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# Harmonic Mappings Related to the $\lambda-$ Spirallike Function With Bounded Radius Rotation

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**Abstract:** In the present paper, we will give some properties of the class of harmonic mappings related $\lambda-$ spirallike functions with bounded radius rotation.

## 1. Introduction

Let $\Omega$ be the family of functions $\phi(z)$ which are regular in the open unit disc $\mathbb{D} = \{z||z| < 1\}$ and satisfying the conditions $\phi(0) = 0, |\phi(z)| < 1$ for all $z \in \mathbb{D}$. Denote by $\mathscr{P}$ the family of functions $p(z) = 1 + p_1z + p_2z^2 + \ldots$ which are regular in $\mathbb{D}$. It is well-know that $p(z)$ in $P$ if and only if

$$p(z) = \frac{1+\phi(z)}{1-\phi(z)} \Leftrightarrow p(z) \prec \frac{1+z}{1-z} \tag{1}$$

for some $\phi(z) \in \Omega$ and every $z \in \mathbb{D}$.
Next, let $A$ be the class of functions $f$ in the open unit disc $\mathbb{D}$, that are normalized with $f(0) = 0, f'(0) = 1$. A function $f(z) \in A$ is called $\lambda-$ spirallike function, if there is a real number $\lambda (|\lambda| < \frac{\pi}{2})$, such that

$$Re\left(e^{i\lambda} z\frac{f'(z)}{f(z)}\right) > 0, z \in \mathbb{D} \tag{2}$$

The class of such functions is denoted by $S^*_\lambda$, and this class was introduced by Spacek [6].
Let $h(z), g(z) \in A$ then we say that $h(z)$ is subordinate to $g(z)$ and we write $h(z) \prec g(z)$, if there exists a function $\phi(z) \in \Omega$ such that $h(z) = g(\phi(z))$ for all $z \in \mathbb{D}$. Specially if $g(z)$ is univalent in $\mathbb{D}$, then $h(z) \prec g(z)$ if and only if $h(0) = g(0), h(\mathbb{D}) \subset g(\mathbb{D})$, implies $h(\mathbb{D}_r) \subset g(\mathbb{D}_r)$, where $\mathbb{D}_r = \{z||z| < r, 0 < r < 1\}$.(Subordination principle [2]).
Moreover, an analytic function $p(z) \in P(k), k \geq 4$ if and only if there exists $p_1(z), p_2(z) \in P$ such that

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z) \tag{3}$$

$$h(z) = \sum_{n=0}^{\infty} a_n z^n, g(z) = \sum_{n=0}^{\infty} b_n z^n, \tag{4}$$

where $a_n, b_n \in C, n = 1, 2, 3, \ldots$. As usual we call $h(z)$ is analytic part of $f$ and $g(z)$ is co-analytic part of $f$. An elegant and complete account of the theory of harmonic mappings are given Duren's monograph [1]. Lewy proved that in 1936 that the harmonic mapping $f$ is locally univalent in $\mathbb{D}$ if and only if its Jacobian $J_f = (|h'(z)|^2 - |g'(z)|^2)$ is different from zero in $\mathbb{D}$. In view of this result locally univalent harmonic mapping in the open unit disc are either sense-preserving if $|g'(z)| < |h'(z)|$ in $\mathbb{D}$ or sense-reversing if $|g'(z)| > |h'(z)|$ in $\mathbb{D}$. Throughout this paper, we will restricted ourselves to the study of sense-preserving harmonic mappings. We also note that $f = h(z) + \overline{g(z)}$ is sense-preserving in $\mathbb{D}$ if and only if $h'(z)$ does not vanish in $\mathbb{D}$ and the second dilatation $w(z) = \frac{g'(z)}{h'(z)}$ has the property $|w(z)| < 1$ for all $z \in \mathbb{D}$. Therefore the class of all sense-preserving harmonic mapping in the open unit disc

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$\mathbb{D}$ with $a_0=0, b_0=0, a_1=1$ will be denoted by $S_H$. Thus $S_H$ contain the standart class $S$ of univalent functions. The family of all mappings $f\in S_H$ with the additional property $g'(0)=0$, i.e., $b_1=0$ is denoted by $S_H^0$. Hence $S\subset S_H^0\subset S_H$[1].
In the present paper we will examine the class

$$\begin{aligned} S_H^*(\alpha,k) &= \Big\{ f=h(z)+\overline{g(z)} | h(z)\in S_\alpha^*(k) \\ &\Leftrightarrow e^{i\alpha} z\frac{h'(z)}{h(z)} = \cos\alpha p(z)+i\sin\alpha, p(z)\in P(k)\Big\}. \quad (1.5) \end{aligned}$$

## 2. Main Results

**Theorem 2.1.** *[1, 4] Let $h(z)$ be an element of $S_\alpha^*(k)$ then*

$$\frac{r}{(1-r)^{A_1}(1+r)^{B_1}} \le |h(z)| \le \frac{r}{(1-r)^{B_1}(1+r)^{A_1}} \quad (5)$$

$$\frac{\sqrt{1+2r^2\cos 2\alpha+r^4}-(k\cos\alpha)r}{(1-r)^{A_1}(1+r)^{B_1}} \le |h'(z)| \le \frac{\sqrt{1+2r^2\cos 2\alpha+r^4}+(k\cos\alpha)r}{(1-r)^{1+B_1}(1+r)^{1+A_1}} \quad (6)$$

*where*

$$A_1=\frac{1}{2}(1-k\cos\alpha+\cos 2\alpha), B_1=\frac{1}{2}(1+k\cos\alpha+\cos 2\alpha).$$

**Theorem 2.2.** *Let $f=h(z)+\overline{g(z)}$ be an element of $S_H^*(\alpha,k)$, then $f=h(z)+\overline{b_1 p(z)h(z)}$ is the solution of non-linear partial differential equation $w(z)=\frac{\overline{f_{\bar z}}}{f_z}$ under the condition $|w(z)|<1$, $w(z)=\frac{\overline{f_{\bar z}}}{f_z}\prec b_1p(z)$ and $p(z)\in\mathscr{P}_k$.*

*Proof.* Since $w(z)\prec b_1p(z)$, then the variability of $\left(\frac{\overline{f_{\bar z}}}{f_z}\right)$ is the closed disc. Using subordination principle

$$\left|\frac{\overline{f_{\bar z}}}{f_z}-\frac{b_1(1+r^2)}{1-r^2}\right| \le \frac{|b_1|kr}{1-r^2}. \quad (7)$$

Therefore we have

$$w(\mathbb{D}_r)=\left\{\frac{g'(z)}{h'(z)}\left|\left|\frac{g'(z)}{h'(z)}-\frac{b_1(1+r^2)}{1-r^2}\right|\le\frac{|b_1|kr}{1-r^2}, 0<r<1\right.\right\}. \quad (8)$$

Now we define the function $\phi(z)$ by the relation

$$\frac{g(z)}{h(z)}=b_1\left[\left(\frac{k}{4}+\frac{1}{2}\right)\frac{1+\phi(z)}{1-\phi(z)}-\left(\frac{k}{4}-\frac{1}{2}\right)\frac{1-\phi(z)}{1+\phi(z)}\right], \quad (9)$$

then $\phi(z)$ is analytic in $\mathbb{D}$ and $\phi(0)=0$. On the other hand, if we take derivative from (9) and after simple calculations, we get

$$\begin{aligned} \frac{g'(z)}{h'(z)} &= b_1\Bigg\{\left[\left(\frac{k}{4}+\frac{1}{2}\right)\frac{1+\phi(z)}{1-\phi(z)}-\left(\frac{k}{4}-\frac{1}{2}\right)\frac{1-\phi(z)}{1+\phi(z)}\right] \\ &+ \left[\left(\frac{k}{4}+\frac{1}{2}\right)\frac{2z\phi'(z)}{(1-\phi(z))^2}+\left(\frac{k}{4}-\frac{1}{2}\right)\frac{2z\phi'(z)}{(1+\phi(z))^2}\right]\frac{h(z)}{zh'(z)}\Bigg\}. \quad (2.6) \end{aligned}$$

One can easily conclude that the subordination

$$\frac{\overline{f_{\bar z}}}{f_z}\prec b_1p(z), \quad p(z)\in\mathscr{P}_k$$

is equivalent to $|\phi(z)|<1$ for all $z\in\mathbb{D}$. Since $h(z)\in S_\alpha^*(k)$ then the boundary value of $\left(z\frac{h'(z)}{h(z)}\right)$ is $\frac{1+(k\cos\alpha)r+e^{-2i\alpha}r^2}{1-r^2}$ and I.S. Jack Lemma says that "Let $\phi(z)$ be analytic in $\mathbb{D}$ with $\phi(0)=0$. If $|\phi(z)|$ attains its maximum value on the circle $|z|=r$ at a point $z$, then we have

$$z\phi'(z)=m\phi(z), \quad m\ge 1."$$

Considering Jack lemma, (10) and the boundary value of $\left(z\frac{h'(z)}{h(z)}\right)$ together, then we get

$$\begin{aligned} \frac{g'(z_0)}{h'(z_0)} &= b_1\Bigg\{\left[\left(\frac{k}{4}+\frac{1}{2}\right)\frac{1+\phi(z)}{1-\phi(z)}-\left(\frac{k}{4}-\frac{1}{2}\right)\frac{1-\phi(z)}{1+\phi(z)}\right] \\ &+ \left[\left(\frac{k}{4}+\frac{1}{2}\right)\frac{2z\phi'(z)}{(1-\phi(z))^2}+\left(\frac{k}{4}-\frac{1}{2}\right)\frac{2z\phi'(z)}{(1+\phi(z))^2}\right]\frac{1-r^2}{1+(k\cos\alpha)r+e^{-2i\alpha}r^2}\Bigg\}. \end{aligned}$$

this shows that $\frac{g'(z_0)}{h'(z_0)}\notin w(\mathbb{D})$ which contradicts with $\frac{\overline{f_{\bar z}}}{f_z}\prec b_1p(z)$, so $|\phi(z)|<1$ for all $z\in\mathbb{D}$. □

**Corollary 2.3.** *Let $f = (h(z) + G(z))$ be the solution of the non-linear partial differential equation $w(z) = \frac{\overline{f_{\bar{z}}}}{f_z}$ under the condition $|w(z)| < 1$, $w(z) = \frac{\overline{f_{\bar{z}}}}{f_z} \prec b_1 p(z)$ and $p(z) \in \mathscr{P}_k$ where*

$$G(z) = \overline{b_1 p(z) h(z)},$$

*then*

$$\frac{|b_1|\, r\left(1 - kr + r^2\right)}{(1-r)^{1+A_1}(1+r)^{1+B_1}} \le |G(z)| \le \frac{|b_1|\, r\left(1 + kr + r^2\right)}{(1-r)^{1+B_1}(1+r)^{1+A_1}}$$

*and*

$$\begin{aligned} & \frac{|b_1|\left(1 - kr + r^2\right)\left(\sqrt{1 + 2r^2\cos 2\alpha + r^4} - (k\cos\alpha r)\right)}{(1-r)^{2+A_1}(1+r)^{2+B_1}} \\ \le \quad & |G(z)| \\ \le \quad & \frac{|b_1|\left(1 + kr + r^2\right)\left(\sqrt{1 + 2r^2\cos 2\alpha + r^4} - (k\cos\alpha r)\right)}{(1-r)^{2+B_1}(1+r)^{2+A_1}} \end{aligned}$$

*Proof.* The proof of this corollary is a simple consequence of Theorem 2.1 and Theorem 2.2. □

**Corollary 2.4.** *Using the Theorem* 2.2 *and following formulas [1]*

$$\mathscr{J}_{f(z)} = |f(z)|^2 - \left|\overline{f_{\bar{z}}}\right|^2,$$

$$\left(\left|\overline{f_{\bar{z}}}\right| - |f(z)|\right)|dz| \le |df| \le \left(\left|\overline{f_{\bar{z}}}\right| + |f(z)|\right)|dz|$$

*we obtain the estimates of $\mathscr{J}_{f(z)}$*

$$\mathscr{J}_{f(z)} = \int_0^r \left(\left|\overline{f_{\bar{\rho}}}\right| - |f(\rho)|\right)|d\rho| \le |f|$$

*and $f = h(z) + \overline{b_1 p(z) h(z)}$.*

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