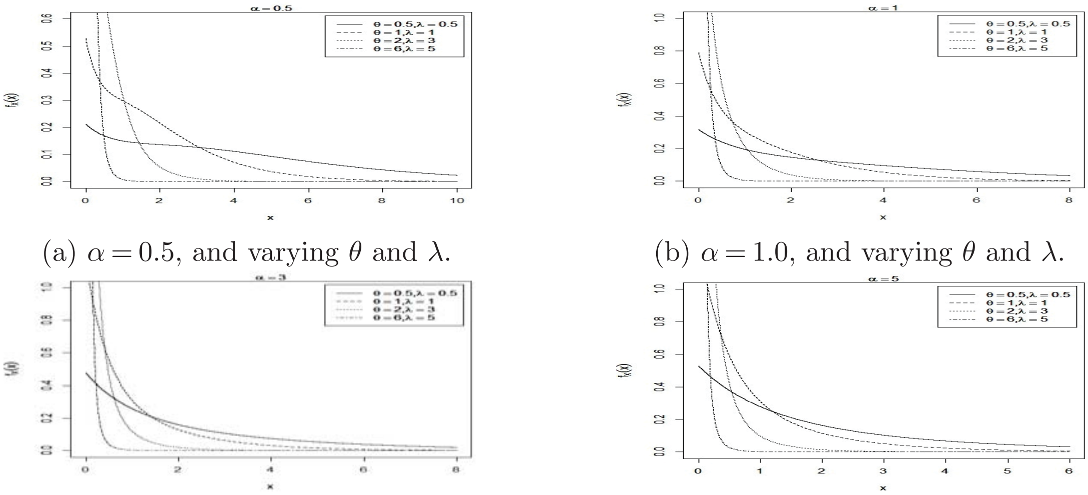
where $K_1(\theta,\lambda) = \frac{\lambda\theta^2}{\left(-1+e^{\lambda}\right)(1+\theta)}$, a function of θ and λ.

We can denote it by $X \sim XGP(\theta,\lambda)$.

(ii) While $\lambda \to 0$ in (2.2), the QXG model is obtained.

(iii) When $\alpha=\theta$ and $\lambda \to 0$ in (2.2), we obtain xgamma distribution with parameter θ (see for more details Sen et al., 2016).

The pdf curves for different values of α, θ and λ are shown in Figure 1.



(a) $\alpha = 0.5$, and varying θ and λ. (b) $\alpha = 1.0$, and varying θ and λ.

(c) $\alpha = 3.0$, and varying θ and λ. (d) $\alpha = 5.0$, and varying θ and λ.

FIGURE 1. The pdf curves of QXGP distribution for various values of α, θ and λ.

3. Properties

The cdf of $QXGP(\alpha,\theta,\lambda)$ is obtained as

$$F(x) = \frac{e^{\lambda} - e^{\frac{\lambda e^{-\theta x}\left(1+\alpha+\theta x+\frac{1}{2}\theta^2 x^2\right)}{(1+\alpha)}}}{e^{\lambda}-1}\Big|_{(x>0)}. \tag{3.1}$$

The corresponding survival function (or reliability function) is given by

$$S(x) = \frac{e^{\frac{\lambda e^{-\theta x}\left(\frac{1}{2}\theta^2 x^2+\theta x+\alpha+1\right)}{(1+\alpha)}} - 1}{e^{\lambda}-1}\Big|_{(x>0)}.$$

The failure rate function (or hazard rate function(hrf)) is, then, derived as

$$h(x) = \frac{f(x)}{S(x)} = \frac{\lambda\theta\left(\frac{1}{2}\theta^2 x^2+\alpha\right) e^{\frac{\lambda e^{-\theta x}\left(\frac{1}{2}\theta^2 x^2+\theta x+\alpha+1\right)}{(1+\alpha)} - \theta x}}{(1+\alpha)\left[e^{\frac{\lambda e^{-\theta x}\left(\frac{1}{2}\theta^2 x^2+\theta x+\alpha+1\right)}{(1+\alpha)}} - 1\right]}\Big|_{(x>0)}.$$

The failure rate curves for different values of α, θ and λ are shown in Figure 2.

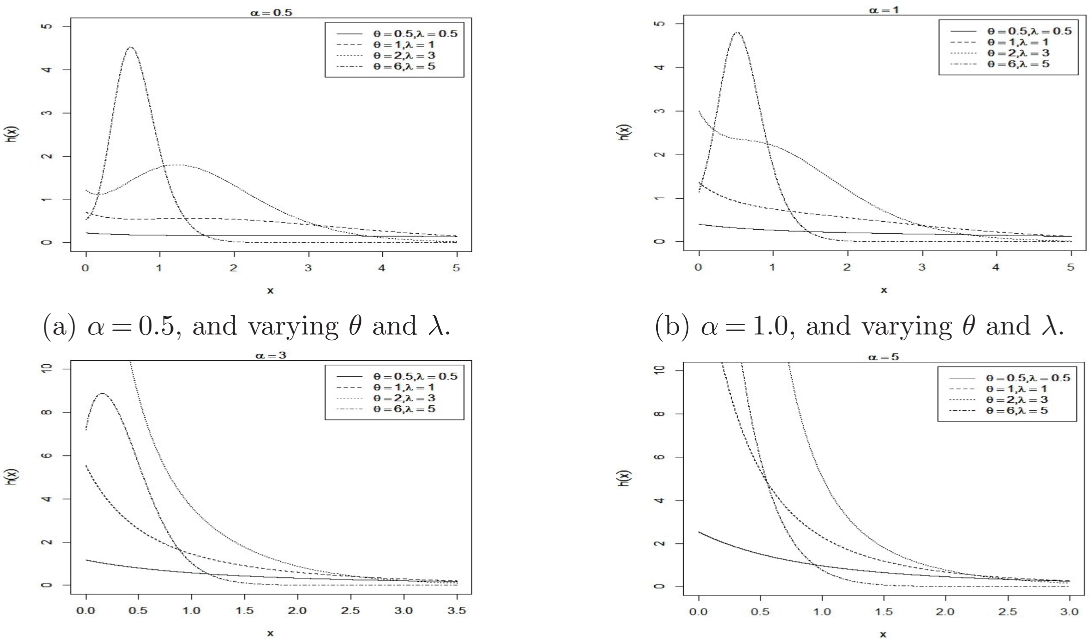
3.1. Moments and related measures

When $X \sim QXGP(\alpha,\theta,\lambda)$, the k^{th} raw moment of X is given by

$$\begin{aligned}\mu_k^{'} &= k\int_0^{\infty} x^{k-1} S(x)dx \\ &= \frac{k}{e^{\lambda}-1}\int_0^{\infty} x^{k-1}\left[e^{\frac{\lambda e^{-\theta x}\left(\frac{1}{2}\theta^2 x^2+\theta x+\alpha+1\right)}{(1+\alpha)}} - 1\right] dx\, \Big|_{(k=1,2,\cdots)}.\end{aligned} \tag{3.2}$$

μ_k's can not expressed in a closed form and hence numerical integration can be applied to fine the mean and other important related measures. The j^{th} order central moment can be obtained by the following relationship.

$$\mu_j = E[(X-\mu)^j] = \sum_{r=0}^{j}\binom{j}{r}\mu_r{}'(-\mu)^{j-r}\ \Big|_{(\,j=2,3,\ldots)},$$



(a) $\alpha = 0.5$, and varying θ and λ. (b) $\alpha = 1.0$, and varying θ and λ.

(c) $\alpha = 3.0$, and varying θ and λ. (d) $\alpha = 5.0$, and varying θ and λ.

FIGURE 2. Failure rate curves of QXGP distribution for various values of α, θ and λ.

where $\mu = E(X)$.
With above formula, the skewness and kurtosis coefficients are respectively given by

$$\sqrt{\beta_1} = \sqrt{\frac{\mu_3^2}{\mu_2^3}} \quad \text{and} \quad \beta_2 = \frac{\mu_4}{\mu_2^2}.$$

The values for mean, variance, $\sqrt{\beta_1}$ and β_2 for selected values of α, θ and λ are shown in Table 1. We note that for fixed values of α and λ, the values of $\sqrt{\beta_1}$ and β_2 do not depend on varying θ.

3.2. Asymptotic distributions of order statistics

Let $X_1, X_2, \ldots, X_{n-1}, X_n$ be a random sample (rs) of size n from $QXGP(\alpha, \theta, \lambda)$, then by the central limit theorem, the mean $(X_1 + X_2 + \ldots + X_n)/n$ approaches to normal distribution as $n \to \infty$.
Sometimes one might be interested in the asymptotics of the extreme order statistics. Let us denote:

$$X_{1:n} = Min\{X_1, X_2, \ldots, X_n\} := \text{Smallest order statistic}$$

and

$$X_{n:n} = Max\{X_1, X_2, \ldots, X_n\} := \text{Largest order statistic.}$$

These extreme order statistics represent the lives of series and parallel systems respectively and have important applications in reliability engineering and system sciences. We have the following theorem on the distributions of extreme order statistics.

THEOREM 1. *If $X_{n:n}$ and $X_{1:n}$ denote, respectively, the largest and smallest order statistics from $QXGP(\alpha, \theta, \lambda)$, then*
(1) $\lim_{n\to\infty} \Pr(X_{n:n} \le tb_n) = e^{-t^{-1}}, t > 0 \mid \left[F^{-1}\left(1 - \frac{1}{n}\right) = b_n\right].$

Table 1. Mean, variance, coefficients of skewness and kurtosis for different values of parameters

(α,θ,λ)	μ	$Var(X)$	$\sqrt{\beta_1}$	β_2
(0.5,0.5,0.5)	4.1906	11.764	1.3027	5.3126
(1.0,0.5,0.5)	3.5504	10.795	1.5115	5.9875
(2.0,0.5,0.5)	2.9321	9.0414	1.8055	7.3156
(5.0,0.5,0.5)	2.3348	6.5830	2.1681	9.5956
(0.5,0.5,1.0)	3.7410	10.523	1.4526	5.9288
(0.5,0.5,2.0)	2.9497	7.9725	1.7761	7.6041
(0.5,0.5,5.0)	1.5026	2.7438	2.5810	14.719
(0.5,1.0,0.5)	2.0953	2.9409	1.3027	5.3126
(0.5,2.0,0.5)	1.0476	0.7352	1.3027	5.3126
(0.5,5.0,0.5)	0.4190	0.1176	1.3027	5.3126
(0.5,0.05,1.0)	37.410	1052.3	1.4526	5.9288
(1.0,5.0,0.5)	0.3550	0.1079	1.5115	5.9875
(5.0,5.0,5.0)	0.0654	0.0092	4.9895	42.528

(2) $\lim_{n\to\infty}\Pr(X_{1:n}\le b_n^* t)=1-e^{-t}, t>0 \mid \left[F^{-1}\left(\frac{1}{n}\right)=b_n^*\right]$.

Proof. We apply the following asymptotic results (see Arnold et al., 2008) for $X_{1:n}$ and $X_{n:n}$.
(1) For the largest order statistic $X_{n:n}$, we have

$$\lim_{n\to\infty}\Pr(X_{n:n}\le a_n+b_n t)=e^{-t^{-d}}, t>0, c>0 \text{ (Fréchet type)},$$

where $a_n=0$ and $b_n=F^{-1}(1-1/n)$ iff $F^{-1}(1)=\infty$ and $\exists$ a constant $d>0$ such that,

$$\lim_{x\to\infty}\frac{1-F(xt)}{1-F(x)}-t^{-d}.$$

From the cdf of $QXGP(\alpha,\theta,\lambda)$ distribution as given in (3.1), letting $F(x)=1$, we can easily see that $F^{-1}(1)=\infty$ and

$$\lim_{x\to\infty}\frac{1-F(xt)}{1-F(x)}-t^{-1}.$$

Therefore, we obtain $d=1, a_n=0$ and $b_n=F^{-1}(1-1/n)$.
(2) For the smallest order statistic $X_{1:n}$, we have

$$\lim_{n\to\infty}\Pr(X_{1:n}\le a_n^*+b_n^* t)=1-e^{-t^c}, t>0, c>0 \text{ (Weibull type)},$$

where $a_n^*=F^{-1}(0)$ and $b_n^*=F^{-1}(1/n)-F^{-1}(0)$ iff $F^{-1}(0)$ is finite,

$$\lim_{\epsilon\to 0^+}\frac{F(F^{-1}(0)+\epsilon t)}{F(F^{-1}(0)+\epsilon)}=t^c \quad \forall t>0, c>0.$$

Letting $F(x)=0$ we see that $F^{-1}(0)=0$ and finite. Moreover,

$$\lim_{\epsilon\to 0^+}\frac{F(0+\epsilon t)}{F(0+\epsilon)}=t.$$

Finally, we have $c=1, a_n^*=0$ and $b_n^*=F^{-1}(1/n)$.
Hence the proof is completed.

3.3. Stochastic ordering

For a positive continuous rv, stochastic ordering is an important tool for judging the comparative behavior. Let us denote the pdf, cdf, hrf amd mean residual life function (mrl) of a positive continuous rv X by $f_X(\cdot), F_X(\cdot), h_X(\cdot)$ and $m_X(\cdot)$, respectively, and those of another positive continuous rv Y by $f_Y(\cdot), F_Y(\cdot), h_Y(\cdot)$ and $m_Y(\cdot)$, respectively. We recall some basic definitions.

Definition 2. A rv X is said to be smaller than a rv Y in the
(i) The stochastic order $(X \leq_{(sto)} Y)$ if $F_X(x) \geq F_Y(x)$, $\forall\, x$.
(ii) The hazard rate order $(X \leq_{(hro)} Y)$ if $h_X(x) \geq h_Y(x)$, $\forall\, x$.
(iii) The mean residual life order $(X \leq_{(mrlo)} Y)$ if $m_X(x) \leq m_Y(x)$, $\forall\, x$.
(iv) The ikelihood ratio order $(X \leq_{(lro)} Y)$ if $\frac{f_X(x)}{f_Y(x)}$ decreases in x.
The below given implications (see Shaked and Shanthikumar, 1994) are well justified:

$$\left[X \leq_{(lro)} Y\right] \Rightarrow \left[X \leq_{(hro)} Y\right] \Rightarrow \left[X \leq_{mrl} Y\right] \text{ and } \left[X \leq_{(hro)} Y\right] \Rightarrow \left[X \leq_{(sto)} Y\right] \tag{3.3}$$

The following theorem shows that the QXGP distributions are ordered with respect to different stochastic orderings.

Theorem 2. *Let $X \sim QXGP(\alpha,\theta,\lambda_1)$ and $Y \sim QXGP(\alpha,\theta,\lambda_2)$. If $\lambda_1 > \lambda_2$ then $\left[X \leq_{(lro)} Y\right]$ and $\left[X \leq_{(hro)} Y\right]$, $\left[X \leq_{(mrlo)} Y\right]$, $\left[X \leq_{(sto)} Y\right]$.*

Proof. For any $x > 0$, the ratio of the densities is given by

$$g(x) = \frac{f_X(x)}{f_Y(x)} = \frac{\lambda_1\left(e^{\lambda_2}-1\right)}{\lambda_2\left(e^{\lambda_1}-1\right)} exp\left\{\frac{(\lambda_1-\lambda_2)e^{-\theta x}\left(1+\alpha+\theta x+\frac{\theta^2}{2}x^2\right)}{(1+\alpha)}\right\}$$

Taking derivative with respect to x, we have

$$g'(x) = -\frac{\theta\lambda_1(\lambda_1-\lambda_2)\left(e^{\lambda_2}-1\right)e^{-\theta x}\left(\alpha+\frac{\theta^2}{2}x^2\right)}{\lambda_2\left(e^{\lambda_1}-1\right)(1+\alpha)} exp\left\{\frac{(\lambda_1-\lambda_2)e^{-\theta x}\left(1+\alpha+\theta x+\frac{\theta^2}{2}x^2\right)}{(1+\alpha)}\right\}$$

Now, $g'(x) < 0$ if $\lambda_1 > \lambda_2$ and hence $X \leq_{lr} Y$ if $\lambda_1 > \lambda_2$. The other orderings are immediate by (3.3). Hence the proof is completed.

4. Maximum likelihood estimation (MLE)

Let $x_1, x_2, \ldots, x_n$ be a rs from the QXGP. Let $\varphi = (\alpha,\theta,\lambda)^T$ be the parameter vector. Then, the log likelihood (**LL**) function for φ, say $\ell(\varphi) = \ell$,

$$\begin{aligned}\ell = & -n\log(1+\alpha) + n\log\theta + n\log\lambda - n\log(e^{\lambda}-1) \\ & + \sum_{i=1}^{n}\log\left(\frac{1}{2}\theta^2x^2+\alpha\right) + \sum_{i=1}^{n}\left[\frac{\lambda e^{-\theta x_i}\left(1+\alpha+\theta x_i+\frac{1}{2}\theta^2x_i^2\right)}{(1+\alpha)} - \theta x_i\right].\end{aligned} \tag{4.1}$$

Equation (10) can be maximized directly via some sub-routine in any packet programs. The score vector components, say $\mathbf{U}(\varphi) = \frac{\partial\ell}{\partial\varphi} = \left(U_\alpha, U_\theta, U_\lambda\right)^T$, are given by λ

$$U_\alpha = -\frac{n}{1+\alpha} + \sum_{i=1}^{n}\left(\frac{1}{2}\theta^2x_i^2+\alpha\right)^{-1} - \lambda\sum_{i=1}^{n}e^{-\theta x_i}\frac{\theta x_i+\frac{1}{2}\theta^2x_i^2}{(1+\alpha)^2},$$

$$U_\theta = \frac{n}{\theta} + \sum_{i=1}^{n}\frac{\theta x_i^2}{\frac{1}{2}\theta^2x^2+\alpha} - \sum_{i=1}^{n}\left[\frac{e^{-\theta x_i}}{(1+\alpha)}\left(\alpha x_i+\frac{1}{2}\theta^2x_i^3\right)+x_i\right]$$

and

$$U_{\lambda} = \frac{n}{\lambda} - \frac{ne^{\lambda}}{e^{\lambda}-1} + \sum_{i=1}^{n} \frac{e^{-\theta x_i}\left(1+\alpha+\theta x_i + \frac{1}{2}\theta^2 x_i^2\right)}{(1+\alpha)}.$$

Setting $U_\alpha = U_\theta = U_\lambda = 0$ and solving them simultaneously we get the MLE $\widehat{\varphi} = (\widehat{\alpha}, \widehat{\theta}, \widehat{\lambda})^T$ of $\varphi = (\alpha, \theta, \lambda)^T$. The likelihood ratio (LR) statistic can be used for comparing the QXGP model with XGP model, which is equivalently to test $H_0 : \alpha = \theta$. For this situaiton, the LR statistic is computed with $w = 2[\ell(\hat{\alpha}, \hat{\theta}, \hat{\lambda}) - \ell(\tilde{\theta}, \tilde{\lambda})]$, where $(\hat{\alpha}, \hat{\theta}, \hat{\lambda})$ are the unrestricted MLEs and $(\tilde{\theta}, \tilde{\lambda})$ are the restricted estimates under H_0. The statistic w is asymptotically (as $n \to \infty$) distributed as χ^2_υ, where υ is difference of two parameter vectors of nested models. For example, $\upsilon = 1$ for above hypothesis test.

5. Simulation study

We can generate a random data from the $QXGP(\alpha, \theta, \lambda)$ distribution using the following simulation algorithm:

1. Generate $M \sim$ zero-truncated Poisson (λ);
2. Generate $U_i \sim uniform\ [Uni(0,1)], i = 1, 2, \ldots, M$;
3. Generate $V_i \sim exponential\ [Exp(\theta)], i = 1, 2, \ldots, M$;
4. Generate $W_i \sim gamma\, [Gam(3, \theta)], i = 1, 2, \ldots, M$;
5. If $U_i \leq \alpha/(1+\alpha)$, then set $Y_i = V_i$, otherwise, set $Y_i = W_i, i = 1, 2, \ldots, M$;
6. Set $X = min(Y_1, Y_2, \ldots, Y_M)$, then X is the required sample.

Here, we give the simulation study based on graphical results to see performance of the maximum likelihood estimations of parameters. We generate $N = 1000$ samples of sizes $n = 20, 21, \ldots, 250$ from QXGP model with the true parameters values $\alpha = 2.2$, $\theta = 1$ and $\lambda = 0.5$. Random numbers procedure has been obtained by using inverse of QXGP cdf. We obtain the empirical mean (em), standard deviations (sd), bias and mean square error (MSE) of the MLEs for this simulation study. The empirical bias and MSE are calculated by (for $h = \alpha, \theta, \lambda$)

$$\widehat{Bias_h} = \tfrac{1}{N} \sum\nolimits_{i=1}^{N} \left(\hat{h}_i - h\right)$$

and

$$\widehat{MSE_h} = \tfrac{1}{N} \sum\nolimits_{i=1}^{N} \left(\hat{h}_i - h\right)^2,$$

respectively. All results and estimations have been calculated by optim-CG routine in the R programme. We give results of this simulation study in Figure 3. This Figure shows that that when the sample size (n) increases, all estimated means approach to true parameter value as well as empirical biases approach to 0. The sds and MSEs also decrease in all the cases, while sample size increases.

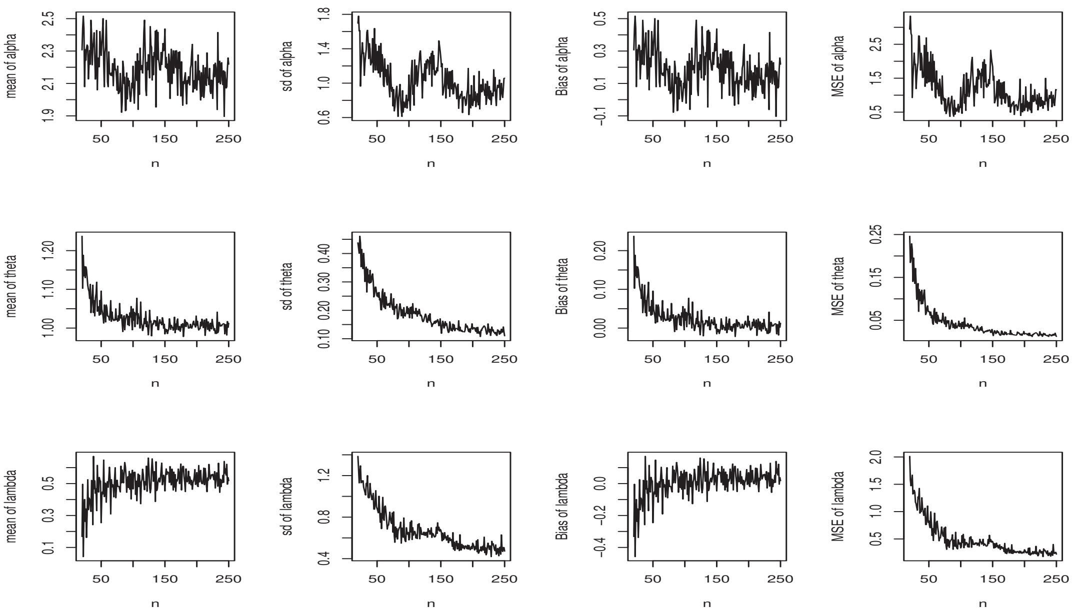


FIGURE 3. The empirical means, sds, biases and MSEs of the estimated parameters versus n

6. Application with real data illustration

We illustrate the flexibility of the QXGP model on the real data set. We also compare this model with the QXG model, XGP model, XG model, exponential Poisson (EP) model (see Kuş (2007)), exponentiated Weibull (EW) model (see Mudholkar and Srivasta (1993)), Weibull Poisson (WP) model (see Lu and Shi (2012)), exponentiated exponential (EE) model (see Gupta and Kundu (1999)) and exponentiated Nadarajah-Haghighi (ENH) model (see Lemonte (2013)) under the estimated log-likelihood values $\hat{\ell}$, Akaike Information Criteria (AIC), corrected Akaike information criterion (CAIC), Cramer von Mises (W^*) and Anderson-Darling (A^*) goodness of-fit statistics for all distribution models. We note that The AIC and CAIC are by given by $AIC = -2\widehat{\ell} + 2p$ and $CAIC = -2\widehat{\ell} + 2pn\left(n-k-1\right)^{-1}$, where p is the number of the estimated model parameters and n is sample size. The W^* and A^* statistics have been described as

$$W^* = \sum\nolimits_{i=1}^{n}\left(\hat{F}\left(x_{(i)}\right) - \frac{i-0.5}{n}\right)^2 + \frac{1}{12n}$$

and

$$A^* = -\sum\nolimits_{i=1}^{n}\frac{2i-1}{n}\left[\ln\hat{F}\left(x_{(i)}\right) + \ln\hat{\bar{F}}\left(x_{(n+1-i)}\right)\right] - n$$

by Evans et al. (2008). Also, one may see Chen and Balakrishnan (1995) for W^* and A^* in detail. It can be seen as the best model which has the smaller the values of the AIC, CAIC, W^* and A^* statistics and the larger the values of $\hat{\ell}$. The real data set is the stress-rupture life of kevlar 49/epoxy strands which are subjected to constant sustained pressure at the 90% stress level until all had failed. This data set was studied by Andrews and Herzberg (1985), Cooray and Ananda (2008), and Paraniaba et al. (2013).

In the applications, the information about the hazard shape can help in selecting a particular model. For this aim, a device called the total time on test (TTT) plot (Aarset, 1987) is useful. The TTT plot is obtained by plotting $T\left(\frac{r}{n}\right)$ against r/n where $T\left(\frac{r}{n}\right) =$

$\left[\sum_{i=1}^{n} y_{(i)} + (-r+n)\, y_{(r)}\right] / \sum_{i=1}^{n} y_{(i)} \;|_{(r=1,\ldots,n)}$ and y_i are the order statistics of the sample. It is convex shape for decreasing hrf and is concave shape for increasing hrf. The TTT plot for the kevlar data in Figure 4 deals with convex-concave-convex shaped. That is it has a firstly bathtub-shaped then decreasing shaped on the other words down-and-up shaped failure rate function. The MLEs

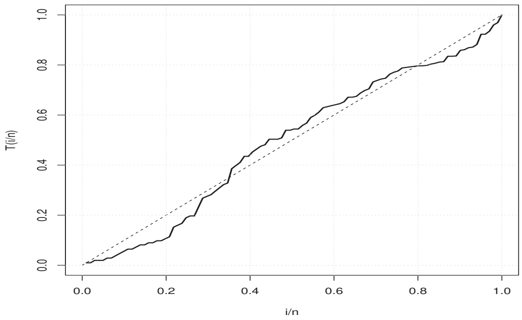


FIGURE 4. TTT plot for the kevlar data

of all models parameters, their standard erros, AIC, CAIC, W^* and A^* statistics are listed in Table 2 for data set. Table 2 shows that the QXGP model could be chosen as the best model among the fitted models under the AIC, CAIC, HQIC, and W^* statistics. We note that to show the likelihood equations have a unique solution, we plot the profiles of the **LL** of α, θ and λ in Figure 5. The WP model is better than QXGP model according to A^* statistics. In this case, the WP model can be choose as the best model.

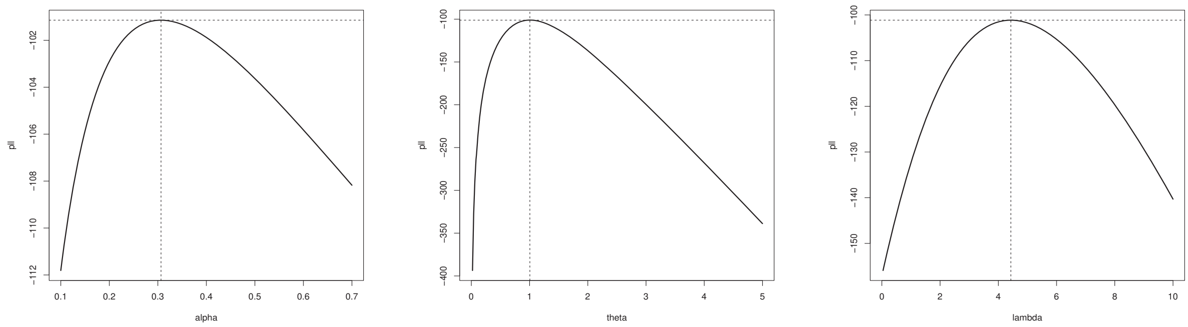


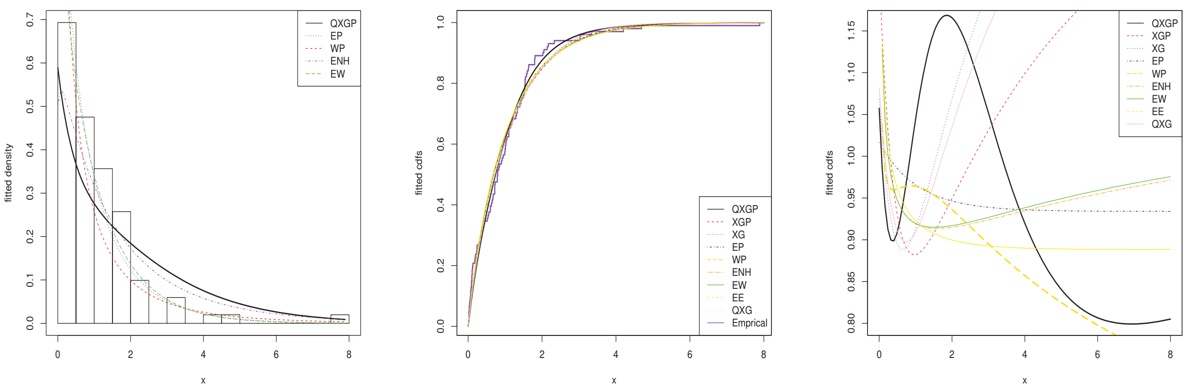
FIGURE 5. The profile of the **LL** function plots

The plots of the fitted densities, cdfs and hrfs of all models are displayed in Figure 6. These plots also shows that the QXGP model provides the good fit to these data compared to the other models. The fitted hrf shape both QXGP and WP models have firstly bathtub-shaped then decreasing shaped (convex-concave-convex).

A comparison of the proposed distribution with some of its sub-models using LR statistics is performed in Table 3. Table 3 shows that QXGP model provides a better representation of the data than the their sub-model based on the LR test at the 6% significance level. Hence, we reject all H_0 hypotheses in favour of the QXGP model.

TABLE 2. MLEs, standard erros of the estimates (in parentheses), $\hat{\ell}$, AIC, $CAIC$, A^* and W^* statistics for the applications models

Model	$\widehat{\alpha}$	$\widehat{\theta}$	$\widehat{\lambda}$	$-\hat{\ell}$	AIC	$CAIC$	A^*	W^*
QXGP	0.3065	1.0051	4.4307	101.1425	208.2849	208.5324	0.9527	0.1168
	(0.1099)	(0.2652)	(1.7112)					
QXG	1.9343	1.6408		104.0904	212.1807	212.1807	1.0947	0.1397
	(2.0215)	(0.4662)						
XGP		1.4839	0.6804	103.7876	211.5751	211.6976	1.0761	0.1673
		(0.3296)	(0.9887)					
XG		1.6978		104.1007	210.2015	210.2419	1.0916	0.1322
		(0.1248)						
EP	0.9340		0.1720	103.4497	210.8994	211.0218	1.2332	0.1742
	(0.1963)		(0.7079)					
WP	0.8059	1.4042	-1.2719	102.3688	210.7376	210.9851	0.9303	0.1514
	(0.1273)	(0.3703)	(1.0984)					
ENH	1.0717	0.7860	0.8473	102.7904	211.5808	211.82834	0.9633	0.1668
	(0.3093)	(0.4094)	(0.1308)					
EW	0.7929	0.8210	1.0604	102.7872	211.5743	211.8218	0.9554	0.1648
	(0.2873)	(0.2654)	(0.2400)					
EE	0.8660		0.8883	102.8200	209.6369	209.7624	1.0215	0.1812
	(0.1097)		(0.1201)					



(a) Fitted pdfs for data set (b) Fitted cdfs for data set (c) Fitted hrfs for data set

FIGURE 6. Fitted pdfs, cdfs and hrfs for data set

TABLE 3. LR statistics for data set

Model	Hypothesis	Test statistics	p-value
QXGP vs XGP	$H_0 : \alpha = \theta$ & $H_1 : H_0$ false	5.2902	0.0214
QXGP vs QXG	$H_0 : \lambda = 0$ & $H_1 : H_0$ false	5.8958	0.0151
QXGP vs XG	$H_0 : \alpha = \theta, \lambda = 0$ & $H_1 : H_0$ false	5.9164	0.0520

7. Conclusions

In this paper, we propose a new three-parameter xgamma-Poisson model, called the quasi xgamma-Poisson (QXGP) distribution, which extends the xgamma-Poisson (XGP), QXG and xgamma distributions . In fact, the QXGP model is motivated by the wide use of the xgamma distribution in many applied areas and also for the fact that the new generalization provides more flexibility to analyze real data. We discuss the MLE of the model parameters. An applications illustrate that the proposed model provides consistently better fit than the other competitive models like QXG, XGP, XG, EW, EE, EP, WP and ENH models.

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