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On $\bar{G}$-J anti-invariant submanifolds of almost complex contact metric manifolds

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Abstract: In this article we studied anti-invariant submanifolds of almost complex contact metric manifolds. We found a relation between Nijenhuis tensor fields of anti-invariant submanifolds and almost complex contact manifolds. We investigated relations between curvature tensors of these manifolds. Moreover, we studied anti-invariant submanifolds of almost complex contact metric manifolds.Some necessary conditions on which a submanifolds of an almost complex contact metric manifolds is $\bar{G}$-J anti-invariant were given. Also we found some characterizations for totally geodesic or umbilical $\bar{G}$-J anti-invariant submanifolds of almost complex contact metric manifolds.

Keywords: Complex contact metric manifolds invariant, submanifolds, anti-invariant submanifolds

1 Introduction

Contact manifolds was first worked by W. M. Boothby and H. C. Wang [1,2,3], and J.W. Gray [14] described an almost contact manifold by reducing the structural group of the tangent bundle to $U(n)X1$. Later S. Sasaki[15] showed the existence of four tensor fields, and introduced the Riemannian metric with regard to the almost contact structure. Tensor calculus has been a powerful and prominent method since the study of contact manifold were initiated to use by these four tensor fields. Two special contact Riemannian manifolds are K-contact Riemannian manifolds and Sasakian manifolds.It can be said that a Sasakian manifold can be regarded as an odd-dimensional analogue of a Kahlerian manifold. Differentiable manifolds were worked by Y. Hatakeyama [5] with almost contact metric structure in 1963. In this work contact metric structure was called with vanishing N^i_j or N^i_{jk} K-contact metric structure or normal contact metric structure respectively. In 1976, D. E. Blair [4] provided necessary and sufficient conditions for normality on almost contact metric manifolds.

Although complex contact manifolds are almost as old as real contact manifolds' modern theory, this subject attracts less attention but recently many examples about this subject have been studied in the literature. B. Korkmaz [9] showed a complex analogue of real contact metric manifolds in her PhD thesis.

The concept of complex contact manifold was found as a result of the works of Kobayashi and Boothby[11,12,13] in late 1950s and the early 1960s. This is just shortly after the Boothby-Wang fibration in real contact geometry. Then in 1965, J. A. Wolf[16] studied homogeneous complex contact manifolds. Ishihara and Konishi[6,7] introduced a notion of normality for complex contact structures. In this development however, the notion of normality seems too strong since it precludes the complex Heisenberg group as one of the canonical examples, although it does include complex projective

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spaces as odd complex dimension as one would expect. Then B. Korkmaz [8,9,10] give a new condition for the normality. As a subject the Riemannian Geometry of complex contact manifolds have just made it debut and it tends to be studied on it. In the literature work we have done, the submanifolds of complex contact metric manifolds have defected not to be studied on and that's why we decided to work on this issue.

Based on these studies, we have investigated submanifolds of almost complex contact metric manifolds. We first search case in which anti-invariant according to J and $\bar{G}$. Namely, we accept $JTM \subset TM^{\perp}$ and $\bar{G}TM \subset TM^{\perp}$. In this study, we studied case in which structure vector fields aren't tangent to TM. In this case, we have found relations between Nijenhuis tensor fields of almost complex contact metric submanifolds of this almost complex contact manifolds with almost complex contact manifolds . Also, we have investigated relation between curvature tensors of these two manifolds. In this study, we have showed that structure vector fields must not be tangent to TM. In this case, we have given condition be total umbilical and totally geodesic $\bar{G}-J$ anti-invariant submanifold of almost complex contact metric manifolds. Also we have given condition be parallel with respect to the induced connection on $\bar{G}-J$ anti-invariant submanifold of structure vector fields.

2 Some fundamental concepts and definitions

2.1 Contact manifolds

Firstly let us present definition of contact manifold. A C^{∞} manifold M^{2n+1} is called a contact manifold if there is a 1-form μ such that

$$\mu \wedge (d\mu)^n \neq 0. \tag{1}$$

In particulary, a contact manifold is routable when (1) inequality is provided. Since $d\mu$ has rank $2n$ on Grassmann algebra $\wedge T_m^*M$ at each point $m \in M$, it is obtained a 1-dimensional subspace,

$$\{W \in T_mM \mid d\mu\,(W, T_mM) = 0\},$$

when $\mu \neq 0$. On the other hand if μ is zero, it is obtained complementary of that subspace. Also, we get a global vector field ξ satisfying

$$d\mu\,(\xi, W) = 0, \mu\,(\xi) = 1.$$

taking ξ_m in this subspace normalized by $\mu\,(\xi_m) = 1$. ξ is called the characterstic vector field of the contact structure [4].

Theorem 1. *Let M^{2n+1} be a contact manifold in widersense. If μ is odd, M^{2n+1} is routable, then M^{2n+1} is contact manifold[4].*

2.2 Almost complex and almost contact structures

A tensor field J of type $(1,1)$ is called an almost complex structure, where $J^2 = -I$. A Riemannian manifold endowed with an almost Complex structure a called almost complex manifold. A Hermitian metric on an almost complex manifold (M, J) is an invariant Riemannian metric under J,i.e.,

$$g\,(JW, JZ) = g\,(W, Z).$$

Pointing out that J is negative-self-adjoint with respect to g, i.e.,

$$g(W,JZ) = -g(JW,Z),$$
$$\Omega(W,Z) = g(W,JZ)$$

defines a 2-form called the fundamental 2-form of the Hermitian structure (M,J,g). A complex manifold M together with J the corresponding almost complex structure is called (M,J,g) Hermitian manifold. If $d\Omega = 0$, the structure is almost Keahlerian. Also note that, every almost complex manifold receives a Hermitian metric g defined by

$$g(W,Z) = k(W,Z) + k(JW,JZ),$$

where k is any Riemaniann metric.

In terms of structure tensors, we can say that M^{2n+1} has an almost contact structure or sometimes $(\phi,\xi,\mu)-$structure if it admits a tensor field ϕ of type $(1,1)$, a vector field ξ and a 1-form μ satisfying

$$\phi^2 = -I + \mu \otimes \xi, \mu(\xi) = 1.$$

Theorem 2. *Let M^{2n+1} be a $(\phi,\xi,\mu)-$structure. Then $\phi\xi = 0$ and $\mu \circ \phi = 0$. Moreover the endomorphism ϕ has rank $2n$[4].*

Definition 1. *Let g be a Riemaniann metric providing*

$$g(\phi W, \phi Z) = g(W,Z) - \mu(W)\mu(Z).$$

A manifold M^{2n+1} with a $(\phi,\xi,\mu)-$structure taking a Riemannann metric g is called an almost contact metric structure and we say that g is compatible metric[4].

2.3 Complex contact manifolds

Let us recall main notation about complex contact manifold for this subject, main reference is B.Korkmaz.

Definition 2. *A complex contact manifold is called a complex manifold of odd complex dimension $2n+1$ together with an open covering $\{\mathbf{O}_\alpha\}$ by coordinate neighborhoods such that:*

(1) *On each $\mathbf{O}_\alpha$ there is a holomorphic 1-form Ψ_α such that*

$$\Psi_\alpha \wedge (d\Psi_\alpha)^n \neq 0;$$

(2) *On $\mathbf{O}_\alpha \cap \mathbf{O}_\beta \neq \varnothing$ there is a non-vanishing holomorphic function $f_{\alpha\beta}$ such that $\Psi_\alpha = f_{\alpha\beta}\Psi_\beta$.*

The subspaces $\{W \in T_m\mathbf{O}_\alpha : \Psi_\alpha(W) = 0\}$ define a non-integrable holomorphic subbundle $\mathscr{H}$ of complex dimension 2n called the complex contact subbundle or horizontal subbundle. The quotient $L = T\ M/\mathscr{H}$ is a complex line bundle over M. For sake of brevity, we will often neglect the subscripts on local tensor fields. Define a local section X of TM, i.e., a section of $T\mathbf{O}$, by $dx(X,W) = 0$ for every $W \in \mathscr{H}, x(X) = 0$. Then such local sections define a global subbundle ϑ by $\vartheta\ |_{\mathbf{O}} = span\{X, JX\}$. Now we get $TM = \mathscr{H} \oplus \vartheta$ and we denote the projection map p from TM to $\mathscr{H}$. We suppose throughout in this study that ϑ is integrable and we call ϑ the vertical subbundle or characteristic subbundle.

Otherwise if M is a complex manifold with almost complex structure J, Hermitian metric g and open covering by coordinate neighborhoods $\{\mathbf{O}_\alpha\}$, M is called a complex almost contact metric manifold, if it provides the following two properties:

(1) On each $\mathbf{O}_\alpha$ there exist 1-forms x_α and $y_\alpha = x_\alpha \circ J$ with orthogonal dual vector fields X_α and $Y_\alpha = -JX_\alpha$ and $(1,1)$ tensor fields G_α and $H_\alpha = G_\alpha J$ such that

$$G_\alpha^2 = H_\alpha^2 = -I + x_\alpha \otimes X_\alpha + y_\alpha \otimes Y_\alpha, \tag{2}$$
$$G_\alpha J = -JG_\alpha, \tag{3}$$
$$G_\alpha X = 0, \tag{4}$$
$$g(W, G_\alpha Z) = -g(G_\alpha W, Z), \tag{5}$$
$$g(X_\alpha, W) = x_\alpha(W), \tag{6}$$
$$x_\alpha(X_\alpha) = 1 \tag{7}$$

(2) On $\mathbf{O}_\alpha \cap \mathbf{O}_\beta \neq \varnothing$,

$$x_\beta = a x_\alpha - b y_\alpha,$$
$$y_\beta = b x_\alpha + a y_\alpha,$$
$$G_\beta = a G_\alpha - b H_\alpha,$$
$$H_\beta = b G_\alpha + a H_\alpha$$

where a and b are functions providing the equality $a^2 + b^2 = 1$[4]

Consequently, on a complex almost contact metric manifold M, the following identities is held:

$$H_\alpha G_\alpha = -G_\alpha H_\alpha = J + x_\alpha \otimes Y_\alpha - y_\alpha \otimes X_\alpha$$

$$JH_\alpha = -H_\alpha J = G_\alpha$$

$$g(H_\alpha W, Z) = -g(W, H_\alpha Z)$$

$$G_\alpha Y_\alpha = H_\alpha X_\alpha = H_\alpha Y_\alpha = 0$$
$$x_\alpha G_\alpha = y_\alpha G_\alpha = x_\alpha H_\alpha = y_\alpha H_\alpha = 0$$
$$JY_\alpha = X_\alpha, g(X_\alpha, Y_\alpha) = 0.$$

Let $(M, \{\omega_\alpha\})$ be a complex contact manifold. We can find a non-vanishing, complex-valued function multiple π_α of ω_α such that on $\mathbf{O}_\alpha \cap \mathbf{O}_\beta$, $\pi_\alpha = h_{\alpha\beta}\pi_\beta$ with $h_{\alpha\beta} : \mathbf{O}_\alpha \cap \mathbf{O}_\beta \to S^1$. Let $\pi_\alpha = x_\alpha - i y_\alpha$. Since ω_α is holomorphic $y_\alpha = x_\alpha J$.

We can locally descriptive of a vector field X providing following properties

(1) for all W in $\mathscr{H}$, $du(X, W) = 0$
(2) $x(X) = 1, y(X) = 0$.

Then we have a global subbundle ϑ locally spanned by X and $Y = -JX$ with $TM = \mathscr{H} \oplus \vartheta$. We say ϑ the vertical subbundle on contact structure. Here we can obtain a local $(1,1)$ tensor G from a complex almost contact metric structure on M such that $(x, y, X, Y, G, H = GJ, g)$[10].

Definition 3. *Let $(M, \{\omega\})$ be a complex contact manifold with the complex structure J and hermitian metric g. (M, x, y, X, Y, g) is called a complex contact metric manifold if*

(1) *There is a local $(1,1)$ tensor g such that $(x, y, X, Y, G, H = GJ, g)$ is a complex almost contact metric structure on M, and*
(2) *$g(W, GZ) = dx(W, Z)$ and $g(W, HZ) = dy(W, Z)$ for all W, Z in $\mathscr{H}$.*

Now, let us define 2-forms $\hat{G}$ and $\hat{H}$ by

$$\hat{G}(W,Z) = g(W,GZ)$$
$$\hat{H}(W,Z) = g(W,HZ).$$

Then

$$\hat{G}(W,Z) = dx(W,Z),$$
$$\hat{H}(W,Z) = dy(W,Z),$$

where W, Z are horizontal vector fields.

Generally, for $\sigma(W) = g(\nabla_W X, Y)$, we get

$$\hat{G} = dx - \sigma \wedge y \quad (8)$$
$$\hat{H} = dx + \sigma \wedge x. \quad (9)$$

[10].

Let p be projection map from TM to $\mathscr{H}$. There is a symmetric operator $h = \frac{1}{2}\mathscr{L}_\xi \phi$ playing an important role in real contact geometry, where ξ is the characteristic vector field, ϕ is the structure tensor of the real contact metric structure and $\mathscr{L}$ represents the Lie- differentiation. Especially, we obtain

$$\nabla_W \xi = -\phi W - \phi h W$$

on a real contact manifold. We define symmetric operator h_X, h_Y from TM to $\mathscr{H}$ in the same way, as follows:

$$h_X = \frac{1}{2} sym(\mathscr{L}_X G) \circ p$$
$$h_Y = \frac{1}{2} sym(\mathscr{L}_Y G) \circ p,$$

where *sym* represents the symmetrization. Then for levi-civita connection ∇ of g, we have

$$h_X G = -G h_X, h_Y H = -H h_Y,$$
$$h_X(X) = h_X(Y) = h_Y(X) = h_Y(Y) = 0,$$

and

$$\nabla_W X = -GW - Gh_X W + \sigma(W)Y, \tag{10}$$
$$\nabla_W Y = -HW - Hh_Y W - \sigma(W)X, \tag{11}$$

where ∇ is Levi-Civita connection of g[10].

Therefore

$$\nabla_X X = \sigma(X)Y, \nabla_Y X = \sigma(Y)Y, \nabla_X Y = -\sigma(X)X, \nabla_Y Y = -\sigma(Y)X. \tag{12}$$

Lemma 1. *$\nabla_X G = \sigma(X)H$ and $\nabla_Y H = -\sigma(Y)G$.*

Let M be a complex contact metric manifold. the authors in [7] defined $(1,2)$ tensors S and T on a complex almost contact manifolds as follows:

$$\begin{aligned} S(W,Z) = {} & [G,G](W,Z) + 2y(Z)HW - 2y(W)HZ \\ & + 2g(W,GZ)X - 2g(W,HZ)Y - \sigma(GW)HZ \\ & + \sigma(GZ)HW + \sigma(W)GHZ - \sigma(Z)GHW \end{aligned} \tag{13}$$

$$\begin{aligned} T(W,Z) = {} & [H,H](W,Z) + 2u(Z)GW - 2x(W)GZ \\ & + 2g(W,HZ)Y - 2g(W,GZ)X + \sigma(HW)GZ - \sigma(HZ)GW - \sigma(W)HGZ + \sigma(Z)HGW \end{aligned} \tag{14}$$

where

$$[G,G](W,Z) = (\nabla_{GW}G)Z - (\nabla_{GZ}G)W - G(\nabla_W G)Z + G(\nabla_Z G)W$$

is the Nijenhuis torsion of G. In [7] the authors introduced concept of normality in which case the two tensor S and T are vanish. One of the important their result is that if M is normal then it is Keahlerian.

Definition 4. *[10] A complex contact metric manifold M is called normal if*

(1) $S(W,Z) = T(W,Z) = 0$ *for all* W,Z *in* $\mathscr{H}$, *and*
(2) $S(X,W) = T(Y,W) = 0$ *for all* W.

In real contact geometry, normality means the vanishing of the operator h. The following proposition gives the parallel result for complex contact geometry.

Proposition 1. *If M is normal, then $h_x = h_y = 0$[10].*

By the above proposition, on a normal complex contact metric manifold we have

$$\nabla_W X = -GW + \sigma(W)Y \tag{15}$$

and

$$\nabla_W Y = -HW - \sigma(W)X \tag{16}$$

3 $G-J$ anti-invariant submanifolds of almost complex contact metric manifolds

Assume that complex contact structure of $\overline{M}$ is defined by $\left(\overline{M},\overline{X},\overline{Y},\overline{x},\overline{y},\overline{g},\overline{H}=\overline{G}J\right)$. If M is a submanifold of $\overline{M}$, that is, according to J and $\overline{G}$,

$$JT_mM \subseteq T_mM^{\perp} \tag{17}$$

and

$$GT_mM \subseteq T_mM^{\perp}. \tag{18}$$

In this case M is $G-J$ anti- invariant submanifold of $\overline{M}$. Then from(17) and (18), we have

$$JT_mM \cap T_mM^{\perp} \neq \varnothing$$

and

$$GT_mM \cap T_mM^{\perp} \neq \varnothing.$$

If we get $JT_mM \cap GT_mM \neq \varnothing$,we define μ orthogonal distribution to $JT_mM \oplus GT_mM$ in $T_mM^{\perp}$. In this case, we can write

$$T_mM^{\perp} = JT_mM \oplus GT_mM \oplus \mu.$$

Since M is anti-invariant submanifold according to G and J , we can get

$$\overline{G}JW = -JGW \tag{19}$$

for all $W \in \Gamma(TM)$. Also we apply $\overline{G}$ to both side of (19), then we can find

$$\overline{G}^2JW = JG^2W$$

and from (2), we obtain

$$\overline{G}^2JW = JG^2W = -JW + \overline{g}(JW,\overline{X})\overline{X} + \overline{g}(JW,\overline{Y})\overline{Y}.$$

If structure vector fields $\overline{X},\overline{Y}$ of $\overline{M}$ are tangent to M, we find $G^2=-I$ for $W \in T_mM$. If structure vector field $\overline{X},\overline{Y}$ of $\overline{M}$ are normal to M, we find $G^2=-I$ for $W \in T_mM$.

Let us assume that the structure vector fields $\overline{X},\overline{Y}$ are normal to M, in this case, we can take

$$\overline{X} = -JX \tag{20}$$

and

$$\overline{Y} = -JY \tag{21}$$

where X,Y are unit vector fields of M.Then by using (20) and (21) , we obtain

$$\overline{Y} = \overline{X}J \Rightarrow Y = XJ.$$

Let x and y be 1-form of M, then we write

$$\overline{x} = xJ, \overline{y} = yJ \tag{22}$$

therefore using (22) , we find

$$y = -xJ.$$

If we use $\bar{H} = \bar{G}J$, we obtain $H = GJ$,also, we find

$$\bar{y}(JW) = -y(W), \bar{x}(JW) = -x(W). \tag{23}$$

If we apply $\bar{G}$ to both side of (19), we find

$$G^2 = -I + x \otimes X + y \otimes Y$$
$$y(Y) = 1, x(X) = 1 \tag{24}$$
$$y(GW) = 0, x(GW) = 0 \tag{25}$$

and

$$GX = GY = HY = HX = 0 \tag{26}$$

via $\bar{G}\bar{X} = 0$. Then we have the following theorem.

Theorem 3. *Let M be a $G - J$ anti invariant submanifold of an almost Complex contact manifold $\bar{M}$, such that $\bar{X}$ and $\bar{Y}$ are normal to M. Then M has almost complex contact structure $(M, X, Y, x, y, H = GJ, g)$.*

From Gauss formula, we can write

$$\bar{\nabla}_W GZ = \left(\bar{\nabla}_W G\right) Z + G\left(\nabla_W Z\right) + GB(W,Z) \tag{27}$$

for all $W, GZ \in \Gamma\left(TM\right)$. Then, if we use

$$\bar{\nabla}_W GZ = -A_{GZ}W + D_W GZ, \tag{28}$$

from (27) and (28), we have

$$\left(\bar{\nabla}_W G\right) Z + G\left(\nabla_W Z\right) + GB(W,Z) = -A_{GZ}W + D_W GZ. \tag{29}$$

Assume that G is parallel according to induced connection, then from (29), we have

$$A_{GZ}W = 0. \tag{30}$$

From Gauss formula

$$\bar{\nabla}_{GW} Z = \nabla_{GW} Z + B(GW,Z)$$

and if we use

$$G\bar{\nabla}_W Z = \nabla_{GW} Z + B(GW,Z)$$

we can obtain

$$B(GW,Z) = GB(W,Z). \tag{31}$$

Also from Weingarten formula, we can write

$$\bar{\nabla}_{JW} GZ = -A_{GZ}JW + D_{JW} GZ.$$

Then from

$$G\nabla_{JW}Z + GB(JW,Z) = -A_{GZ}JW + D_{JW}GZ,$$

we can find

$$A_{GZ}JW = 0 \tag{32}$$

and from

$$\begin{aligned} \bar{\nabla}_{GZ}JW &= -A_{JW}GZ + D_{GZ}JW \\ J\nabla_{GZ}W + JB(GZ,W) &= -A_{JW}GZ + D_{GZ}JW, \end{aligned}$$

we have

$$A_{JW}GZ = 0. \tag{33}$$

The following equations give the relations between the curvature tensor fields of M and $\bar{M}$. Let R and $\bar{R}$ be curvature tensor fields of M and $\bar{M}$ respectively. Then we write

$$\begin{aligned} \bar{R}(JW,\bar{G}JW,\bar{G}JW,JW) &= \bar{g}\left(\bar{R}(JW,\bar{G}JW)\bar{G}JW,JW\right) \\ &= \bar{g}\left(\bar{\nabla}_{JW}\bar{\nabla}_{JGW}JGW,JW\right) - \bar{g}\left(\bar{\nabla}_{JGW}\bar{\nabla}_{JW}JGW,JW\right) \\ &- \bar{g}\left(\bar{\nabla}_{[JW,JGW]}JGW,JW\right). \end{aligned}$$

Here, using Gauss and Weingarten formulas, we can obtain

$$\begin{aligned} \bar{R}(JW,\bar{G}JW,\bar{G}JW,JW) &= -\bar{g}\left(\bar{\nabla}_{JW}A_{JGW}JGW,JW\right) + \bar{g}(\bar{\nabla}_{JW}D_{JGW}JGW,JW) \\ &+ \bar{g}\left(\bar{\nabla}_{JW}A_{JGW}JW,JW\right) - \bar{g}(\bar{\nabla}_{JW}D_{JW}JGW,JW) \\ &+ \bar{g}(A_{JGW}[JW,JGW],JW) - \bar{g}\left(D_{[JW,JGW]}JGW,JW\right) \end{aligned}$$

and if we use (27)-(33), then we can obtain

$$\begin{aligned} \bar{R}(JW,\bar{G}JW,\bar{G}JW,JW) &= R^{\perp}(JW,\bar{G}JW,\bar{G}JW,JW) - \bar{g}(A_{JW}JW,A_{JGW}JGW) \\ &+ \bar{g}(A_{JW}JGW,A_{JGW}JW). \end{aligned} \tag{34}$$

Also from Weingarten formula, we get

$$\bar{\nabla}_{JGW}JGW = -A_{JGW}JGW + D_{JGW}JGW$$

and from

$$\bar{\nabla}_{JGW}JGW = \left(\bar{\nabla}_{JGW}J\right)GW + J\bar{\nabla}_{JGW}GW,$$

we have

$$A_{JGW}JGW = 0. \tag{35}$$

If we use (35) in (34), we can find

$$\bar{R}(JW,\bar{G}JW,\bar{G}JW,JW) = R^{\perp}(JW,\bar{G}JW,\bar{G}JW,JW) + \bar{g}(A_{JW}JGW,A_{JGW}JW)$$

Theorem 4. *Let M be a $G-J$ anti invariant submanifold with almost complex contact metric structure of almost contact metric manifold $\bar{M}$. The relation between curvature tensor fields of M and $\bar{M}$ is given by*

$$\bar{R}\left(JW,\bar{G}JW,\bar{G}JW,JW\right)=R^{\perp}\left(JW,\bar{G}JW,\bar{G}JW,JW\right)+\bar{g}\left(A_{JW}JGW,A_{JGW}JW\right)$$

where A is shape operator of M.

Now we investigate relation between Nijenhuis tensor fields of M and $\bar{M}$. N and $\bar{N}$ be Nijenhuis tensor fields of M and $\bar{M}$ respectively.Assume that

$$[JW,JZ]=J[W,Z]. \tag{36}$$

Then we have

$$\bar{N}(W,Z)=\left[\bar{G}W,\bar{G}Z\right]-\bar{G}\left[\bar{G}W,Z\right]-\bar{G}\left[W,\bar{G}Z\right]+\bar{G}^{2}[W,Z],$$
$$\bar{N}(JW,JZ)=\left[\bar{G}JW,\bar{G}JZ\right]-\bar{G}\left[\bar{G}JW,JZ\right]-\bar{G}\left[JW,\bar{G}JZ\right]+\bar{G}^{2}[JW,JZ].$$

Here, by using (19) and (36),we have

$$\begin{aligned}\bar{N}(JW,JZ)&=[-JGW,-JGZ]-\bar{G}[-JGW,JZ]-\bar{G}[JW,-JGZ]+\bar{G}^{2}J[W,Z]\\&=J[GW,GZ]-JG[GW,Z]-JG[W,GZ]+JG^{2}[W,Z]\\&=JN(W,Z).\end{aligned}$$

Therefore, we have the following theorem

Theorem 5. *Let M be a $G-J$ anti invariant submanifold with almost complex contact metric structure of almost contact metric manifold $\bar{M}$. The relation between Nijenhuis tensor fields of M and $\bar{M}$ is given by*

$$\bar{N}(JW,JZ)=JN(W,Z).$$

Now we investigate relation between S and $\bar{S}$ tensor fields of M and $\bar{M}$ respectively. If we apply $\bar{G}$ to both side of

$$\bar{H}^{2}JW=-JW+\bar{x}(JW)X+\bar{y}(JW)Y,$$

then, we obtain

$$\bar{H}JW=-JHW. \tag{37}$$

Also, we find

$$\bar{G}\bar{H}JZ=JGHZ \tag{38}$$

and from

$$\bar{G}JW=-\bar{H}W, \tag{39}$$

we obtain

$$\bar{G}JW=-GW \tag{40}$$

and

$$\sigma\left(\bar{G}JW\right)=-\sigma(GW). \tag{41}$$

If we utilize (22) and (37)-(41) in

$$\begin{aligned}\bar{S}(JW,JZ) &= \bar{N}(JW,JZ) + 2y(JZ)\bar{H}JW - 2\bar{y}(JW)\bar{H}JZ + 2\bar{g}\left(JW,\bar{G}JZ\right)\bar{X} \\ &- 2\bar{g}(JW,\bar{H}JZ)\bar{Y} - \sigma\left(\bar{G}JW\right)\bar{H}JZ + \sigma\left(\bar{G}JZ\right)\bar{H}JW \\ &+ \sigma(JW)\bar{G}\bar{H}JZ - \sigma(JZ)\bar{G}\bar{H}JW,\end{aligned}$$

we find

$$\begin{aligned}\bar{S}(JW,JZ) &= JN(W,Z) + 2y(Z)JHW - 2y(W)JHZ + 2\bar{g}(JW,JGZ)JX \\ &- 2\bar{g}(JW,JHZ)JY - \sigma(JGW)JHZ + \sigma(JGZ)JHW \\ &+ \sigma(JW)JGHZ - \sigma(JZ)JGHW.\end{aligned}$$

Then, we have

$$\bar{S}(JW,JZ) = J\begin{pmatrix} N(W,Z) + 2y(Z)HW - 2y(W)HZ + 2\bar{g}(W,GZ)X \\ -2\bar{g}(W,HZ)Y - \sigma(GW)HZ + \sigma(GZ)HW \\ \sigma(W)GHZ - \sigma(Z)GHW \end{pmatrix}.$$

Since $\bar{g}$ is Hermitian metric, we get

$$\bar{S}(JW,JZ) = J\bar{S}(W,Z).$$

Theorem 6. *Let M be a $G-J$ anti invariant submanifold with almost complex contact metric structure of almost contact metric manifold $\bar{M}$. The relation between Nijenhuis tensor fields of M and $\bar{M}$ is given by*

$$\bar{S}(JW,JZ) = J\bar{S}(W,Z).$$

Theorem 7. *Let M be a n-dimensional submanifold of $2n+2$- dimensional normal almost complex contact metric manifold $\bar{M}$. If the structure vector fields $\bar{X},\bar{Y}$ are normal to M, then M is totally geodesic if and only if M is $G-J$ anti invariant submanifold of $\bar{M}$.*

Proof. Since the structure vector fields $\bar{X},\bar{Y}$ are normal to M, Weingarten formula implies

$$\bar{\nabla}_W J\bar{X} = \nabla_W X = -A_X W + D_W X = -\bar{G}W + \sigma(W)Y.$$

From here, we find

$$\bar{g}\left(\bar{\nabla}_W J\bar{X}, Z\right) = -\bar{g}\left(J\bar{X}, \bar{\nabla}_W Z\right) = -\bar{g}\left(\bar{G}W, Z\right) = \bar{g}(X, B(W,Z)) = 0 \Rightarrow B(W,Z) = 0$$

for any W,Z on M.

Since M is $G-J$ anti-invariant submanifold of $\bar{M}$, from

$$\bar{\nabla}_W \bar{G}JZ = -\bar{\nabla}_W JGZ = A_{JGZ}W - D_W JGZ \tag{42}$$

we have

$$JA_{GZ}W - JD_W GZ = A_{JGZ}W - D_W JGY \tag{43}$$

and again if (42) is calculated, we find

$$-J((\bar{\nabla}_W G)Z + G\bar{\nabla}_W Z) = A_{JGZ}W - D_W JGZ.$$

If G is parallel, then we have

$$-JG\bar{\nabla}_W Z = A_{JGZ}W - D_W JGZ. \tag{44}$$

The same way, we find

$$\bar{\nabla}_W \bar{G}Z = -JG\nabla_W Z - JGB(W,Z) \tag{45}$$
$$\bar{\nabla}_Z \bar{G}W = -JG\nabla_Z W - JGB(Z,W) \tag{46}$$

If (43)-(44) is used in (45)-(46) , we can obtain

$$A_{JGZ}W = A_{JGW}Z.$$

Also, from

$$\bar{\nabla}_W GZ = -A_{GZ}W + D_W GZ,$$

if G is parallel, we can obtain

$$A_{GZ}W = 0.$$

In the same way, from

$$\bar{\nabla}_W JZ = -A_{JZ}W + D_W JZ$$
$$\bar{\nabla}_Z JW = -A_{JW}Z + D_Z JW$$

we can find

$$A_{JZ}W = A_{JW}Z = 0.$$

Namely, $A = 0$. From here we get the following corollary

Corollary 1. *Let M be a $G-J$ anti invariant submanifold with almost complex contact metric structure of almost contact metric manifold $\bar{M}$. Then M is totally geodesic submanifold and totally umbilical submanifold of $\bar{M}$. Also we have*

$$\bar{\nabla}_X X = -A_X X + D_X X = \sigma(X)Y \Rightarrow A_X X = 0$$
$$\bar{\nabla}_Y Y = -A_Y Y + D_Y Y = -\sigma(Y)X \Rightarrow A_Y Y = 0$$

for X,Y is structure vector fields of M.

4 Conclusion

In this study, submanifolds of almost complex contact metric manifolds are investigated. Firstly, the case of being anti-invariant according to J and $\bar{G}$ is obtained. The main point in this study is that the structure vector fields are not tangent to TM. In this case, we have found relations between Nijenhuis tensor fields of almost complex contact metric submanifolds of this almost complex contact manifolds with almost complex contact manifolds as well as relation between curvature tensors of these two manifolds. Next, we have showed that structure vector fields must not be tangent to TM. In this case, conditions to be total umbilical and totally geodesic $\overline{G}-J$ anti-invariant submanifold of almost complex contact metric manifolds are given. Finally we give condition to be parallel with respect to the induced connection on $\overline{G}-J$ anti-invariant submanifold of structure vector fields.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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