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Common fixed point results for generalized $\alpha-\psi-$ contractions in $0-\sigma-$complete metric-like space

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Abstract: In this paper, we establish some common fixed point theorems for mappings satisfying an $\alpha-\psi-$contractive condition in $0-\sigma-$complete metric-like space. Our results extend and generalize many existing results in the literature. Moreover, the present results are supported by two illustrative examples.

Keywords: Common fixed point, weakly compatible, $0-\sigma-$complete, metric-like spaces.

1 Introduction and preliminaries

In 1994, Matthews [7] established the concept of partial metric spaces as follows.

Definition 1. [7] *A partial metric on a nonempty set X is a function $p: X\times X\to\mathbb{R}^+$ such that for all $x, y, z\in X$,*

(p1) $x=y \Leftrightarrow p(x,x)=p(x,y)=p(y,y)$,
(p2) $p(x,x)\leq p(x,y)$,
(p3) $p(x,y)=p(y,x)$,
(p4) $p(x,y)\leq p(x,z)+p(z,y)-p(z,z)$.

A partial metric space is a pair (X,p) such that X is a nonemty set and p is a partial metric on X.

The notion of metric-like spaces which is an interesting generalization of partial metric space (see, e.g., [6-7]) and dislocated metric space (see [2-5]) was introduced by Amini-Harandi [1].

Definition 2. [1] *A mapping $\sigma: X\times X\to[0,+\infty)$, where X is a nonempty set, is said to be metric-like on X if for any x, y, $z\in X$, the following three conditions hold true:*

(σ1) $\sigma(x,y)=0\Rightarrow x=y$;
(σ2) $\sigma(x,y)=\sigma(y,x)$;
(σ3) $3\ \sigma(x,z)\leq\sigma(x,y)+\sigma(y,z)$.

The pair (X,σ) is called a metric-like space. Then a metric-like on X satisfies all of the conditions of a metric except that $\sigma(x,x)$ may be positive for $x\in X$.

Each metric-like σ on X generates a topology τ_σ on X whose base is the family of open $\sigma-$balls $D_\sigma(x,\varepsilon)=\{a\in X: |\sigma(x,a)-\sigma(x,x)|<\varepsilon\}$ for all $x\in X$ and $\varepsilon>0$.

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Definition 3. [1] *Let (X,σ) be a metric-like space, and let $\{x_n\}$ be any sequence in X and $x\in X$. Then*

(a) *a sequence $\{x_n\}$ is convergent to x with respect to τ_σ, if $\lim_{n\to\infty}\sigma(x_n,x)=\sigma(x,x)$;*
(b) *a sequence $\{x_n\}$ is a $\sigma-$Cauchy sequence in (X,σ) if $\lim_{n,m\to\infty}\sigma(x_n,x_m)$ exists and is finite;*
(c) *(X,σ) is called $\sigma-$complete if for every $\sigma-$Cauchy sequence $\{x_n\}$ in X there exists $x\in X$ such that $\lim_{n,m\to\infty}\sigma(x_n,x_m)=\lim_{n\to\infty}\sigma(x_n,x)=\sigma(x,x)$.*

The notion of $0-\sigma-$complete metric-like spaces was initiated by Shukla et al. [8] in 2013 as a new generalization of metric-like space. Recently, Fadail et al. [12] presented some fixed point results of maps that satisfy (F,ψ,φ)-weak contractive condition in $0-\sigma-$complete metric-like space.

Definition 4. [8] *Let (X,σ) be a metric-like space. A sequence $\{x_n\}$ in X is called a $0-\sigma-$Cauchy sequence if $\lim_{n,m\to\infty}\sigma(x_n,x_m)=0$. The space (X,σ) is said to be $0-\sigma-$complete if every $0-\sigma-$Cauchy sequence in X converges with respect to τ_σ to a point $x\in X$ such that $\sigma(x,x)=0$.*

Remark. [8]

(1) It is clear that every $0-\sigma-$Cauchy sequence is a $\sigma-$Cauchy sequence in (X,σ) and every $\sigma-$complete metric-like space is $0-\sigma-$complete. Also, every $0-$complete partial metric space is a $0-\sigma-$complete metric-like spaces.
(2) It is not hard to see that, if $\sigma(x_n,x)\to\sigma(x,x)=0$, then $\sigma(x_n,y)\to\sigma(x,y)$ for all $y\in X$.

Definition 5. [9] *Let f and g be self maps of a set X. If $w=fx=gx$ for some $x\in X$, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g. The pair f,g of self maps is weakly compatible if they commute at their coincidence points.*

Proposition 1. *Let f and g be weakly compatible self maps of a set X. If f and g have a unique point of coincidence $w=fx=gx$, then w is the unique common fixed point of f and g.*

In 2012, Samet et al. [10] introduced the notion of $\alpha-admissible$ mappings as follows.

Definition 6. [10] *Let $f:X\to X$ and $\alpha:X\times X\to[0,\infty)$ be given mappings. We say that f is $\alpha-admissible$ if for all x, $y\in X$, we have*

$$\alpha(x,y)\geq 1\Rightarrow\alpha(fx,fy)\geq 1.$$

In 2013, Shahi et al. [11] introduced the new concept of $\alpha-admissible$ with respect to (abbreviated as w.r.t.) g.

Definition 7. [11] *Let $f,g:X\to X$ and $\alpha:X\times X\to[0,\infty)$. We say that f is $\alpha-admissible$ w.r.t g if for all $x,y\in X$, we have*

$$\alpha(gx,gy)\geq 1\Rightarrow\alpha(fx,fy)\geq 1.$$

Remark. It is obvious that every $\alpha-admissible$ mapping is $\alpha-admissible$ w.r.t g when $g=I$ (see [11, Example 3.2-3.4]).

Definition 8. [10] *Let Ψ be the family of functions $\psi:[0,+\infty)\to[0,+\infty)$ satisfying the following conditions:*

(i) *ψ is nondecreasing. item $limits_{n=1}^{+\infty}\psi^n(t)<\infty$ for all $t>0$, where ψ^n is the n^{th} iterate of ψ. Note that if $\psi\in\Psi$, we have $\psi(t)<t$ for all $t>0$.*

Lemma 1. *Let (X,σ) be a $0-\sigma-$complete metric-like space, $f,g:X\to X$ be mappings such that the following condition is satisfied.*

$$\alpha(gx,gy)\sigma(fx,fy)\leq\psi(M(gx,gy)) \text{ for all } x,y\in X, \tag{1}$$

where $\psi\in\Psi$ and $\alpha:X\times X\to[0,\infty)$ and

$$M(gx,gy)=\max\left\{\sigma(gx,gy),\sigma(gx,fx),\sigma(gy,fy),\frac{\sigma(gx,fy)+\sigma(gy,fx)}{4}\right\}. \tag{2}$$

If f and g have a point of coincidence $z\in X$ and $\alpha(gu,gu)\geq 1$, then $\sigma(z,z)=0$.

Proof. Let $z\in X$ be the point of coincidence of f and g and u be the corresponding coincidence point, that is, $gu=fu=z$. Suppose to the contrary that $\sigma(z,z)>0$. From (1) and (2), we get that

$$\begin{aligned}
\sigma(z,z)&=\sigma(fu,fu)\\
&\leq\alpha(gu,gu)\sigma(fu,fu)\\
&\leq\psi(M(gu,gu))\\
&=\psi\left(\max\left\{\sigma(gu,gu),\sigma(gu,fu),\sigma(gu,fu),\frac{\sigma(gu,fu)+\sigma(gu,fu)}{4}\right\}\right)\\
&=\psi\left(\max\left\{\sigma(z,z),\sigma(z,z),\sigma(z,z),\frac{\sigma(z,z)+\sigma(z,z)}{4}\right\}\right)\\
&=\psi(\sigma(z,z)) \qquad\qquad \text{(Using definition of } \psi)\\
&<\sigma(z,z)
\end{aligned}$$

which is a contradiction. Hence, $\sigma(z,z)=0$.

2 Main Results

The following theorem is a generalization and improvement of Theorem 2.2 of Aydi and Karapinar [5].

Theorem 1. *Let (X,σ) be a $0-\sigma-$complete metric-like space. Suppose the mappings $f,g:X\to X$ satisfy*

$$\alpha(gx,gy)\sigma(fx,fy)\leq\psi(M(gx,gy)) \text{ for all } x,y\in X, \tag{3}$$

where $\psi\in\Psi$ and $\alpha:X\times X\to[0,\infty)$ and

$$M(gx,gy)=\max\left\{\sigma(gx,gy),\sigma(gx,fx),\sigma(gy,fy),\frac{\sigma(gx,fy)+\sigma(gy,fx)}{4}\right\}. \tag{4}$$

Suppose that

(i) *f is $\alpha-$admissible w.r.t g;*
(ii) *there exists $x_0\in X$ such that $\alpha(gx_0,fx_0)\geq 1$;*
(iii) *If $\{gx_n\}$ is a sequence in X such that $\alpha(gx_n,gx_{n+1})\geq 1$ for all n and $gx_n\to gz\in g(X)$ as $n\to\infty$, then there exists a subsequence $\{gx_{n(k)}\}$ of $\{gx_n\}$ such that $\alpha(gx_{n(k)},gz)\geq 1$ for all k;*
(iv) *either $\alpha(gu,gu_*)\geq 1$ or $\alpha(gu_*,gu)\geq 1$ whenever $gu=fu$ and $gu_*=fu_*$.*

Also suppose $f(X)\subset g(X)$ and $f(X)$ or $g(X)$ is a closed subset of X. Then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point z and $\sigma(z,z)=0=\sigma(fz,fz)=\sigma(gz,gz)$.

Proof. By (ii), there exists $x_0\in X$ such that $\alpha(gx_0,fx_0)\geq 1$. Let x_0 be an arbitrary point in X. Choose a point $x_1\in X$ such that $fx_0=gx_1$. This can be done, since the range of g contains the range of f. Continuing this process, having chosen $x_n\in X$, we obtain x_{n+1} in X such that

$$fx_n=gx_{n+1} \text{ for all } n\in\mathbb{N}. \tag{5}$$

As f is $\alpha-$*admissible* w.r.t g, we get

$$\alpha(gx_0,fx_0)=\alpha(gx_0,gx_1)\geq 1\Rightarrow\alpha(fx_0,fx_1)=\alpha(gx_1,gx_2)\geq 1.$$

By repeating the process above, we derive that

$$\alpha(gx_n,gx_{n+1})\geq 1 \text{ for all } n\in\mathbb{N}. \tag{6}$$

Consider the two possible cases. Suppose that $gx_n=gx_{n+1}$ for some $n\in\mathbb{N}$. Therefore, $gx_n=fx_n$ is a point of coincidence and then the proof is finished. Hence, suppose that $gx_n\neq gx_{n+1}$ for all $n\in\mathbb{N}_0$. By (3), (4) and (6), we have

$$\begin{aligned}\sigma(gx_{n+1},gx_n)&=\sigma(fx_n,fx_{n-1})\\&\leq\alpha(gx_n,gx_{n-1})\sigma(fx_n,fx_{n-1})\\&\leq\psi(M(gx_n,gx_{n-1})),\end{aligned} \tag{7}$$

for all $n\geq 1$, where

$$\begin{aligned}M(gx_n,gx_{n-1})&=\max\left\{\begin{array}{c}\sigma(gx_n,gx_{n-1}),\sigma(gx_n,fx_n),\\\sigma(gx_{n-1},fx_{n-1}),\frac{\sigma(gx_n,fx_{n-1})+\sigma(gx_{n-1},fx_n)}{4}\end{array}\right\}\\&\leq\max\left\{\begin{array}{c}\sigma(gx_n,gx_{n-1}),\sigma(gx_n,gx_{n+1}),\\\frac{3\sigma(gx_n,gx_{n+1})+\sigma(gx_{n-1},gx_n)}{4}\end{array}\right\}\\&=\max\{\sigma(gx_n,gx_{n-1}),\sigma(gx_n,gx_{n+1})\}.\end{aligned} \tag{8}$$

Due to monotonicity of the function ψ and using the inequalities (5), (7) and (8), we get

$$\sigma(gx_{n+1},gx_n)\leq\psi(\max\{\sigma(gx_n,gx_{n-1}),\sigma(gx_n,gx_{n+1})\}) \tag{9}$$

for all $n\geq 1$. If for some $n\geq 1$, we get $\sigma(gx_n,gx_{n-1})\leq\sigma(gx_n,gx_{n+1})$, by (9), we have

$$\sigma(gx_{n+1},gx_n)\leq\psi(\sigma(gx_n,gx_{n+1}))<\sigma(gx_n,gx_{n+1})$$

a contradiction. Hence, for all $n\geq 1$, we get

$$\max\{\sigma(gx_n,gx_{n-1}),\sigma(gx_n,gx_{n+1})\}=\sigma(gx_n,gx_{n-1}). \tag{10}$$

By virtue of (9) and (10), we get for all $n \geq 1$ that

$$\sigma(gx_{n+1}, gx_n) \leq \psi(\sigma(gx_n, gx_{n-1})). \tag{11}$$

Continuing this process inductively, we have

$$\sigma(gx_{n+1}, gx_n) \leq \psi^n(\sigma(gx_0, gx_1)) \text{ for all } n \geq 1. \tag{12}$$

By using (σ3) and (12), for all $k \geq 1$, we have

$$\begin{aligned}\sigma(gx_{n+k}, gx_n) &\leq \sigma(gx_{n+k}, gx_{n+k-1}) + \cdots + \sigma(gx_{n+1}, gx_n)\\ &\leq \sum_{p=n}^{n+k-1} \psi^p(\sigma(gx_0, gx_1))\\ &\leq \sum_{p=n}^{+\infty} \psi^p(\sigma(gx_0, gx_1)) \to 0 \text{ as } n \to \infty.\end{aligned} \tag{13}$$

We get that $\{gx_n\}$ is a $0-\sigma-Cauchy$ sequence in the $0-\sigma-$complete metric-like space (X, σ). Since $\{fx_n\} = \{gx_{n+1}\} \subseteq g(X)$ and $g(X)$ is closed in $0-\sigma-$complete metric-like space (X, σ), there exists $u \in X$ such that

$$\lim_{n\to\infty} \sigma(gx_n, gu) = \lim_{n,m\to\infty} \sigma(gx_n, gx_m) = \sigma(gu, gu) = 0. \tag{14}$$

Now, we show that u is a coincidence point of f and g. On the contrary, assume that $\sigma(gu, fu) > 0$. Since by assumption (iii) and (14), we have $\alpha\left(gx_{n(k)}, gu\right) \geq 1$ for all k, by using (σ3) and (3), (4) we get

$$\begin{aligned}&\sigma(gu, fu)\\ &\leq \sigma\left(gu, fx_{n(k)}\right) + \sigma\left(fx_{n(k)}, fu\right)\\ &\leq \sigma\left(gu, fx_{n(k)}\right) + \alpha\left(gx_{n(k)}, gu\right)\sigma\left(fx_{n(k)}, fu\right)\\ &\leq \sigma\left(gu, fx_{n(k)}\right)\\ &+\psi\left(\max\left\{\begin{array}{c}\sigma\left(gx_{n(k)}, gu\right), \sigma\left(gx_{n(k)}, fx_{n(k)}\right),\\ \sigma(gu, fu), \frac{\sigma\left(gx_{n(k)}, fu\right)+\sigma\left(gu, fx_{n(k)}\right)}{4}\end{array}\right\}\right).\end{aligned} \tag{15}$$

Taking the limit as $k \to \infty$ in (15) and Remark 1, we conclude

$$\begin{aligned}\sigma(gu, fu) &\leq \psi\left(\max\left\{\sigma(gu, fu), \frac{\sigma(gu, fu)}{4}\right\}\right)\\ &= \psi(\sigma(gu, fu)) < \sigma(gu, fu)\end{aligned}$$

which is a contradiction. Thus, we get that $\sigma(gu, fu) = 0$, that is, $gu = fu = z$. This show that f and g have a coincidence point.

Uniqueness: Assume that there exists another point of coincidence z_* of f and g and u_* is the corresponding point, that is, $gu_* = fu_* = z_*$. Then by Lemma 1, we have $\sigma(z_*,z_*) = 0$. Therefore, we have

$$\begin{aligned}
\sigma(z,z_*) &= \sigma(fu, fu_*) \\
&\leq \alpha(gu,gu_*)\sigma(fu, fu_*) \\
&\leq \psi(M(gu,gu_*)) \\
&= \psi\left(\max\left\{\sigma(gu,gu_*), \sigma(gu, fu), \sigma(gu_*, fu_*), \frac{\sigma(gu, fu_*)+\sigma(gu_*, fu)}{4}\right\}\right) \\
&= \psi\left(\max\left\{\sigma(z,z_*), \sigma(z,z), \sigma(z_*,z_*), \frac{\sigma(z,z_*)+\sigma(z_*,z)}{4}\right\}\right) \\
&= \psi(\sigma(z,z_*)) \\
&< \sigma(z,z_*),
\end{aligned}$$

which is a contradiction. Thus, we have $\sigma(z,z_*) = 0$, that is, $z = z_*$. Hence, the coincidence point of f and g is unique. From the Proposition 1, f and g have the unique common fixed point. In the case when $f(X)$ is closed set in (X,σ) the proof is similar.

From Theorem 1, if we choose $g = I$ the identity mapping on X, we deduce the following corollary.

Corollary 1. *Let (X,σ) be a $0-\sigma-$complete metric-like space. Suppose the mapping $f : X \to X$ satisfy*

$$\alpha(x,y)\sigma(fx, fy) \leq \psi(M(x,y)) \text{ for all } x, y \in X, \tag{16}$$

where $\psi \in \Psi$ and $\alpha : X \times X \to [0,\infty)$ and

$$M(x,y) = \max\left\{\sigma(x,y), \sigma(x, fx), \sigma(y, fy), \frac{\sigma(x, fy)+\sigma(y, fx)}{4}\right\}. \tag{17}$$

Suppose that

(i) *f is $\alpha-$admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$;*
(iii) *If $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1}) \geq 1$ for all n and $x_n \to z \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)},z) \geq 1$ for all k.*
(iv) *either $\alpha(u,u_*) \geq 1$ or $\alpha(u_*,u) \geq 1$ whenever $u = fu$ and $u_* = fu_*$.*

Then f has a unique fixed point $\omega \in X$ and $\sigma(\omega,\omega) = 0$.

From Corrollary 1, if the function $\alpha : X \times X \to [0,\infty)$ is such that $\alpha(x,y) = 1$ for all $x, y \in X$, we deduce the following corollary.

Corollary 2. *Let (X,σ) be a $0-\sigma-$complete metric-like space. Suppose the mapping $f : X \to X$ satisfy*

$$\sigma(fx, fy) \leq \psi(M(x,y)) \text{ for all } x, y \in X, \tag{18}$$

where $\psi \in \Psi$ and $M(x,y)$ is defined by (17). Then f has a fixed point.

The following theorem is a generalization and improvement of Theorem 2.5 of Aydi and Karapinar [5] ($\sigma-$completeness (respectively, $\alpha - admissible$) of space is replaced by $0-\sigma-$completeness ((respectively, $\alpha - admissible$ w.r.t g)).

Theorem 2. *Let (X,σ) be a $0-\sigma-$complete metric-like space. Suppose the mappings $f, g : X \to X$ satisfy*

$$\alpha(gx,gy)\sigma(fx,fy) \leq \psi(M(gx,gy)) \text{ for all } x,y \in X, \tag{19}$$

where $\psi \in \Psi$ and $\alpha : X \times X \to [0,\infty)$ and

$$M(gx,gy) = \max\{\sigma(gx,gy), \sigma(gx,fx), \sigma(gy,fy)\}. \tag{20}$$

Suppose that

(i) *f is $\alpha - admissible$ w.r.t g;*
(ii) *there exists $x_0 \in X$ such that $\alpha(gx_0, fx_0) \geq 1$;*
(iii) *If $\{gx_n\}$ is a sequence in X such that $\alpha(gx_n, gx_{n+1}) \geq 1$ for all n and $gx_n \to gz \in g(X)$ as $n \to \infty$, then there exists a subsequence $\{gx_{n(k)}\}$ of $\{gx_n\}$ such that $\alpha(gx_{n(k)}, gz) \geq 1$ for all k.*
(iv) *either $\alpha(gu,gu_*) \geq 1$ or $\alpha(gu_*,gu) \geq 1$ whenever $gu = fu$ and $gu_* = fu_*$.*

Also suppose $f(X) \subset g(X)$ and $f(X)$ or $g(X)$ is a closed subset of X. Then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point z and $\sigma(z,z) = 0 = \sigma(fz,fz) = \sigma(gz,gz)$.

Proof. By the given assumption and the proof of Theorem 1, we construct the sequence $\{x_n\}$ in X such that

$$fx_n = gx_{n+1} \text{ for all } n \in \mathbb{N}. \tag{21}$$

Since also f is $\alpha - admissible$ w.r.t g, we obtain that

$$\alpha(gx_n, gx_{n+1}) \geq 1 \text{ for all } n \in \mathbb{N}. \tag{22}$$

Consider the two possible cases. Assume that $gx_n = gx_{n+1}$ for some $n \in \mathbb{N}$. Therefore, $gx_n = fx_n$ is a point of coincidence and in that case the proof is completed. Thus, assume that $gx_n \neq gx_{n+1}$ for all $n \in \mathbb{N}_0$. From (19), (20) and (22), we have

$$\begin{aligned}
&\sigma(gx_{n+1}, gx_n) \\
&= \sigma(fx_n, fx_{n-1}) \\
&\leq \alpha(gx_n, gx_{n-1})\sigma(fx_n, fx_{n-1}) \\
&\leq \psi(M(gx_n, gx_{n-1})) \\
&= \max\{\sigma(gx_n, gx_{n-1}), \sigma(gx_n, fx_n), \sigma(gx_{n-1}, fx_{n-1})\} \\
&= \max\{\sigma(gx_n, gx_{n-1}), \sigma(gx_n, gx_{n+1})\}
\end{aligned} \tag{23}$$

for all $n \geq 1$. By virtue of monotonicity of the function ψ and from (21) and (23), we have

$$\sigma(gx_{n+1}, gx_n) \leq \psi(\max\{\sigma(gx_n, gx_{n-1}), \sigma(gx_n, gx_{n+1})\}) \tag{24}$$

for all $n \geq 1$. If for some $n \geq 1$, we get $\sigma(gx_n, gx_{n-1}) \leq \sigma(gx_n, gx_{n+1})$, from (24), we get

$$\begin{aligned}\sigma(gx_{n+1}, gx_n) &\leq \psi(\sigma(gx_n, gx_{n+1})) \\ &< \sigma(gx_n, gx_{n+1})\end{aligned}$$

a contradiction. Thus, for all $n \geq 1$, we have

$$\max\{\sigma(gx_n, gx_{n-1}), \sigma(gx_n, gx_{n+1})\} = \sigma(gx_n, gx_{n-1}). \tag{25}$$

Owing to (24) and (25), we have for all $n \geq 1$ that

$$\sigma(gx_{n+1}, gx_n) \leq \psi(\sigma(gx_n, gx_{n-1})). \tag{26}$$

Continuing this process inductively, we obtain

$$\sigma(gx_{n+1}, gx_n) \leq \psi^n(\sigma(gx_0, gx_1)) \text{ for all } n \geq 1. \tag{27}$$

By using triangular inequality and (27), for all $k \geq 1$, we get

$$\begin{aligned}\sigma(gx_{n+k}, gx_n) &\leq \sigma(gx_{n+k}, gx_{n+k-1}) + \cdots + \sigma(gx_{n+1}, gx_n) \\ &\leq \sum_{p=n}^{n+k-1} \psi^p(\sigma(gx_0, gx_1)) \\ &\leq \sum_{p=n}^{+\infty} \psi^p(\sigma(gx_0, gx_1)) \to 0 \text{ as } n \to \infty.\end{aligned} \tag{28}$$

We obtain that $\{gx_n\}$ is a $0-\sigma-$*Cauchy* sequence in the $0-\sigma-$complete metric-like space (X, σ). As $\{fx_n\} = \{gx_{n+1}\} \subseteq g(X)$ and $g(X)$ is closed in $0-\sigma-$complete metric-like space (X, σ), there exists $u \in X$ such that

$$\lim_{n\to\infty} \sigma(gx_n, gu) = \lim_{n,m\to\infty} \sigma(gx_n, gx_m) = \sigma(gu, gu) = 0. \tag{29}$$

Next, we shall prove that u is a coincidence point of f and g. Assume, in contrast to, that $\sigma(gu, fu) > 0$. Resulting from assumption (iii) and (29), we obtain $\alpha\left(gx_{n(k)}, gu\right) \geq 1$ for all k, by using triangular inequality and (19), (20) we have

$$\begin{aligned}&\sigma(gu, fu) \\ &\leq \sigma\left(gu, fx_{n(k)}\right) + \sigma\left(fx_{n(k)}, fu\right) \\ &\leq \sigma\left(gu, fx_{n(k)}\right) + \alpha\left(gx_{n(k)}, gu\right)\sigma\left(fx_{n(k)}, fu\right) \\ &\leq \sigma\left(gu, fx_{n(k)}\right) \\ &+ \psi\left(\max\left\{\sigma\left(gx_{n(k)}, gu\right), \sigma\left(gx_{n(k)}, fx_{n(k)}\right), \sigma(gu, fu)\right\}\right).\end{aligned} \tag{30}$$

Letting $k \to \infty$ in (30) and Remark 1, we obtain that

$$\sigma(gu, fu) \leq \psi(\sigma(gu, fu)) < \sigma(gu, fu)$$

which is a contradiction. Thus, we get that $\sigma(gu, fu) = 0$, that is, $gu = fu = z$. This show that f and g have a coincidence point.

Suppose that there exists another point of coincidence z_* of f and g and u_* is the corresponding point, that is, $gu_* = fu_* = z_*$. Then by Lemma 1, we have $\sigma(z_*, z_*) = 0$. Therefore, we have

$$\begin{aligned}\sigma(z,z_*) &= \sigma(fu, fu_*)\\ &\leq \alpha(gu,gu_*)\sigma(fu,fu_*)\\ &\leq \psi(M(gu,gu_*))\\ &= \psi(\max\{\sigma(gu,gu_*),\sigma(gu,fu),\sigma(gu_*,fu_*)\})\\ &= \psi(\max\{\sigma(z,z_*),\sigma(z,z),\sigma(z_*,z_*)\})\\ &= \psi(\sigma(z,z_*))\\ &< \sigma(z,z_*),\end{aligned}$$

a contradiction. Therefore, we get $\sigma(z,z_*) = 0$, in other words, $z = z_*$. Thus, the coincidence point of f and g is unique. From the Proposition 1, f and g have the unique common fixed point. In the case when $f(X)$ is closed set in (X,σ) the proof is similar.

From Theorem 2, if we choose $g = I$ the identity mapping on X, we deduce the following corollary.

Corollary 3. *Let (X,σ) be a $0-\sigma-$complete metric-like space. Suppose the mapping $f : X \to X$ satisfy*

$$\alpha(x,y)\sigma(fx,fy) \leq \psi(M(x,y)) \text{ for all } x, y \in X, \tag{31}$$

where $\psi \in \Psi$ and $\alpha : X \times X \to [0,\infty)$ and

$$M(x,y) = \max\{\sigma(x,y),\sigma(x,fx),\sigma(y,fy)\} \tag{32}$$

Suppose that

(i) *f is $\alpha-$admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$;*
(iii) *If $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1}) \geq 1$ for all n and $x_n \to z \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)},z) \geq 1$ for all k.*
(iv) *either $\alpha(u,u_*) \geq 1$ or $\alpha(u_*,u) \geq 1$ whenever $u = fu$ and $u_* = fu_*$.*

Then f has a unique fixed point $\omega \in X$ and $\sigma(\omega,\omega) = 0$.

From Corollary 3, if the function $\alpha : X \times X \to [0,\infty)$ is such that $\alpha(x,y) = 1$ for all $x, y \in X$, we deduce the following corollary.

Corollary 4. *Let (X,σ) be a $0-\sigma-$complete metric-like space. Suppose the mapping $f : X \to X$ satisfy*

$$\sigma(fx,fy) \leq \psi(M(x,y)) \text{ for all } x, y \in X, \tag{33}$$

where $\psi \in \Psi$ and $M(x,y)$ is defined by (32). Then f has a fixed point.

Taking $M(x,y)=\sigma(x,y)$ in (32), we have the following results.

Corollary 5. *Let (X,σ) be a $0-\sigma-$complete metric-like space. Suppose the mapping $f:X\to X$ satisfy*

$$\sigma(fx,fy)\leq\psi(\sigma(x,y)) \text{ for all } x,y\in X, \tag{34}$$

where $\psi\in\Psi$. Then f has a fixed point.

3 Examples

We give an illustrative example wherein one demostrates Theorem 1 on the existence and uniqueness of a common fixed point.

Example 1. Let $X=\{0,2,4\}$. Define $\sigma:X\times X\to[0,+\infty)$ as follows:

$$\begin{aligned}&\sigma(0,0)=0,\ \sigma(2,2)=4,\ \sigma(4,4)=2,\\&\sigma(0,2)=\sigma(2,0)=8,\\&\sigma(0,4)=\sigma(4,0)=4,\\&\sigma(2,4)=\sigma(4,2)=5.\end{aligned}$$

Then (X,σ) is a $0-\sigma-$complete metric-like space. Given $f,g:X\to X$ as

$$f0=0,\ f2=4 \text{ and } f4=0$$

and

$$g0=0,\ g2=2 \text{ and } g4=4.$$

Take $\psi(t)=\frac{2}{3}t$ for each $t\geq 0$. Define the mapping $\alpha:X\times X\to[0,+\infty)$ by

$$\alpha(x,y)=\begin{cases}1 & \text{if } x=0,\\ 0 & \text{otherwise}\end{cases}.$$

First we shall show that f is $\alpha-admissible$ w.r.t g. Let $x,y\in X$ such that $\alpha(gx,gy)\geq 1$. This implies that $gx=0$ and since $f0=0$, $\alpha(fx,fy)=1$ for each $y\in X$. Hence, f is $\alpha-admissible$ w.r.t g.

We need to consider three cases:

Case 1: If $x=0$ and $y=0$, we have

$$\alpha(gx,gy)\sigma(fx,fy)=\sigma(fx,fy)=0.$$

Case 2: If $x=0$ and $y=2$, we have

$$\begin{aligned}\alpha\left(g0,g2\right)\sigma\left(f0,f2\right)=\sigma\left(f0,f2\right)=\sigma\left(0,4\right)=4\leq\frac{2}{3}8=\frac{2}{3}\sigma\left(g0,g2\right)=\psi\left(\sigma\left(g0,g2\right)\right)\\ \leq\psi\left(\max\left\{\sigma\left(g0,g2\right),\sigma\left(g0,f0\right),\sigma\left(g2,f2\right),\frac{\sigma\left(g0,f2\right)+\sigma\left(g2,f0\right)}{4}\right\}\right)\\ =\psi\left(M\left(g0,g2\right)\right).\end{aligned}$$

Case 3: If $x=0$ and $y=4$, we have

$$\alpha\left(gx,gy\right)\sigma\left(fx,fy\right)=\sigma\left(fx,fy\right)=0.$$

It is also clear that assumption (iii) and (iv) of Theorem 1 is satisfied. Consequently, f and g have a coincidence point. Here, 0 is a coincidence point of f and g. Also, obviously all the assumptions of Theorem 2 are satisfied. In this example, 0 is the unique common fixed point of f and g.

In what follows, we give the following example making effective our obtained results.

Example 2. Let $X=[0,\infty)\cap\mathbb{Q}$ and $\sigma:X\times X\to[0,+\infty)$ be defined by

$$\sigma\left(x,y\right)=\begin{cases}2x & \text{if } x=y\\ \max\left\{x,y\right\} & \text{otherwise}\end{cases}.$$

Then (X,σ) is a $0-\sigma-$complete metric-like space (for more details, see [8]). Define the mappings $f,g:X\to X$ by

$$fx=\begin{cases}0 & \text{if } x=3,\\ \frac{2x}{7} & \text{otherwise,}\end{cases}\quad\text{and } gx=\begin{cases}1 & \text{if } x=3,\\ \frac{x}{2} & \text{otherwise}\end{cases}.$$

Consider $\psi:[0,+\infty)\to[0,+\infty)$ defined by

$$\psi\left(t\right)=\begin{cases}\frac{2t}{3} & \text{if } t\in\left[0,1\right],\\ \frac{4}{5} & \text{otherwise}\end{cases}.$$

Now, we define the mapping $\alpha:X\times X\to[0,+\infty)$ by

$$\alpha\left(x,y\right)=\begin{cases}1 & \text{if } x,y\in\left[0,1\right],\\ 0 & \text{otherwise}\end{cases}.$$

First, let $x,y\in X$ such that $\alpha\left(gx,gy\right)\geq1$, so $gx,gy\in[0,1]$. In this case,

$$\alpha\left(fx,fy\right)=\alpha\left(\frac{2x}{7},\frac{2y}{7}\right)=1;$$

that is, f is $\alpha-admissible$ w.r.t g.

We distinguish three cases:

Case 1: If $x, y \in [0,1]$ and $x > y$, then we get

$$\begin{aligned}\alpha(gx,gy)\sigma(fx,fy) &= \sigma(fx,fy)\\ &= \max\{fx,fy\}\\ &= \frac{2x}{7} \leq \frac{2}{3}.\frac{x}{2}\\ &= \frac{2}{3}\sigma(gx,gy)\\ &\leq \frac{2}{3}M(gx,gy) = \psi(M(gx,gy)).\end{aligned}$$

Case 2: If $x, y \in [0,1]$ and $x = y$, then we get

$$\begin{aligned}\alpha(gx,gy)\sigma(fx,fy) &= \sigma(fx,fy)\\ &= 2.\frac{2x}{7} \leq \frac{2}{3}.2.\frac{x}{2}\\ &= \frac{2}{3}\sigma(gx,gy)\\ &\leq \frac{2}{3}M(gx,gy) = \psi(M(gx,gy)).\end{aligned}$$

Case 3: If $x, y \in [0,1]$ and $x < y$, then we get

$$\begin{aligned}\alpha(gx,gy)\sigma(fx,fy) &= \sigma(fx,fy)\\ &= \max\{fx,fy\}\\ &= \frac{2y}{7} \leq \frac{2}{3}.\frac{y}{2}\\ &= \frac{2}{3}\sigma(gx,gy)\\ &\leq \frac{2}{3}M(gx,gy) = \psi(M(gx,gy)).\end{aligned}$$

It is also clear that assumption (iii) and (iv) of Theorem 1 is satisfied. Consequently, f and g have a unique common fixed point, that is to say 0. Note that (X,σ) is not a $\sigma-$complete metric-like space. Hence, Theorem 2.2 of Aydi and Karapinar [5] is not applicable.

4 Conclusion

Our theorems and corolaries which include the corresponding results announced in Samet et al. (2012) as special cases fundamentally improve and generalize the results of Aydi and Karapinar (2015) in the following sense.

(i) Extend from $\sigma-$completeness to $0-\sigma-$completeness.

(ii) Extend the mappings from $\alpha-$ *admissible* mapping to $\alpha-$ *admissible* w.r.t g mapping.

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