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HERMITE-HADAMARD TYPE INEQUALITIES FOR GENERALIZED (k,h) –FRACTIONAL INTEGRALS

ABDULLAH AKKURT, H. FEHMI GIDERGELMEZ, SEDA KILINÇ, AND HÜSEYIN YILDIRIM

Abstract. In this paper, the authors have obtained some new developments of Hermite-Hadamard type inequalities for generalized fractional integrals defined by Akkurt et al. [1]. With the help of this fractional integral definition, some new original results have been obtained, which include many studies in the literature.

1. Introduction

Integral inequalities play a fundamental role in the theory of differential equations, functional analysis and applied sciences. Important development in this theory has been achieved for the last two decades. For these, see [4], [5] and the references there in. Moreover, the study of fractional type inequalities is also of vital importance. Also see [1]-[12] for further information and applications.

Now we will give fundamental definitions and notations for fractional integrals.

Definition 1.1. Let $a,b\in\mathbb{R}$, $a<b$, and $\alpha>0$. For $f\in L_1(a,b)$

$$(J_{a^+}^{\alpha}f)(x)=\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}f(t)dt,\ \alpha>0,\ x>a \tag{1.1}$$

and

$$(J_{b^-}^{\alpha}f)(x)=\frac{1}{\Gamma(\alpha)}\int_x^b(t-x)^{\alpha-1}f(t)dt,\ b>0,\ b>x. \tag{1.2}$$

These integrals are called right-sided Riemann-Liouville fractional integral and left-sided Riemann-Liouville fractional integral respectively [4], [5].

This integrals is motivated by the well known Cauchy formula:

$$\int_a^x d\tau_1\int_a^{\tau_1}d\tau_2...\int_a^{\tau_{n-1}}f(\tau_n)d\tau_n=\frac{1}{\Gamma(n)}\int_a^x(x-\tau)^{n-1}f(\tau)d\tau. \tag{1.3}$$

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Definition 1.2. Let (a,b) be a finite interval of the real line $\mathbb{R}$ and $\Re(\alpha)>0$. Also let $h(x)$ be an increasing and positive monotone function on $(a,b]$, having a continuous derivative $h^{'}(x)$ on (a,b). The left- and right-sided fractional integrals of a function f with respect to another function h on $[a,b]$ are defined by [4]

$$\left(J^{\alpha}_{a^+,h}f\right)(x):=\frac{1}{\Gamma(\alpha)}\int_a^x\left[h(x)-h(t)\right]^{\alpha-1}h^{'}(t)\,f(t)\,dt,\ x\geq a, \tag{1.4}$$

and

$$\left(J^{\alpha}_{b^-,h}f\right)(x):=\frac{1}{\Gamma(\alpha)}\int_x^b\left[h(t)-h(x)\right]^{\alpha-1}h^{'}(t)\,f(t)\,dt,\ x\leq b. \tag{1.5}$$

For (1.4) and (1.5)

$$\left(J^{\alpha}_{a^+,h}f\right)(a)=\left(J^{\alpha}_{b^-,h}f\right)(b)=0.$$

If we take $h(x)=x$ in (1.4) and (1.5) integral formulas, we have

$$J^{\alpha}_{a^+,h}=J^{\alpha}_{a^+}\text{ and }J^{\alpha}_{b^-,h}=J^{\alpha}_{b^-}.$$

Also if we choose $h(x)=\dfrac{x^{\rho+1}}{\rho+1}$ for $\rho\geq 0$, then the equalities (1.4) and (1.5) will be

$$(J^{\alpha}_{a^+,\rho}f)(x)=\frac{(\rho+1)^{1-\alpha}}{\Gamma(\alpha)}\int_a^x(x^{\rho+1}-t^{\rho+1})^{\alpha-1}t^{\rho}f(t)dt,\ x>a \tag{1.6}$$

and

$$(J^{\alpha}_{b^-,\rho}f)(x)=\frac{(\rho+1)^{1-\alpha}}{\Gamma(\alpha)}\int_x^b(t^{\rho+1}-x^{\rho+1})^{\alpha-1}t^{\rho}f(t)dt,\ x<b \tag{1.7}$$

respectively [8].

In [8], Katugampola gave a new fractional integration which generalized Riemann-Liouville fractional integrals. This (1.6) and (1.7) generalizations is based on the following equality,

(1.8)

$$\int_a^x\tau_1^{\rho}d\tau_1\int_a^{\tau_1}\tau_2^{\rho}d\tau_2\ldots\int_a^{\tau_{n-1}}\tau_n^{\rho}f(\tau_n)d\tau_n=\frac{(\rho+1)^{1-n}}{(n-1)!}\int_a^x(x^{\rho+1}-\tau^{\rho+1})^{n-1}\tau^{\rho}f(\tau)d\tau.$$

Definition 1.3. Let $\alpha>0$ and $x>0$, defined by [7], [10]

$$\left({}_kJ^{\alpha}f\right)(x)=\frac{1}{k\Gamma_k(\alpha)}\int_0^x(x-t)^{\frac{\alpha}{k}-1}f(t)dt. \tag{1.9}$$

Where $k-$gamma function is defined by

$$\Gamma_k(x)=\int_0^{\infty}t^{\frac{x}{k}-1}e^{-\frac{t^k}{k}}dt,\ x>0.$$

and

$$B_k(x,y)=\frac{1}{k}\int_0^1t^{\frac{x}{k}-1}\left(1-t\right)^{\frac{y}{k}-1}dt.$$

Also

$$B_k(x,y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)} \text{ and } B_k(x,y) = \frac{1}{k}B_k(\frac{x}{k}, \frac{y}{k}).$$

Definition 1.4. Let (a,b) be a finite interval of the real line $\mathbb{R}$ and $\Re(\alpha) > 0$. Also let $h(x)$ be an increasing and positive monotone function on $(a,b]$, having a continuous derivative $h'(x)$ on (a,b). The left- and right-sided fractional integrals of a function f with respect to another function h on $[a,b]$ are defined by [1]

$$\left({}_kJ^{\alpha}_{a^+,h}f\right)(x) := \frac{1}{k\Gamma_k(\alpha)} \int_a^x [h(x) - h(t)]^{\frac{\alpha}{k}-1} h'(t) f(t)\, dt, \tag{1.10}$$

and

$$\left({}_kJ^{\alpha}_{b^-,h}f\right)(x) := \frac{1}{k\Gamma_k(\alpha)} \int_x^b [h(t) - h(x)]^{\frac{\alpha}{k}-1} h'(t) f(t)\, dt, \tag{1.11}$$

for $k > 0$, $\Re(\alpha) > 0$. If we take $h(x) = x$ in (1.10) and (1.11) integral formulas, we will obtain

$$({}_kJ^{\alpha}_{a^+}f)(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t)dt, \ x > a$$

$$({}_kJ^{\alpha}_{b^-}f)(x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^b (t-x)^{\frac{\alpha}{k}-1} f(t)\, dt, \ b > x.$$

Note that when $k \to 1$, then it reduces to the classical Riemann-Liouville fractional integral.

Also if we choose $h(x) = \frac{x^{\rho+1}}{\rho+1}$ for $\rho \in \mathbb{R}/\{-1\}$, then the equalities (1.10) and (1.11) will be

$$({}^{\rho}_{k}J^{\alpha}_{a^+}f)(x) = \frac{(\rho+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^x (x^{\rho+1} - t^{\rho+1})^{\frac{\alpha}{k}-1} t^{\rho} f(t)dt, \ x > a \tag{1.12}$$

and

$$({}^{\rho}_{k}J^{\alpha}_{b^-}f)(x) = \frac{(\rho+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_x^b (t^{k+1} - x^{k+1})^{\alpha-1} t^k f(t)dt, \ x < b. \tag{1.13}$$

This kind of generalized fractional integrals are studied in [11].

Now, let's define a general fractional integral which is quite useful. This definition includes many fractional integral definitions.

Definition 1.5. Let $h : [a^\rho, b^\rho] \to R$ be an increasing and positive monotone function, having a continuous derivative $h'(x)$ on (a^ρ, b^ρ). The left and right sided fractional of f with respect to the function h on $[a^\rho, b^\rho]$ of order $\alpha > 0$ is defined by

$$ {}^\rho I_{a^+}^\alpha f(b^\rho) = \frac{1}{k\Gamma_k(\alpha)} \int_a^b \frac{h'(t^\rho) f(t^\rho) t^{\rho-1}}{[h(b^\rho) - h(t^\rho)]^{1-\frac{\alpha}{k}}} dt \tag{1.14} $$

and

$$ {}^\rho I_{b^-}^\alpha f(a^\rho) = \frac{1}{k\Gamma_k(\alpha)} \int_a^b \frac{h'(t^\rho) f(t^\rho) t^{\rho-1}}{[h(t^\rho) - h(a^\rho)]^{1-\frac{\alpha}{k}}} dt. \tag{1.15} $$

If by using the change of varible $s^\rho = \dfrac{t^\rho - a^\rho}{b^\rho - a^\rho}$ and $s = \dfrac{t^\rho - b^\rho}{a^\rho - b^\rho}$ respectively, then we have the following definition:

$$ I_{a^+}^\alpha f(b^\rho) = \frac{b^\rho - a^\rho}{k\Gamma_k(\alpha)} \int_0^1 \frac{h'((1-s^\rho)a^\rho + b^\rho s^\rho) f((1-s^\rho)a^\rho + b^\rho s^\rho)}{[h(b^\rho) - h((1-s^\rho)a^\rho + b^\rho s^\rho)]^{1-\frac{\alpha}{k}}} s^{\rho-1} ds, \tag{1.16} $$

and

$$ I_{b^-}^\alpha f(a^\rho) = \frac{b^\rho - a^\rho}{k\Gamma_k(\alpha)} \int_0^1 \frac{h'(s^\rho a^\rho + (1-s^\rho)b^\rho) f((s^\rho a^\rho + (1-s^\rho)b^\rho)}{[h(s^\rho a^\rho + (1-s^\rho)b^\rho) - h(a^\rho)]^{1-\frac{\alpha}{k}}} s^{\rho-1} ds. \tag{1.17} $$

In this work, the authors obtained some new Hermite-Hadamard type inequalities for generalized fractional integrals.

The main aim of this work is to establish a new fractional integral inequality for (k, h) −Riemann-Liouville fractional integral. Using the technique of [6] a key role in our study.

2. Main Results

Theorem 2.1. *Let $h : [a^\rho, b^\rho] \to R$ be an increasing and positive monotone function, having a continuous derivative $h'(x)$ on (a^ρ, b^ρ). Let $f : I \to \mathbb{R}$ be a convex function, then the following inequality holds;*

$$f\left(\frac{a^\rho + b^\rho}{2}\right) \leq \frac{\rho\Gamma_k(\alpha+1)}{4[h(b^\rho) - h(a^\rho)]^{\frac{\alpha}{k}}}[I^\alpha_{a^+}F(b^\rho) + I^\alpha_{b^-}F(a^\rho)] \leq \frac{f(a^\rho) + f(b^\rho)}{2}.$$

Proof. Since f is a strongly convex function on $[a^\rho, b^\rho]$ for $t = \frac{1}{2}$

$$f\left(\frac{x^\rho + y^\rho}{2}\right) \leq \frac{f(x^\rho) + f(y^\rho)}{2} \tag{2.1}$$

obtained.

For $s^\rho \in [0,1]$, $x^\rho = s^\rho a^\rho + (1 - s^\rho)b^\rho$ and $y^\rho = (1 - s^\rho)a^\rho + b^\rho s^\rho$, Then since f is convex, therefore

$$f\left(\frac{a^\rho + b^\rho}{2}\right) \leq \frac{f(s^\rho a^\rho + (1 - s^\rho)b^\rho) + f((1 - s^\rho)a^\rho + b^\rho s^\rho)}{2} \tag{2.2}$$

Since $F(x^\rho) = f(x^\rho) + f(a^\rho + b^\rho - x^\rho)$, then substituting $F((1 - s^\rho)a^\rho + b^\rho s^\rho) = f((1 - s^\rho)a^\rho + b^\rho s^\rho) + f(s^\rho a^\rho + (1 - s^\rho)b^\rho)$, we have

$$f\left(\frac{a^\rho + b^\rho}{2}\right) \leq \frac{F((1 - s^\rho)a^\rho + b^\rho s^\rho)}{2}. \tag{2.3}$$

Also, $F(s^\rho a^\rho + (1 - s^\rho)b^\rho) = f(s^\rho a^\rho + (1 - s^\rho)b^\rho) + f((1 - s^\rho)a^\rho + b^\rho s^\rho)$, we have

$$f\left(\frac{a^\rho + b^\rho}{2}\right) \leq \frac{F(s^\rho a^\rho + (1 - s^\rho)b^\rho)}{2}. \tag{2.4}$$

Multiplying both sides of inequality 2.3 with $\frac{b^\rho - a^\rho}{k\Gamma_k(\alpha)} \frac{h'((1 - s^\rho)a^\rho + b^\rho s^\rho)s^{\rho-1}}{[h(b^\rho) - h\left((1 - s^\rho)a^\rho + b^\rho s^\rho\right)]^{1-\frac{\alpha}{k}}}$ and integrating with respect to s from 0 to 1, we obtain

$$\begin{aligned} & f\left(\frac{a^\rho + b^\rho}{2}\right) \frac{b^\rho - a^\rho}{k\Gamma_k(\alpha)} \int_0^1 \frac{h'((1 - s^\rho)a^\rho + b^\rho s^\rho)s^{\rho-1}}{[h(b^\rho) - h\left((1 - s^\rho)a^\rho + b^\rho s^\rho\right)]^{1-\frac{\alpha}{k}}} ds \\ \leq \; & \frac{b^\rho - a^\rho}{2k\Gamma_k(\alpha)} \left[\int_0^1 \frac{h'((1 - s^\rho)a^\rho + b^\rho s^\rho)F((1 - s^\rho)a^\rho + b^\rho s^\rho)}{[h(b^\rho) - h\left((1 - s^\rho)a^\rho + b^\rho s^\rho\right)]^{1-\frac{\alpha}{k}}} s^{\rho-1} ds\right]. \end{aligned} \tag{2.5}$$

Substituting $u = h(b^\rho) - h((1 - s^\rho)a^\rho + b^\rho s^\rho)$, in the left hand side of (2.5), we get

$$\begin{aligned} & f\left(\frac{a^\rho + b^\rho}{2}\right) \frac{b^\rho - a^\rho}{k\Gamma_k(\alpha)} \int_0^1 \frac{h'((1 - s^\rho)a^\rho + b^\rho s^\rho)s^{\rho-1}}{[h(b^\rho) - h\left((1 - s^\rho)a^\rho + b^\rho s^\rho\right)]^{1-\frac{\alpha}{k}}} ds \\ = \; & f\left(\frac{a^\rho + b^\rho}{2}\right) \frac{[h(b^\rho) - h(a^\rho)]^{\frac{\alpha}{k}}}{\rho\Gamma_k(\alpha+1)}. \end{aligned}$$

Hence,

$$f\left(\frac{a^\rho+b^\rho}{2}\right)\frac{\left[h(b^\rho)-h(a^\rho)\right]^{\frac{\alpha}{k}}}{\rho\Gamma_k(\alpha+1)}$$

$$\leq \frac{b^\rho-a^\rho}{2k\Gamma_k(\alpha)}\int_0^1 \frac{h'((1-s^\rho)a^\rho+b^\rho s^\rho)F((1-s^\rho)a^\rho+b^\rho s^\rho)s^{\rho-1}}{\left[h(b^\rho)-h\left((1-s^\rho)a^\rho+b^\rho s^\rho\right)\right]^{1-\frac{\alpha}{k}}}ds.$$

Finally, we get

$$f\left(\frac{a^\rho+b^\rho}{2}\right)\frac{\left[h(b^\rho)-h(a^\rho)\right]^{\frac{\alpha}{k}}}{\rho\Gamma_k(\alpha+1)}\leq\frac{1}{2}I^{\alpha}_{a^+}F(b^\rho). \tag{2.6}$$

Multiplying both sides of above inequality 2.4 with $\frac{b^\rho-a^\rho}{k\Gamma_k(\alpha)}\frac{h'(s^\rho a^\rho+(1-s^\rho)b^\rho)s^{\rho-1}}{\left[h(s^\rho a^\rho+(1-s^\rho)b^\rho)-h(a^\rho)\right]^{1-\frac{\alpha}{k}}}$ and integrating with respect to s from 0 to 1, we obtain

$$\begin{aligned}&\frac{b^\rho-a^\rho}{k\Gamma_k(\alpha)}f\left(\frac{a^\rho+b^\rho}{2}\right)\int_0^1\frac{h'(s^\rho a^\rho+(1-s^\rho)b^\rho)s^{\rho-1}}{\left[h(s^\rho a^\rho+(1-s^\rho)b^\rho)-h(a^\rho)\right]^{1-\frac{\alpha}{k}}}ds\\&\leq\frac{b^\rho-a^\rho}{2\Gamma_k(\alpha)}\int_0^1\left[\frac{h'(s^\rho a^\rho+(1-s^\rho)b^\rho)s^{\rho-1}}{\left[h(s^\rho a^\rho+(1-s^\rho)b^\rho)-h(a^\rho)\right]^{1-\frac{\alpha}{k}}}F(s^\rho a^\rho+(1-s^\rho)b^\rho)ds\right]\end{aligned} \tag{2.7}$$

Substituting $u=h(s^\rho a^\rho+(1-s^\rho)b^\rho)-h(a^\rho)$, in the left hand side of (2.7) we get

$$\frac{b^\rho-a^\rho}{k\Gamma_k(\alpha)}f\left(\frac{a^\rho+b^\rho}{2}\right)\int_0^1\frac{h'(s^\rho a^\rho+(1-s^\rho)b^\rho)s^{\rho-1}}{\left[h(s^\rho a^\rho+(1-s^\rho)b^\rho)-h(a^\rho)\right]^{1-\frac{\alpha}{k}}}ds$$

$$=f\left(\frac{a^\rho+b^\rho}{2}\right)\frac{\left[h(b^\rho)-h(a^\rho)\right]^{\frac{\alpha}{k}}}{\rho\Gamma_k(\alpha+1)}.$$

Hence,

$$f\left(\frac{a^\rho+b^\rho}{2}\right)\frac{\left[h(b^\rho)-h(a^\rho)\right]^{\frac{\alpha}{k}}}{\rho\Gamma_k(\alpha+1)}$$

$$\leq\frac{b^\rho-a^\rho}{2\Gamma_k(\alpha)}\int_0^1\frac{h'(s^\rho a^\rho+(1-s^\rho)b^\rho)F(s^\rho a^\rho+(1-s^\rho)b^\rho)s^{\rho-1}}{\left[h(s^\rho a^\rho+(1-s^\rho)b^\rho)-h(a^\rho)\right]^{1-\frac{\alpha}{k}}}ds.$$

Finally, we get

$$f\left(\frac{a^\rho+b^\rho}{2}\right)\frac{\left[h(b^\rho)-h(a^\rho)\right]^{\frac{\alpha}{k}}}{\rho\Gamma_k(\alpha+1)}\leq\frac{1}{2}I^{\alpha}_{a^+}F(b^\rho).$$

$$f\left(\frac{a^\rho+b^\rho}{2}\right)\frac{\left[h(b^\rho)-h(a^\rho)\right]^{\frac{\alpha}{k}}}{\rho\Gamma_k(\alpha+1)}\leq\frac{1}{2}I^{\alpha}_{b^-}F(a^\rho). \tag{2.8}$$

Adding 2.6 and 2.8, we have

$$2f\left(\frac{a^\rho+b^\rho}{2}\right)\frac{\left[h(b^\rho)-h(a^\rho)\right]^{\frac{\alpha}{k}}}{\rho\Gamma_k(\alpha+1)}\leq\frac{1}{2}\left[I^{\alpha}_{a^+}F(b^\rho)+I^{\alpha}_{b^-}F(a^\rho)\right]. \tag{2.9}$$

Now, multiplying both sides of above inequality 2.9 with $\frac{\rho\Gamma_k(\alpha+1)}{2[h(b^\rho)-h(a^\rho)]^{\frac{\alpha}{k}}}$, we have

$$f\left(\frac{a^\rho+b^\rho}{2}\right) \leq \frac{\rho\Gamma_k(\alpha+1)}{4[h(b^\rho)-h(a^\rho)]^{\frac{\alpha}{k}}}[I^\alpha_{a^+}F(b^\rho)+I^\alpha_{b^-}F(a^\rho)], \tag{2.10}$$

and the first inequality is proved.

For the proof of the second inequality in (2.1) we first note that if f is a convex function on $[a^\rho, b^\rho]$, then we have

$$f(s^\rho a^\rho+(1-s^\rho)b^\rho) \quad \leq s^\rho f(a^\rho)+(1-s^\rho)f(b^\rho)$$

and

$$f((1-s^\rho)a^\rho+s^\rho b^\rho) \quad \leq (1-s^\rho)f(a^\rho)+s^\rho f(b^\rho)$$

adding this inequalities we have

$$f(a^\rho s^\rho+(1-s^\rho)b^\rho)+f((1-s^\rho)a^\rho+s^\rho b^\rho) \leq f(a^\rho)+f(b^\rho)$$

Here, Since $F(a^\rho s^\rho+(1-s^\rho)b^\rho) = f(a^\rho s^\rho+(1-s^\rho)b^\rho)+f((1-s^\rho)a^\rho+s^\rho b^\rho)$, then we have

$$F(a^\rho s^\rho+(1-s^\rho)b^\rho) \leq f(a^\rho)+f(b^\rho) \tag{2.11}$$

And, Since $F((1-s^\rho)a^\rho+s^\rho b^\rho) = f((1-s^\rho)a^\rho+s^\rho b^\rho)+f(a^\rho s^\rho+(1-s^\rho)b^\rho)$, then we have

$$F((1-s^\rho)a^\rho+s^\rho b^\rho) \leq f(a^\rho)+f(b^\rho). \tag{2.12}$$

Multiplying both sides of the inequality (2.12) with $\frac{b^\rho-a^\rho}{k\Gamma_k(\alpha)}\frac{h'((1-s^\rho)a^\rho+s^\rho b^\rho)s^{\rho-1}}{[h(b^\rho)-h((1-s^\rho)a^\rho+b^\rho s^\rho)]^{1-\frac{\alpha}{k}}}$ and integrating with respect to s from 0 to 1, we obtain

$$\begin{aligned} &\frac{b^\rho-a^\rho}{k\Gamma_k(\alpha)}\int_0^1 \frac{h'((1-s^\rho)a^\rho+s^\rho b^\rho)F((1-s^\rho)a^\rho+s^\rho b^\rho)}{[h(b^\rho)-h((1-s^\rho)a^\rho+b^\rho s^\rho)]^{1-\frac{\alpha}{k}}}s^{\rho-1}ds \\ &\leq [f(a^\rho)+f(b^\rho)]\frac{b^\rho-a^\rho}{k\Gamma_k(\alpha)}\int_0^1 \frac{h'((1-s^\rho)a^\rho+s^\rho b^\rho)s^{\rho-1}}{[h(b^\rho)-h((1-s^\rho)a^\rho+b^\rho s^\rho)]^{1-\frac{\alpha}{k}}}ds. \end{aligned} \tag{2.13}$$

Substituting $u = h(b^\rho)-h((1-s^\rho)a^\rho+b^\rho s^\rho)$, the right hand side of (2.13) we get

$$\begin{aligned} &[f(a^\rho)+f(b^\rho)]\frac{b^\rho-a^\rho}{k\Gamma_k(\alpha)}\int_0^1 \frac{h'((1-s^\rho)a^\rho+s^\rho b^\rho)s^{\rho-1}}{[h(b^\rho)-h((1-s^\rho)a^\rho+b^\rho s^\rho)]^{1-\frac{\alpha}{k}}}ds \\ &= [f(a^\rho)+f(b^\rho)]\frac{[h(b^\rho)-h(a^\rho)]^{\frac{\alpha}{k}}}{\rho\Gamma_k(\alpha+1)}. \end{aligned}$$

Hence

$$\begin{aligned} &\frac{b^\rho-a^\rho}{k\Gamma_k(\alpha)}\int_0^1 \frac{h'((1-s^\rho)a^\rho+s^\rho b^\rho)F((1-s^\rho)a^\rho+s^\rho b^\rho)}{[h(b^\rho)-h((1-s^\rho)a^\rho+b^\rho s^\rho)]^{1-\frac{\alpha}{k}}}s^{\rho-1}ds \\ &\leq [f(a^\rho)+f(b^\rho)]\frac{[h(b^\rho)-h(a^\rho)]^{\frac{\alpha}{k}}}{\rho\Gamma_k(\alpha+1)}. \end{aligned}$$

Finally, we get

$$I^\alpha_{a^+}F(b^\rho) \leq [f(a^\rho)+f(b^\rho)]\frac{[h(b^\rho)-h(a^\rho)]^\alpha}{\Gamma_k(\alpha+1)}. \tag{2.14}$$

Multiplying both sides of above inequality 2.11 with $\frac{b^\rho - a^\rho}{k\Gamma_k(\alpha)} \frac{h'(a^\rho s^\rho + (1-s^\rho)b^\rho)s^{\rho-1}}{[h(a^\rho s^\rho + (1-s^\rho)b^\rho) - h((a^\rho)]^{1-\frac{\alpha}{k}}}$ and integrating with respect to s from 0 to 1, we obtain

$$\frac{b^\rho - a^\rho}{k\Gamma_k(\alpha)} \int_0^1 \frac{h'(a^\rho s^\rho + (1-s^\rho)b^\rho)F(a^\rho s^\rho + (1-s^\rho)b^\rho)s^{\rho-1}}{[h(a^\rho s^\rho + (1-s^\rho)b^\rho) - h(a^\rho)]^{1-\frac{\alpha}{k}}} ds$$

$$\leq [f(a^\rho) + f(b^\rho)] \frac{b^\rho - a^\rho}{k\Gamma_k(\alpha)} \int_0^1 \frac{h'(a^\rho s^\rho + (1-s^\rho)b^\rho)s^{\rho-1}}{[h(a^\rho s^\rho + (1-s^\rho)b^\rho) - h(a^\rho)]^{1-\frac{\alpha}{k}}} ds.$$

Similarly, we have

$$I_{b^-}^\alpha F(a^\rho) \leq [f(a^\rho) + f(b^\rho)] \frac{[h(b^\rho) - h(a^\rho)]^{\frac{\alpha}{k}}}{\rho\Gamma_k(\alpha+1)}. \tag{2.15}$$

Adding 2.14 and 2.15, we have

$$[I_{a^+}^\alpha F(b^\rho) + I_{b^-}^\alpha F(a^\rho)] \leq 2[f(a^\rho) + f(b^\rho)] \frac{[h(b^\rho) - h(a^\rho)]^{\frac{\alpha}{k}}}{\rho\Gamma_k(\alpha+1)} \tag{2.16}$$

Now, multiplying both sides of above inequality 2.16 with $\frac{\rho\Gamma_k(\alpha+1)}{4[h(b^\rho) - h(a^\rho)]^{\frac{\alpha}{k}}}$, we have

$$\frac{\rho\Gamma_k(\alpha+1)}{4[h(b^\rho) - h(a^\rho)]^{\frac{\alpha}{k}}} [I_{a^+}^\alpha F(b^\rho) + I_{b^-}^\alpha F(a^\rho)] \leq \frac{f(a^\rho) + f(b^\rho)}{2} \tag{2.17}$$

and the second inequality is proved.

If combined 2.9 and 2.17

$$f\left(\frac{a^\rho + b^\rho}{2}\right) \leq \frac{\rho\Gamma_k(\alpha+1)}{4[h(b^\rho) - h(a^\rho)]^{\frac{\alpha}{k}}} [I_{a^+}^\alpha F(b^\rho) + I_{b^-}^\alpha F(a^\rho)] \leq \frac{f(a^\rho) + f(b^\rho)}{2}$$

The proof is completed. □

Corollary 2.2. *If we take $\rho \to 1$ in Theorem 2.1, we have*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma_k(\alpha+1)}{4[h(b) - h(a)]^{\frac{\alpha}{k}}} [I_{a^+}^\alpha F(b) + I_{b^-}^\alpha F(a)] \leq \frac{f(a) + f(b)}{2}.$$

Corollary 2.3. *If we take $h(x) = x$ in Theorem 2.1, we have*

$$f\left(\frac{a^\rho + b^\rho}{2}\right) \leq \frac{\rho\Gamma_k(\alpha+1)}{4\left(b^\rho - a^\rho\right)^{\frac{\alpha}{k}}} [I_{a^+}^\alpha F(b^\rho) + I_{b^-}^\alpha F(a^\rho)] \leq \frac{f(a^\rho) + f(b^\rho)}{2}.$$

Corollary 2.4. *If we take $\rho \to 1$ and $h(x) = x$ in Theorem 2.1, we have*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma_k(\alpha+1)}{4\left(b - a\right)^{\frac{\alpha}{k}}} [I_{a^+}^\alpha F(b) + I_{b^-}^\alpha F(a)] \leq \frac{f(a) + f(b)}{2}.$$

Corollary 2.5. *If we take $\alpha = \rho = k = 1$ and $h(x) = x$ in Theorem 2.1, then we have the classical Hermite-Hadamard inequality,*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)\, dx \leq \frac{f(a) + f(b)}{2}.$$

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Department of Mathematics, Kahramanmaraş Sütcü İmam University, Kahramanmaraş,Turkey
E-mail address: abdullahmat@gmail.com, hfgidergelmez@hotmail.com, sedaaa_kilinc@hotmail.com, hyildir@ksu.edu.tr