IKONION JOURNAL OF MATHEMATICS

YEAR: 2019 VOLUME: 1 ISSUE: 1

SOME BOUNDS FOR ECCENTRIC VERSION OF HARMONIC INDEX OF GRAPHS

Yaşar Nacaroğlu*

* Kahramanmaras Sutcu Imam University, Department of Mathematics, 46100, Kahramanmaras, Turkey, E-mail: yasarnacaroglu@ksu.edu.tr

(Received: 19.12.2018, Accepted: 21.01.2019,Published Online: 30.01.2019)

Abstract

The harmonic index of a graph G is defined as the sum $H(G)=\sum_{ij\in E(G)}\frac{2}{d_G(i)+d_G(j)}$, where $d_G(i)$ is the degree of a vertex i in G. In this paper, we examined eccentric version of harmonic index of graphs.

Keywords: Topological index; Graph Parameters; Harmonic Index.

1. Introduction

Let *G* be a simple connected graph with vertex set *V(G)* and edge set *E(G)*. The degree of a vertex *u* in a graph *G* is number of incident edges to the vertex. The degree of a vertex *i* is denoted by $d_G(i)$. The maximum degree is denoted by Δ. The minimum degree is denoted by δ.

The distance between i and j vertices, denoted $d_G(i,j)$ is the length of a shortest path between them. The eccentricity $\varepsilon_G(i)$ of a vertex *i* in a connected graph is its distance to a vertex fatrhest from *i*. The radius of a connected graph, denoted *r(G)* is its minimum eccentricity. The diameter of a connected graph, denoted *D(G)* is maximum eccentricity. For other undefined notations and terminology from graph theory, the readers are referred to [5].

One of the oldest topological indices, the first and second Zagreb indices were defined by [7,8]. The first and second Zagreb indices are defined as

$$M_1(G)=\sum_{i\in V(G)} d_G^2(i) \quad \text{and} \quad M_2(G)=\sum_{ij\in E(G)} d_G(i)d_G(j).$$

An alternative expression for the first Zagreb index is [1]

$$M_1(G)=\sum_{ij\in E(G)} (d_G(i)+d_G(j)).$$

The harmonic index was defined in [3] as

$$H(G)=\sum_{ij\in E(G)} \frac{2}{d_G(i)+d_G(j)}.$$

Ghorbani et al. [4] and Vukičević et al. [12] defined the first and the second Zagreb eccentricity indices by

$$E_1(G) = \sum_{i \in V(G)} \varepsilon_G^2(i) \quad \text{and} \quad E_2(G) = \sum_{ij \in E(G)} \varepsilon_G(i)\varepsilon_G(j).$$

In 1997, The eccentricity connectivity index of a graph *G* was introduced by Sharma et al. [11]. The eccentric connectivity index is defined as

$$\xi^c(G) = \sum_{i \in V(G)} d_G(i)\varepsilon_G(i) = \sum_{ij \in E(G)} (\varepsilon_G(i)+\varepsilon_G(j))\,.$$

In 2000, Gupta et al. [6] introduced the connective eccentricity index, which is defined to be

$$\xi^{ce}(G) = \sum_{i \in V(G)} \frac{d_G(i)}{\varepsilon_G(i)}.$$

The eccentric version of the harmonic index have been defined in [2] as follows.

$$H_4(G) = \sum_{ij \in E(G)} \frac{2}{\varepsilon_G(i) + \varepsilon_G(j)}.$$

In this paper, we are concerned with the upper and lower bounds of $H_4(G)$ which depend on some of the parameters *n, m, r, D* etc.

2. Main Results

In this section, we give some upper and lower bounds for the eccentric harmonic index.

Theorem 2.1. Let *G* be a simple connected graph with *n* vertices, *m* edges, *r* radius and *D* diameter. Then

$$\frac{m}{D} \leq H_4(G) \leq \frac{m}{r}. \tag{1}$$

Equality holds on both sides if and only if *G* is self centered graph.

Proof. We know that $2r \leq \varepsilon_{(G)}(i) + \varepsilon_G(j) \leq 2D$ for all $ij \in E(G)$. Then we have

$$H_4(G) = \sum_{ij \in E(G)} \frac{2}{\varepsilon_G(i) + \varepsilon_G(j)}$$

$$\leq \sum_{ij \in E(G)} \frac{2}{2r} = \frac{m}{r}.$$

In an analogous manner,

$$H_4(G) = \sum_{ij \in E(G)} \frac{2}{\varepsilon_G(i) + \varepsilon_G(j)}$$

$$\geq \sum_{ij \in E(G)} \frac{2}{2D} = \frac{m}{D}.$$

Now suppose that equality holds in (1). Then all the above inequalities must become equalities. Thus we get $\varepsilon_{(G)}(i) = \varepsilon_G(j)$ for all of $ij \in E(G)$. So we conclude that *G* is self centered graph.

Conversely, if G is self centered graph, it is easy to see that equalities (1) hold.

Proposition 2.2. [13] Let G be a connected graph with $n \geq 3$ vertices. Then for all $i \in V(G)$ we have

$$\varepsilon_G(i) \leq n - d_G(i), \tag{2}$$

with equality if and only if $K_n - ke$, for $k = 0,1,2, \dots, \left\lfloor \frac{n}{2} \right\rfloor$, or $G = P_4$.

Theorem 2.3. Let G be connected graph of order n with maximum degree Δ. Then

$$H_4(G) \geq \frac{m}{n - \Delta}. \tag{3}$$

The equality holds if and only if G is regular self centered graph.

Proof. By applying Proposition 2.2, we get

$$\begin{aligned} H_4(G) &= \sum_{ij \in E(G)} \frac{2}{\varepsilon_G(i) + \varepsilon_G(j)} \\ &\geq \sum_{ij \in E(G)} \frac{2}{2n - (d_G(i) + d_G(j))} \\ &\geq \sum_{ij \in E(G)} \frac{2}{2n - 2\Delta} = \frac{m}{n - \Delta}. \end{aligned}$$

Suppose that equality holds in the above inequality. Then $\varepsilon_G(i) = n - d_G(i)$ ve $d_G(i) = \Delta$ for all $i \in V(G)$. So by Proposition 2.2 we conclude that $G \cong K_n$ or $G \cong C_4$.

Conversely, if $G \cong K_n$ or $G \cong C_4$, it is easy see that equality (3) holds.

Theorem 2.4. Let G be a connected graph with n vertices and m edges. Let k be the number of vertices with eccentricity 1 in graph G. Then

$$H_4(G) = \frac{6m + k(2n + k - 3)}{12}.$$

Proof. $K = \{i_1, i_2, \dots, i_k\}$ be the set of vertices with eccentricity 1. Then we have $e(i) = 2$ for any $i \in V(G) \setminus K$. From the definition eccentric-harmonic index, we get

$$\begin{aligned} H_4(G) = & \sum_{\substack{ij \in E(G) \\ i,j \in K}} \frac{2}{\varepsilon_G(i) + \varepsilon_G(j)} + \sum_{\substack{ij \in E(G) \\ i \in K, j \in V(G) \setminus K}} \frac{2}{\varepsilon_G(i) + \varepsilon_G(j)} + \sum_{\substack{ij \in E(G) \\ i,j \in V(G) \setminus K}} \frac{2}{\varepsilon_G(i) + \varepsilon_G(j)} \\ &= \sum_{\substack{ij \in E(G) \\ i,j \in K}} 1 + \sum_{\substack{ij \in E(G) \\ i \in K, j \in V(G) \setminus K}} \frac{2}{3} + \sum_{\substack{ij \in E(G) \\ i,j \in V(G) \setminus K}} \frac{1}{2} \\ &= \frac{6m + k(2n + k - 3)}{12}. \end{aligned}$$

So as desired.

Lemma 2.5. (*Radon Inequality*)[10] For every real numbers $p > 0, x_k \geq 0, a_k > 0$, for $1 \leq k \leq n$, the following inequality holds true:

$$\sum_{k=1}^{n} \frac{x_k^{p+1}}{a_k^p} \geq \frac{(\sum_{k=1}^{n} x_k)^{p+1}}{(\sum_{k=1}^{n} a_k)^p}.$$

The equality holds if and only if $\frac{x_1}{a_1} = \frac{x_2}{a_2} = \cdots = \frac{x_n}{a_n}$.

Theorem 2.6. For any graph *G* we have

$$H_4(G) \geq \frac{2m^2}{\xi^c(G)}, \tag{4}$$

with equality holds if and only if $\varepsilon_G(i) + \varepsilon_G(j)$ is constant for all $ij \in E(G)$.

Proof. Using Lemma 2.5 we get

$$\begin{aligned} H_4(G) &= \sum_{ij \in E(G)} \frac{\left(\sqrt{2}\right)^2}{\varepsilon_G(i) + \varepsilon_G(j)} \\ &\geq \sum_{ij \in E(G)} \frac{\left(\sum_{ij \in E(G)} \sqrt{2}\right)^2}{\sum_{ij \in E(G)} (\varepsilon_G(i) + \varepsilon_G(j))} \\ &\geq \frac{2m^2}{\xi^c(G)} \quad . \end{aligned}$$

Suppose that equality holds in the above inequality. In this case by Lemma 2.5, $\varepsilon_G(i) + \varepsilon_G(j)$ becomes constant for all $ij \in E(G)$.

Conversely, if $\varepsilon_G(i) + \varepsilon_G(j)$ is constant for all $ij \in E(G)$, we can easily see that equality hold in (4).

Theorem 2.7. For any graph *G* we have

$$H_4(G) \leq \frac{\xi^{ce}(G)}{2}, \tag{5}$$

with equality holds if and only if *G* is self centered graph.

Proof. From arithmetic harmonic mean inequality we have

$$\begin{aligned} H_4(G) &= \sum_{ij \in E(G)} \frac{2}{\varepsilon_G(i) + \varepsilon_G(j)} \\ &\leq \frac{1}{2} \sum_{ij \in E(G)} \left(\frac{1}{\varepsilon_G(i)} + \frac{1}{\varepsilon_G(j)} \right) \\ &= \frac{1}{2} \sum_{i \in V(G)} \frac{d_G(i)}{\varepsilon_G(i)} = \frac{\xi^{ce}(G)}{2}. \end{aligned}$$

Suppose that equality holds in the above inequality. Then for every $ij \in E(G)$, $\varepsilon_G(i) = \varepsilon_G(j)$. Thus one can easily see that the equality holds in (5) if and only if *G* is self centered graph.

Conversely let G be self centered graph. Then by applying $\varepsilon_G(i) = \varepsilon_G(j) = r$ for all $ij \in E(G)$ we get

$$H_4(G) = \sum_{ij \in E(G)} \frac{2}{\varepsilon_G(i) + \varepsilon_G(j)} = \frac{m}{r}$$

and

$$\frac{\xi^{ce}(G)}{2} = \frac{1}{2} \sum_{i \in V(G)} \frac{d_G(i)}{\varepsilon_G(i)} = \frac{1}{2} \sum_{i \in V(G)} \frac{d_G(i)}{r} = \frac{m}{r}.$$

This completes the theorem.

Theorem 2.8. For any graph G we have

$$H_4(G) \geq \frac{2m^2 r}{E_2(G) + mr^2}, \tag{6}$$

with equality holds if and only if $\varepsilon_G(i) + \varepsilon_G(j)$ is constant for all $ij \in E(G)$.

Proof. Since $\varepsilon_G(i), \varepsilon_G(j) \geq r$, we have $(\varepsilon_G(i) - r)(\varepsilon_G(j) - r) \geq 0$. Then we get

$$\frac{\varepsilon_G(i)\varepsilon_G(j) + r^2}{r} \geq \varepsilon_G(i) + \varepsilon_G(j) .$$

The equality holds $\varepsilon_G(i) = r$ or $\varepsilon_G(j) = r$ or $\varepsilon_G(i) = \varepsilon_G(j) = r$ for all $ij \in E(G)$. By applying Lemma 2.5 we get

$$\begin{aligned} H_4(G) &= \sum_{ij \in E(G)} \frac{2}{\varepsilon_G(i) + \varepsilon_G(j)} \\ &\geq \sum_{ij \in E(G)} \frac{2r}{\varepsilon_G(i)\varepsilon_G(j) + r^2} = \sum_{ij \in E(G)} \frac{\left(\sqrt{2r}\right)^2}{\varepsilon_G(i)\varepsilon_G(j) + r^2} \\ &\geq \frac{\left(\sum_{ij \in E(G)} \sqrt{2r}\right)^2}{\sum_{ij \in E(G)} \varepsilon_G(i)\varepsilon_G(j) + r^2} = \frac{2m^2 r}{E_2(G) + mr^2}. \end{aligned}$$

Now suppose that equality holds in (6). Then all the inequalities in the above argument must be equalities. By Lemma 2.5 we have $\varepsilon_G(i) + \varepsilon_G(j)$ is constant for all $ij \in E(G)$.

Conversely if $\varepsilon_G(i) + \varepsilon_G(j)$ is constant for all $ij \in E(G)$, it is easy to see that equality (6) holds.

Theorem 2.9. For any graph G we have

$$H_4(G) \leq \frac{\sqrt{(m-1)(mr^2 + 1) + 1}}{r}, \tag{7}$$

with equality holds if and only if $G \cong K_n$.

Proof. From definition of the eccentric harmonic index and the relation $\frac{2}{\varepsilon_G(i) + \varepsilon_G(j)} \leq 1$, we get the following conclusion.

$$H_4^2(G) = \left(\sum_{ij\in E(G)} \frac{2}{\varepsilon_G(i) + \varepsilon_G(j)} \right)^2$$

$$= \sum_{ij\in E(G)} \frac{4}{(\varepsilon_G(i) + \varepsilon_G(j))^2} + 2 \sum_{\substack{ij\in E(G)\\ ij\neq kl}} \left(\frac{2}{\varepsilon_G(i)+\varepsilon_G(j)} \cdot \frac{2}{\varepsilon_G(k)+\varepsilon_G(l)} \right)$$

$$H_4^2(G) \leq \sum_{ij\in E(G)} \frac{4}{(\varepsilon_G(i) + \varepsilon_G(j))^2} + 2 \sum_{\substack{ij\in E(G)\\ ij\neq kl}} 1.$$

$$\leq \frac{m}{r^2} + m(m-1).$$

So we achieve the desired result. Now suppose that equality holds in (7). Then all the inequalities in the above argument must be equalities. In this case, for all $ij \in E(G)$ should be $\varepsilon_G(i) = \varepsilon_G(j) = 1$. Then the equality holds if and only if $G \cong K_n$.

Conversely, if $G \cong K_n$ then it is easy to see that equality (7) holds.

Lemma 2.10. (*Schwetzers Inequality*) Let $x_1, x_2, \dots, x_n$ be positive real numbers such that $1 \leq i \leq n$ holds $m \leq x_i \leq M$. Then

$$\left(\sum_{i=1}^{n} x_i \right) \left(\sum_{i=1}^{n} \frac{1}{x_i} \right) \leq \frac{n^2(m+M)^2}{4nM}. \tag{8}$$

Equality holds in the (8) only when *n* is even, and the if and only if $x_1 = x_2 = \cdots = x_{\frac{n}{2}} = m$ and $x_{\frac{n}{2}+1} = \cdots = x_n = M$.

Theorem 2.11. For any graph *G* we have

$$H_4(G) \leq \frac{m^2(D+r)^2}{2\xi^c(G)Dr}, \tag{9}$$

with equality holds if and only if *G* is self centered graph.

Proof. Since $2r \leq \varepsilon_G(i) + \varepsilon_G(j) \leq 2D$ for all $ij \in E(G)$, using (8) we have

$$\sum_{ij\in E(G)} (\varepsilon_G(i) + \varepsilon_G(j)) \sum_{ij\in E(G)} \frac{1}{\varepsilon_G(i) + \varepsilon_G(j)} \leq \frac{m^2(2r+2D)^2}{4(2r)(2D)}$$

$$\sum_{ij\in E(G)} \frac{1}{\varepsilon_G(i) + \varepsilon_G(j)} \leq \frac{m^2(r+D)^2}{4\xi^c(G)Dr}$$

$$H_4(G) \leq \frac{m^2(D+r)^2}{2\xi^c(G)Dr}.$$

The equality holds if and only if *G* is self centered graph. We get the required result.

References

[1] Došlić, T. (2008) Vertex weighted Wiener polynomials for composite graphs. Ars Mathematica Contemporanea, 1; 66–80.
[2] Ediz, S., Farahani, M. R. and Imran, M. (2017) On novel harmonic indices of certain nanotubes. International Journal of Advanced Biotechnology and Research, 8(4); 277–282.
[3] Fajtlowicz, S. (1987) On conjectures of graffiti II. Congressus Numerantium, 60; 189-197.
[4] Ghorbani, M. and Hosseinzade, M.A. (2012) A new version of Zagreb indices. Filomat, 26; 93-100.
[5] Gross, J.L. and Yellen, J. (2004) Handbook of graph theory, Chapman Hall, CRC Press.
[6] Gupta, S., Singh, M. and Madan, A.K.(2000) Connective eccentricity index: a novel topological descriptor for predicting biological activity. Journal of Molecular Graphics and Modelling, 18; 18-25.
[7] Gutman, I. and Trinajstić, N. (1972) Graph Theory and Molecular Orbitals. Total pi-Electron Energy of Alternant Hydrocarbons. Chemical Physics Letters, 17: 535–538.
[8] Gutman, I., Ruˇsˇci´c, B., Trinajsti´c, N. and Wilkox, C.F. (1975) Graph Theory and Molecular Orbitals. XII. Acyclic Polyenes. The Journal of Chemical Physics, 62(9):3399–3405.
[9] Mitrinovic, D.S. (1970) Analytic Inequalities, Springer.
[10] Radon, J. (1913) Uber die absolut additiven Mengenfunktionen. Wiener Sitzungsber, 122; 1295–1438.
[11] Sharma, V., Goswami, R. and Madan, A.K. (1997) Eccentric connectivity index: A novel highly discriminating topological descriptor for structure property and structure-activity studies. Journal of Chemical Information and Modeling, 37(2); 273–282.
[12] Vukicevic, D. and Graovac, A. (2010) Note on the comparison of the first and second normalized Zagreb eccentricity indices. Acta Chimica Slovenica, 57; 524-528.
[13] Zhou, B. and Du, Z. (2010) On Eccentric Connectivity Index.MATCH Communications in Mathematical and in Computer Chemistry, 63; 181–198.