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ON GEOMETRİC PROPERTIES OF WEIGHTED LEBESGUE SEQUENCE SPACES

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Abstract

In this paper we introduce some geometrical and topological properties of weighted Lebesgue sequence spaces $l_{p,w}$ as a generalization of the Lebesgue sequences spaces l_p , where w a weighted sequence.

Keywords: Striclty Convexity; Uniformly Convexity ; Weighted Lebesgue Sequence Spaces.

1. Introduction

If $1 \leq p < \infty$, then l_p will denote the space of sequences of real numbers $x = (x_n)$ such that $\sum_{n=1}^{\infty} |x_n|^p < \infty$ [2,8] . A weight sequence $w = w(n) = w_n$ is a positive decreasing sequence such that $w(1) = 1$, $\lim_{n\to\infty} w_n = 0$ and $\sum_{n=1}^{\infty} w_n$ divergent. The weighted Lebesgue sequence space $l_{p,w}$ for $0 < p < \infty$ is defined as follows:

$$l_{p,w} = \left\{ x = (x_n): \sum_{n=1}^{\infty} w_n |x_n|^p < \infty, (x_n) \subset \mathbb{R} \right\}$$

and

$$\|x\|_{p,w} = \left(\sum_{n=1}^{\infty} w_n |x_n|^p \right)^{1/p} \tag{1}$$

where $p \geq 1$.

In other words, the weighted sequence space is defined the weight as a multiplier. That is $x \in l_{p,w} \Leftrightarrow x w^{1/p} \in l_p$ weighted sequence spaces $l_{p,w}$ which is considered by author in [9],[10] . It is known that $l_{p,w}$ a Banach space.

A Banach space X is said to be *strictly convex* if $x, y \in X$with $\|x\| = 1, \|y\| = 1$ and $x \neq y$, then $\|(1-\lambda)x + \lambda y\| < 1$ for all $\lambda \in (0,1)$. A Banach space X is said to be *uniformly convex* if the conditions

$$\|x\| \leq 1, \|y\| \leq 1 \text{ and } \|x - y\| \geq \varepsilon \text{ imply } \left\| \frac{x+y}{2} \right\| \leq 1 - \delta \tag{2}$$

holds for all $x, y \in X$. The number

$$\delta(\varepsilon) = inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| = 1, \|y\| = 1\, , \|x-y\| \geq \varepsilon\right\} \tag{3}$$

is called the *modulus of convexity*. If $\varepsilon_1 < \varepsilon_2$, then $\delta(\varepsilon_1) < \delta(\varepsilon_2)$ and $\delta(0) = 0$ since $x = y$ if $\varepsilon = 0$[1]. Recently there has been a lot of interest in investigating geometric properties of sequence spaces besides topological. The geometric properties of different sequence spaces are discusssed by some authors. Agarwal, O'regan&Sahu [1] and Castillo&Rafeiro [2] have studied the strict convexity and uniform convexity properties of sequence spaces l_p where $1 < p < \infty$. Savaş, Karakaya and Şimşek [11] have studied some geometric properties of l(p)- type new sequence spaces. Oğur, O [7] has studied some geometric properties of weighted function spaces $L_{p,w}(G)$ where $1 < p < \infty$. In this paper, we introduce some geometric properties of topological of weighted sequence spaces $l_{p,w}$ as a generalization of the l_p .

We will need some auxiliary lemmas to prove that the spaces $l_{p,w}$ are uniformly convex whenever $1 < p < \infty$.

Proposition 1. (Hölder Inequality) Let $x = (x_n) \in l_p, y = (y_n) \in l_q$ and $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\sum_{k=1}^{\infty} |x_k y_k| \leq \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p} + \left(\sum_{k=1}^{\infty} |y_k|^q\right)^{1/q} \tag{4}$$

Proposition 2. (Minkowski Inequality) Let $x = (x_n),\ y = (y_n) \in l_p$, If $p \in [1, \infty)$, then

$$\left(\sum_{k=1}^{\infty} (|x_k| + |y_k|)^p\right)^{1/p} \leq \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p} + \left(\sum_{k=1}^{\infty} |y_k|^p\right)^{1/p} \tag{5}$$

If $p \in (0,1)$, then

$$\left(\sum_{k=1}^{\infty} (|x_k| + |y_k|)^p\right)^{1/p} \geq \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p} + \left(\sum_{k=1}^{\infty} |y_k|^p\right)^{1/p} \tag{6}$$

We need some lemmas dealing with inequalities.

Lemma 1. Let $0 < p < 1$, we have

$$(a+b)^p \leq a^p + b^p \tag{7}$$

for $a \geq 0$, $b \geq 0$ [8].

Lemma 2. If $p \geq 1$ and $a, b > 0$, then

$$(a+b)^p \leq 2^{p-1}(a^p + b^p) \tag{8}$$

[6].

2. Main Results

Proposition 3. Let $w = (w_k)$ a weighted sequence and $w_k > 1$ for all $k \in \mathbb{N}$. Then $l_{p,w} \subset l_p$. Also if $0 < p < q < \infty$, $l_{p,w} \subsetneq l_{q,w}$ for $w_k > 1$.

Proof. It can be easily seen that $l_{p,w} \subset l_p$ and $l_{p,w} \subset l_{q,w}$ for $0 < p < q < \infty$ .To show that $l_{p,w} \neq l_{q,w}$, we take the sequences $x_k = k^{-1/2p}$ and $w_k = \frac{1}{\sqrt{k}}$ for all $k \in \mathbb{N}$ with $1 \leq p < q < \infty$. Since $p < q$, we have $\frac{q}{p} > 1$ and $\frac{q}{2p} + \frac{1}{2} > 1$. We write

$$\sum_{k=1}^{\infty} w_k |x_k|^q = \sum_{k=1}^{\infty} \frac{1}{k^{1/2}} . \frac{1}{k^{q/2p}} = \sum_{k=1}^{\infty} \frac{1}{k^{q/2p+1/2}} < \infty$$

The last series is convergent since it is a hyper-harmonic series with exponent bigger than 1, therefore $x \in l_{q,w}$. On the other hand

$$\sum_{k=1}^{\infty} w_k |x_k|^p = \sum_{k=1}^{\infty} \frac{1}{k^{1/2}} . \frac{1}{k^{1/2}} = \sum_{k=1}^{\infty} \frac{1}{k}$$

and $x \notin l_{p,w}$.

Proposition 4. The space $l_{p,w}$ is seperable whenever $1 \leq p < \infty$ and w a weighted sequence.

Proof. Let M be the set of all sequences of the form $q = (q_1, q_2, \cdots, q_n, 0,0, \cdots)$ where $n \in \mathbb{N}$ and $q_k \in \mathbb{Q}$. We will show that M is dense in $l_{p,w}$. Since $\sum_{k=1}^{\infty} |x_k|^p w_k$ there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\sum_{k=n+1}^{\infty} |x_k|^p w_k < \frac{\varepsilon^p}{2}$$

for all $\varepsilon > 0$. Since $\overline{\mathbb{Q}} = \mathbb{R}$, we have that for each (x_k) there exists a rational q_k such that

$$|x_k - q_k| < \frac{\varepsilon}{\sqrt[p]{2^n}}$$

hence

$$\sum_{k=1}^{n} |x_k - q_k| w_k < \frac{\varepsilon^p}{2K}$$

where $K = maks\{w_1, w_2, \cdots, w_n\}$. We write

$$\|x - q\|_{p,w}^p = \sum_{k=1}^{n} |x_k - q_k|^p w_k + \sum_{k=n+1}^{\infty} |x_k|^p w_k < \varepsilon^p$$

and so $\|x - q\|_{p,w} < \varepsilon$. This shows that M is dense in $l_{p,w}$.

Theorem 1. The space $l_{p,w}$ is convex, whenever $0 < p < \infty$.

Proof. This show that $tx + (1-t)y \in l_{p,w}$ for $x = (x_n), y = (y_n) \in l_{p,w}$ and $t \in [0,1]$. Let us distinguish two cases:

First case $p \geq 1$. By Lemma 2 and Minkowski's inequality , we write

$$\sum_{n=1}^{\infty} |tx_n + (1-t)y_n|^p w_n = \sum_{n=1}^{\infty} \left|(tx_n + (1-t)y_n){w_n}^{1/p}\right|^p$$

$$= \left[\left(\sum_{n=1}^{\infty} \left|(tx_n + (1-t)y_n){w_n}^{1/p}\right|^p\right)^{1/p}\right]^p$$

$$\leq \left[\left(\sum_{n=1}^{\infty} \left|(tx_n){w_n}^{1/p}\right|^p\right)^{1/p} + \left(\sum_{n=1}^{\infty} \left|\big((1-t)y_n\big){w_n}^{1/p}\right|^p\right)^{1/p}\right]^p$$

$$\leq 2^{p-1}\left[\sum_{n=1}^{\infty} \left|(tx_n){w_n}^{1/p}\right|^p + \sum_{n=1}^{\infty} \left|\big((1-t)y_n\big){w_n}^{1/p}\right|^p\right]$$

$$= 2^{p-1}\sum_{n=1}^{\infty} |tx_n|^p w_n + 2^{p-1}\sum_{n=1}^{\infty} |(1-t)y_n|^p\, w_n$$

$$= 2^{p-1}|t|^p\sum_{n=1}^{\infty} |x_n|^p w_n + 2^{p-1}|1-t|^p\sum_{n=1}^{\infty} |y_n|^p w_n$$

$$< \infty$$

which shows that $tx + (1-t)y \in l_{p,w}$ for $p \geq 1$.

Second case $0 < p < 1$. Let $x = (x_n), y = (y_n) \in l_{p,w}$ and $t \in [0,1]$. By Lemma 1, we have

$$\sum_{n=1}^{\infty} |tx_n + (1-t)y_n|^p w_n = \sum_{n=1}^{\infty} \left|(tx_n + (1-t)y_n){w_n}^{1/p}\right|^p$$

$$\leq \sum_{n=1}^{\infty} \left|(tx_n){w_n}^{1/p}\right|^p + \sum_{n=1}^{\infty} \left|\big((1-t)y_n\big){w_n}^{1/p}\right|^p$$

$$= \sum_{n=1}^{\infty} |tx_n|^p w_n + \sum_{n=1}^{\infty} \left|\big((1-t)y_n\big)\right|^p w_n$$

$$= |t|^p\sum_{n=1}^{\infty} |x_n|^p w_n + |1-t|^p\sum_{n=1}^{\infty} |y_n|^p\, w_n < \infty$$

This completes the proof. It is known that the space l_p is strictly convex for $p \geq 1$ [1].

Theorem 2. The space $l_{p,w}$ is strictly convex for $p \geq 1$.

Proof. Let $x = (x_n), y = (y_n) \in l_{p,w}$ with $x \neq y$, $\|x\|_{p,w} = 1, \|y\|_{p,w} = 1$ and $0 < p < 1$. Then $\left\|xw^{\frac{1}{p}}\right\|_p = 1$, $\left\|yw^{\frac{1}{p}}\right\|_p = 1$. Since l_p is strictly convex for $p \geq 1$, we have

$$\left\|(1-t)xw^{\frac{1}{p}} + tyw^{\frac{1}{p}}\right\|_p = \left\|\big((1-t)x + ty\big)w^{\frac{1}{p}}\right\|_p < 1.$$

Hence

$$\|(1-t)x+ty\|_{p,w} = \left(\sum_{n=1}^{\infty}\left|\left((1-t)x+ty\right)w^{\frac{1}{p}}\right|^{p}\right)^{1/p}$$

$$= \left\|\left((1-t)x+ty\right)w^{\frac{1}{p}}\right\|_{p} < 1$$

We will need the following inequality.

Lemma 3. Let $p \geq 2$. We have

$$(|a+b|^p + |a-b|^p)^{1/p} \leq (|a+b|^2 + |a-b|^2)^{1/2} \tag{9}$$

for all $a, b \in \mathbb{R}$ [2].

Lemma 4. Let $2 \leq p < \infty$ and $x, y \in l_p$, we have

$$\|x+y\|_p^p + \|x-y\|_p^p \leq 2^{p-1}\left(\|x\|_p^p + \|y\|_p^p\right) \tag{10}$$

[1].

Proposition 5. If $2 \leq p < \infty$, then we have

$$\|x+y\|_{p,w}^p + \|x-y\|_{p,w}^p \leq 2^{p-1}\left(\|x\|_{p,w}^p + \|y\|_{p,w}^p\right) \tag{11}$$

for $x = (x_n)$, $y = (y_n) \in l_{p,w}$.

Proof. Let $x, y \in l_{p,w}$. Then $xw^{\frac{1}{p}}, yw^{\frac{1}{p}} \in l_p$. By Lemma 4, we write

$$\|x+y\|_{p,w}^p + \|x-y\|_{p,w}^p = \left\|xw^{\frac{1}{p}} + yw^{\frac{1}{p}}\right\|_p^p + \left\|xw^{\frac{1}{p}} - yw^{\frac{1}{p}}\right\|_p^p$$

$$\leq 2^{p-1}\left(\left\|xw^{\frac{1}{p}}\right\|_p^p + \left\|yw^{\frac{1}{p}}\right\|_p^p\right)$$

$$= 2^{p-1}\left(\|x\|_{p,w}^p + \|y\|_{p,w}^p\right)$$

Theorem 3. The space $l_{p,w}$ is uniformly convex for $2 \leq p < \infty$.

Proof . Let $x = (x_n), y = (y_n) \in l_{p,w}$ with

$$\|x\|_{p,w} \leq 1, \|y\|_{p,w} \leq 1 \text{ and } \|x-y\|_{p,w} \geq \varepsilon$$

By Proposition 5, we have

$$\|x+y\|_{p,w}^p \leq 2^{p-1}\left(\|x\|_{p,w}^p + \|y\|_{p,w}^p\right) - \|x-y\|_{p,w}^p$$

$$\leq 2^{p-1}.2 - \varepsilon^p$$

$$= 2^p\left(1 - \left(\frac{\varepsilon}{2}\right)^p\right)$$

so it follows that $\left\|\frac{x+y}{2}\right\|_{p,w}^{p} \leq 1-\left(\frac{\varepsilon}{2}\right)^{p}$ and hence we get $\left\|\frac{x+y}{2}\right\|_{p,w} \leq 1-\delta$ such that

$$\delta(\varepsilon)=1-\left(1-\left(\frac{\varepsilon}{2}\right)^{p}\right)^{1/p}$$

Lemma 6. Let $1<p\leq 2$ and $q=\frac{p}{p-1}$, then

$$|a+b|^{q}+|a-b|^{q} \leq 2(|a|^{p}+|b|^{p})^{q-1} \tag{12}$$

for all real numbers a and b [3].

Lemma 7. $1<p\leq 2$ and $q=\frac{p}{p-1}$, we have

$$\|x+y\|_{p}^{q}+\|x-y\|_{p}^{q} \leq 2\left(\|x\|_{p}^{p}+\|y\|_{p}^{p}\right)^{q-1} \tag{13}$$

for all $x,y\in l_{p}$ [5].

Proposition 6. If $1<p\leq 2$, then

$$\|x+y\|_{p,w}^{q}+\|x-y\|_{p,w}^{q} \leq 2\left(\|x\|_{p,w}^{p}+\|y\|_{p,w}^{p}\right)^{q-1} \tag{14}$$

for $x=(x_n)$, $y=(y_n)\in l_{p,w}$ and $q=\frac{p}{p-1}$.

Proof . Let $x=(x_n), y=(y_n)\in l_{p,w}$ and by the Minkowski's inequality for $0<r<1$, we have

$$\left(\sum_{n=1}^{\infty}|a_n|^{r}\right)^{1/r}+\left(\sum_{n=1}^{\infty}|b_n|^{r}\right)^{1/r} \leq \left(\sum_{n=1}^{\infty}|a_n+b_n|^{r}\right)^{1/r} \tag{15}$$

If $1<p\leq 2$, we replace r by $\frac{p}{q}$ in Equation (15) , for $a_n=\left|\left((x_n+y_n){w_n}^{1/p}\right)\right|^{q}$, $b_n=\left|(x_n-y_n){w_n}^{1/p}\right|^{q}$, then by Lemma 6 we get

$$\left(\sum_{n=1}^{\infty}\left|(x_n+y_n){w_n}^{1/p}\right|^{p}\right)^{q/p}+\left(\sum_{n=1}^{\infty}\left|(x_n-y_n){w_n}^{1/p}\right|^{p}\right)^{q/p}$$

$$\leq\left[\sum_{n=1}^{\infty}\left(\left|(x_n+y_n){w_n}^{1/p}\right|^{q}+\left|(x_n-y_n){w_n}^{1/p}\right|^{q}\right)^{p/q}\right]^{q/p}$$

$$=\left[\sum_{n=1}^{\infty}\left(\left|x_n{w_n}^{1/p}+y_n{w_n}^{1/p}\right|^{q}+\left|x_n{w_n}^{1/p}-y_n{w_n}^{1/p}\right|^{q}\right)^{p/q}\right]^{q/p}$$

$$\leq\left(\sum_{n=1}^{\infty}\left[2\left(\left|x_n{w_n}^{1/p}\right|^{p}+\left|y_n{w_n}^{1/p}\right|^{p}\right)^{q-1}\right]^{p/q}\right)^{q/p}$$

$$=2\left[\sum_{n=1}^{\infty}\left(\left|x_n{w_n}^{1/p}\right|^{p}+\left|y_n{w_n}^{1/p}\right|^{p}\right)\right]^{q/p}$$

$$= 2\left[\sum_{n=1}^{\infty}|x_n|^p w_n + \sum_{n=1}^{\infty}|y_n|^p w_n\right]^{q/p}$$

where $q = \frac{p}{p-1} \Rightarrow q - 1 = \frac{q}{p}$. Thus , we obtain

$$\|x+y\|_{p,w}^q + \|x-y\|_{p,w}^q \leq 2\left(\|x\|_{p,w}^p + \|y\|_{p,w}^p\right)^{q-1}$$

Theorem 4. The space $l_{p,w}$ is uniformly convex for $1 < p \leq 2$.

Proof. Let $x = (x_n), y = (y_n) \in l_{p,w}$, $1 < p \leq 2$ with

$$\|x\|_{p,w} \leq 1, \|y\|_{p,w} \leq 1 \text{ and } \|x-y\|_{p,w} \geq \varepsilon$$

Then by the Proposition 6, we have

$$\begin{aligned}\|x+y\|_{p,w}^q &\leq 2\left[\|x\|_{p,w}^p + \|y\|_{p,w}^p\right]^{q-1} - \|x-y\|_{p,w}^q \\ &\leq 2.2^{q-1} - \varepsilon^q \\ &= 2^q\left(1 - \left(\frac{\varepsilon}{2}\right)^q\right)\end{aligned}$$

Hence, we write

$$\left\|\frac{x+y}{2}\right\|_{p,w} \leq \left(1 - \left(\frac{\varepsilon}{2}\right)^q\right)^{1/q}$$

where $\delta(\varepsilon) = 1 - \left(1 - \left(\frac{\varepsilon}{2}\right)^q\right)^{1/q}$.

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