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Generalized (k,μ)-Space forms and Ricci solitons

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Abstract
In this paper, we study Ricci-semisymmetric and Ricci pseudo-symmetric generalized (k,μ)-space forms along with characterization of generalized (k,μ)-space forms satisfying the curvature conditions $Q(g,S)=0$ and $Q(S,R)=0$. Further, we study Ricci solitons in generalized (k,μ)-space forms and obtained some interesting results.

Keywords and 2010 Mathematics Subject Classification
Keywords: Generalized (k,μ)-Space form, Ricci-semisymmetric, Ricci pseudosymmetric, Ricci solitons, shrinking, expanding, steady.
MSC: 53D10, 53D15

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1. Introduction

In [1], the authors generalized the notion of Sasakian space form defined generalized Sasakian space form as a contact metric manifold (M,ϕ,ξ,η,g) whose curvature tensor R satisfies

$$\begin{aligned} R(X,Y)Z &= f_1\{g(Y,Z)X-g(X,Z)Y\} \\ &+ f_2\{g(X,\phi Z)\phi Y-g(Y,\phi Z)\phi X+2g(X,\phi Y)\phi Z\} \\ &+ f_3\{\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X+g(X,Z)\eta(Y)\xi-g(Y,Z)\eta(X)\xi\}, \end{aligned} \tag{1}$$

for any vector fields X,Y,Z, where f_1,f_2,f_3 are smooth functions on M.
As a generalization of the notion of (k,μ)-space form, Carriazo et al [4] introduced generalized (k,μ)-space form as a contact metric manifold (M,ϕ,ξ,η,g) whose curvature tensor R satisfies

$$\begin{aligned} R(X,Y)Z &= f_1\{g(Y,Z)X-g(X,Z)Y\} \\ &+ f_2\{g(X,\phi Z)\phi Y-g(Y,\phi Z)\phi X+2g(X,\phi Y)\phi Z\} \\ &+ f_3\{\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X+g(X,Z)\eta(Y)\xi-g(Y,Z)\eta(X)\xi\} \\ &+ f_4\{g(Y,Z)hX-g(X,Z)hY+g(hY,Z)X-g(hX,Z)Y\} \\ &+ f_5\{g(hY,Z)hX-g(hX,Z)hY+g(\phi hX,Z)\phi hY-g(\phi hY,Z)\phi hX\} \\ &+ f_6\{\eta(X)\eta(Z)hY-\eta(Y)\eta(Z)hX+g(hX,Z)\eta(Y)\xi-g(hY,Z)\eta(X)\xi\}, \end{aligned} \tag{2}$$

where f_1,f_2,f_3,f_4,f_5,f_6 are smooth functions on M and $2h=L_\xi\phi$, L is the usual Lie derivative. They proved that the generalized Sasakian space form and the generalized (k,μ)-space form share some properties and identities in common. Further the authors established that the generalized (k,μ)-space forms reduce to generalized (k,μ) spaces for $k=f_1-f_3$, $\mu=f_4-f_6$ and to (k,μ) spaces greater than or equal to 5 with $k=-f_6$ and $\mu=1-f_6$. (k,μ)-space form have been studied widely by several authors like [3, 13, 7, 19, 18, 21, 23] and various others.
Let (M,g) be a Riemannian manifold with the Riemannian metric ∇. A tensor field $F:\chi(M)\times\chi(M)\times\chi(M)\longrightarrow\chi(M)$ of type $(1,3)$ is said to be curvature-like if it has the properties of R. For example, the tensor R given by

$$R(X,Y)Z=(X\wedge_A Y)Z=A(Y,Z)X-A(X,Z)Y, \tag{3}$$

where $X,Y,Z \in \chi(M), \chi(M)$ is the set of all differentiable vector fields on M, A is the symmetric $(0,2)$-tensor, R is the Riemannian curvature tensor of type $(1,3)$ and ∇ is the Levi-Civita connection. For a $(0,k)$-tensor field T, $k \leq 1$, on (M,g), we define the tensor $R \cdot T$ and $Q(g,T)$ by

$$(R(X,Y)\cdot T)(X_1,X_2,.....,X_k) = -T(R(X,Y)X_1,X_2,X_3,.....,X_k) - T(X_1,R(X,Y)X_2,X_3,.....,X_k) \\ ...... - T(X_1,X_2,.....,R(X,Y)X_k) \tag{4}$$

and

$$Q(g,T)(X_1,X_2,.....,X_k,Y) = -T((X\wedge Y)X_1,X_2,X_3,.....,X_k) - T(X_1,(X\wedge Y)X_2,X_3,.....,X_k) \\ ...... - T(X_1,X_2,.....,(X\wedge Y)X_k), \tag{5}$$

respectively [24]. If the tensors $(R\cdot S)$ and $Q(g,S)$ are linearly dependent, then M is called Ricci pseudo-symmetric [24]. Which is equivalent to

$$(R\cdot S) = fQ(g,S), \tag{6}$$

holding on the set $U_S = \{x \in M : S \neq 0 \text{ at } x\}$, where f is some function on U_S. Also if the tensors $R\cdot R$ and $Q(S,R)$ are linearly dependent, then M is said to be Ricci generalized pseudo-symmetric [24]. This is equivalent to

$$R\cdot R = fQ(S,R). \tag{7}$$

In [12], Kowalczyk studied semi-Riemannian manifolds satisfying $Q(S,R) = 0$ and $Q(g,S) = 0$, where S,R are the Ricci tensor and curvature tensor respectively. De et al. [6, 14] studied Ricci pseudo-symmetric and Ricci generalized pseudo-symmetric P-sasakian manifolds and generalized (k,μ)-paracontact metric manifolds.
Ricci soliton, introduced by Hamilton [8] are natural generalizations of the Einstein metrics and is defined on a Riemannian manifold (M,g). A Ricci soliton (g,V,λ) defined on (M,g) as

$$(L_V g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) = 0, \tag{8}$$

where L_V denotes the Lie-derivative of Riemannian metric g along a vector field V, λ be a consant and X,Y are arbitrary vector fields on M. A Ricci soliton is said to shrinking or steady or expanding to the extent that λ is negative, zero or positive respectively. Ricci solitons have been considered broadly with regards to contact geometry; we may refer to [22, 5, 20, 9, 16, 17, 15, 11] and references therein.

The paper is organized as follows: The section 2 contains some basic results on almost contact geometry and generalized (k,μ)-space forms. Section 3 deals with the curvature conditions like Ricci-semisymmetric, Ricci pseudo-symmetric, $Q(g,S) = 0$ and $Q(S,R) = 0$ on generalized (k,μ)-space forms. Also we study Ricci solitons in generalized (k,μ)-space forms and obtained some interesting results.

2. Preliminaries

In this section, we recall some general definitions and fundamental equations are presented which will be utilized later. A $(2n+1)$-dimensional smooth manifold M is said to be contact if it has a global 1-form η such that $\eta\wedge(d\eta)^n \neq 0$ on M. Given a contact 1-form η there always exists a unique vector field ξ such that $(d\eta)(\xi,X) = 0$. Polarization of $d\eta$ on the contact subbundle D (defined by $D = 0$), yields a Riemannian metric g and a $(1,1)$-tensor field ϕ such that

$$\phi^2 X = -X + \eta(X)\xi, \;\; \phi\xi = 0, \;\; g(X,\xi) = \eta(X), \;\; \eta(\xi) = 1, \;\; \eta\circ\phi = 0, \tag{9}$$

$$g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{10}$$

$$g(X,\phi Y) = d\eta(X,Y), \quad g(X,\phi Y) = -g(Y,\phi X), \tag{11}$$

for all vector fields X,Y on M. In a contact metric manifold, we characterize a $(1,1)$ tensor field h by $h = \frac{1}{2}L_\xi\phi$, where L signifies the Lie differentiation. At this point h is symmetric and satisfies $h\phi = -\phi h$. Likewise we have $Tr\cdot h = Tr\cdot\phi h = 0$

and $h\xi = 0$.
Moreover, if ∇ signifies the Riemannian connection of g, then the following relation holds:

$$\nabla_X \xi = -\phi X - \phi hX. \tag{12}$$

In a (k,μ)-contact metric manifold the following relations hold [2] [10];

$$h^2 = (k-1)\phi^2, \quad k \leq 1, \tag{13}$$

$$(\nabla_X \phi)Y = g(X+hX,Y)\xi - \eta(Y)(X+hX), \tag{14}$$

$$\begin{aligned}(\nabla_X h)Y &= [(1-k)g(X,\phi Y) - g(X,\phi hY)]\xi \\ &- \eta(Y)[(1-k)\phi X + \phi hX] - \mu\eta(X)\phi hY.\end{aligned} \tag{15}$$

Also in a $(2n+1)$-dimensional generalized (k,μ)-space form, the following relations hold.

$$\begin{aligned}R(X,Y)\xi &= (f_1-f_3)\{\eta(Y)X - \eta(X)Y\} \\ &+ (f_4-f_6)\{\eta(Y)hX - \eta(X)hY\},\end{aligned} \tag{16}$$

$$\begin{aligned}QX &= \{2nf_1+3f_2-f_3\}X + \{(2n-1)f_4-f_6\}hX \\ &- \{3f_2+(2n-1)f_3\}\eta(X)\xi,\end{aligned} \tag{17}$$

$$\begin{aligned}S(X,Y) &= \{2nf_1+3f_2-f_3\}g(X,Y) + \{(2n-1)f_4-f_6\}g(hX,Y) \\ &- \{3f_2+(2n-1)f_3\}\eta(X)\eta(Y),\end{aligned} \tag{18}$$

$$S(X,\xi) = 2n(f_1-f_3)\eta(X), \tag{19}$$

$$r = 2n\{(2n+1)f_1+3f_2-2f_3\}, \tag{20}$$

where Q is the Ricci operator, S is the Ricci tensor and r is the scalar curvature of $M(f_1.....,f_6)$.

3. Generalized (k,μ)-Space forms and Ricci solitons

A generalized (k,μ)-space form is said to be Ricci-semisymmetric if its Ricci tensor S satisfies the condition $R\cdot S = 0$. Then we have

$$S(R(X,Y)U,V) + S(U,R(X,Y)V) = 0. \tag{21}$$

Taking $X = U = \xi$ in the equation (21), we get

$$S(R(\xi,Y)\xi,V) + S(\xi,R(\xi,Y)V) = 0. \tag{22}$$

Using (16) and (19) in (22), we obtain

$$\begin{aligned}&(f_1-f_3)\{2n(f_1-f_3)g(Y,V) - S(Y,V)\} \\ &+(f_4-f_6)\{2n(f_1-f_3)g(hY,V) - S(hY,V)\} = 0.\end{aligned} \tag{23}$$

Replacing Y by hY in (23) and using (13), we get

$$\begin{aligned}&(f_1-f_3)\{2n(f_1-f_3)g(hY,V) - S(hY,V)\} \\ &-(k-1)(f_4-f_6)\{2n(f_1-f_3)g(Y,V) - S(Y,V)\} = 0.\end{aligned} \tag{24}$$

Eliminating $g(hY,V)$ and $S(hY,V)$ from (23) and (24), we get

$$\{(k-1)(f_4-f_6)^2 + (f_1-f_3)^2\}\{2n(f_1-f_3)g(Y,V) - S(Y,V)\} = 0. \tag{25}$$

Now for $k=1$, either $f_1 = f_3$ or $S(Y,V) = 2n(f_1-f_3)g(Y,V)$.
On the other hand for $k<1$, either $S(Y,V) = 2n(f_1-f_3)g(Y,V)$ or

$$(f_1-f_3)^2 = (1-k)(f_4-f_6)^2. \tag{26}$$

Then from (26), we have $f_1 = f_3$ implies $f_4 = f_6$.
Thus from the above discussions we state the following:

Theorem 1. *If a $(2n+1)$-dimensional generalized (k,μ)-space form $M(f_1.....,f_6)$ with $f_1 \neq f_3$ is Ricci-semisymmetric, then space form is an Einstein manifold.*

Suppose the generalized (k,μ)-space form satisfying the curvature condition $Q(S,R)=0$. Then we have

$$(X \wedge_S Y \cdot R)(U,V)W = 0. \tag{27}$$

Using (7) in (27), we obtain

$$\begin{aligned} &S(Y,R(U,V)W)X - S(X,R(U,V)W)Y - S(Y,U)R(X,V)W \\ &+ S(X,U)R(Y,V)W - S(Y,V)R(U,X)W + S(X,V)R(U,Y)W \\ &- S(Y,W)R(U,V)X + S(X,W)R(U,V)Y = 0. \end{aligned} \tag{28}$$

Replacing $X = U = \xi$ in (28), we get

$$\begin{aligned} &S(Y,R(\xi,V)W)\xi - S(\xi,R(\xi,V)W)Y - S(Y,\xi)R(\xi,V)W \\ &+ S(\xi,\xi)R(Y,V)W - S(Y,V)R(\xi,\xi)W + S(\xi,V)R(\xi,Y)W \\ &- S(Y,W)R(\xi,V)\xi + S(\xi,W)R(\xi,V)Y = 0. \end{aligned} \tag{29}$$

Using (16) and (19) in (29), we obtain

$$\begin{aligned} &-(f_1-f_3)\eta(W)S(Y,V)\xi - (f_4-f_6)\eta(W)S(Y,hV)\xi \\ &-2n(f_1-f_3)^2 g(V,W)Y - 2n(f_1-f_3)(f_4-f_6)g(V,hW)Y \\ &+2n(f_1-f_3)R(Y,V)W + 2n(f_1-f_3)^2 g(Y,W)\eta(V)\xi \\ &+2n(f_1-f_3)(f_4-f_6)g(Y,hW)\eta(V)\xi - (f_1-f_3)S(Y,W)\eta(V)\xi \\ &-2n(f_1-f_3)(f_4-f_6)\eta(V)\eta(W)hY + (f_1-f_3)S(Y,W)V \\ &+(f_4-f_6)S(Y,W)hV + 2n(f_1-f_3)^2 g(V,Y)\eta(W)\xi \\ &+2n(f_1-f_3)(f_4-f_6)g(V,hY)\eta(W)\xi = 0. \end{aligned} \tag{30}$$

Taking inner product with Z, we obtain

$$\begin{aligned} &-(f_1-f_3)\eta(W)S(Y,V)\eta(Z) - (f_4-f_6)\eta(W)S(Y,hV)\eta(Z) \\ &-2n(f_1-f_3)^2 g(V,W)g(Y,Z) - 2n(f_1-f_3)(f_4-f_6)g(V,hW)g(Y,Z) \\ &+2n(f_1-f_3)g(R(Y,V)W,Z) + 2n(f_1-f_3)^2 g(Y,W)\eta(V)\eta(Z) \\ &+2n(f_1-f_3)(f_4-f_6)g(Y,hW)\eta(V)\eta(Z) - (f_1-f_3)S(Y,W)\eta(V)\eta(Z) \\ &-2n(f_1-f_3)(f_4-f_6)\eta(V)\eta(W)g(hY,Z) + (f_1-f_3)S(Y,W)g(V,Z) \\ &+(f_4-f_6)S(Y,W)g(hV,Z) + 2n(f_1-f_3)^2 g(V,Y)\eta(W)\eta(Z) \\ &+2n(f_1-f_3)(f_4-f_6)g(V,hY)\eta(W)\eta(Z) = 0. \end{aligned} \tag{31}$$

Let $\{e_i\}, i = 1,2,3,........,(2n+1)$ be a local orthonormal basis in the tangent space T_PM at each point $p \in M$. Taking $V = W = e_i$ in (31) and summing over $i = 1,2,3,........,(2n+1)$, then we have

$$\begin{aligned} &(2n+1)(f_1-f_3)\{2n(f_1-f_3)g(Y,Z) - S(Y,Z)\} \\ &+(f_4-f_6)\{2n(f_1-f_3)g(hY,Z) - S(hY,Z)\} = 0. \end{aligned} \tag{32}$$

Replacing Y by hY in (32) and using (13), we get

$$\begin{aligned} &(2n+1)(f_1-f_3)\{2n(f_1-f_3)g(hY,Z) - S(hY,Z)\} \\ &-(k-1)(f_4-f_6)\{2n(f_1-f_3)g(Y,Z) - S(Y,Z)\} = 0. \end{aligned} \tag{33}$$

Multiplying (32) by $(2n+1)(f_1-f_3)$ and (33) by (f_4-f_6) and subtracting from (32) to (33), we get

$$\{(k-1)(f_4-f_6)^2 + (2n+1)^2(f_1-f_3)^2\}\{2n(f_1-f_3)g(Y,Z) - S(Y,Z)\} = 0. \tag{34}$$

Now for $k=1$, either $f_1=f_3$ or $S(Y,V)=2n(f_1-f_3)g(Y,V)$.
On the other hand for $k<1$, either $S(Y,V)=2n(f_1-f_3)g(Y,V)$ or

$$(2n+1)^2(f_1-f_3)^2=(1-k)(f_4-f_6)^2. \tag{35}$$

Then from (35), we have $f_1=f_3$ implies $f_4=f_6$.
Thus we can state the following:

Theorem 2. *If a $(2n+1)$-dimensional generalized (k,μ)-space form $M(f_1.....,f_6)$ with $f_1\neq f_3$ satisfying the condition $Q(S,R)=0$, then the space form is an Einstein manifold.*

Suppose, we consider Ricci pseudo-symmetric generalized (k,μ)-space form $M(f_1.....,f_6)$, that is, the manifold satisfying the curvature condition $R\cdot S=fQ(g,S)$, then we have from (6)

$$(R(X,Y)\cdot S)(U,V)=fQ(g,S)(X,Y;U,V), \tag{36}$$

which is equivalent to

$$(R(X,Y)\cdot S)(U,V)=f((X\wedge_g Y\cdot S)(U,V)). \tag{37}$$

Using (6) in (37), we get

$$\begin{aligned}&-S(R(X,Y)U,V)-S(U,R(X,Y)V)\\&=f\{-g(Y,U)S(X,V)+g(X,U)S(Y,V)-g(Y,V)S(U,X)+g(X,V)S(U,Y)\}.\end{aligned} \tag{38}$$

Replacing $X=U=\xi$ in (38), we obtain

$$\begin{aligned}&S(R(\xi,Y)\xi,V)+S(\xi,R(\xi,Y)V)\\&=f\{g(Y,\xi)S(\xi,V)-g(\xi,\xi)S(Y,V)+g(Y,V)S(\xi,\xi)-g(\xi,V)S(\xi,Y)\}.\end{aligned} \tag{39}$$

Using (9), (16) and (19) in (39), we get

$$\begin{aligned}&(f_1-f_3-f)\{2n(f_1-f_3)g(Y,V)-S(Y,V)\}\\&+(f_4-f_6)\{2n(f_1-f_3)g(hY,V)-S(hY,V)\}=0.\end{aligned} \tag{40}$$

Replacing Y by hY in (40) and using (13), we get

$$\begin{aligned}&(f_1-f_3-f)\{2n(f_1-f_3)g(hY,V)-S(hY,V)\}\\&-(k-1)(f_4-f_6)\{2n(f_1-f_3)g(Y,V)-S(Y,V)\}=0.\end{aligned} \tag{41}$$

Multiplying (40) by (f_1-f_3-f) and (41) by (f_4-f_6) and subtracting from (40) to (41), we obtain

$$\{(k-1)(f_4-f_6)^2+(f_1-f_3-f)^2\}\{2n(f_1-f_3)g(Y,V)-S(Y,V)\}=0. \tag{42}$$

Now for $k=1$, either $f=f_1-f_3$ or $S(Y,V)=2n(f_1-f_3)g(Y,V)$.
On the other hand for $k<1$, either $S(Y,V)=2n(f_1-f_3)g(Y,V)$ or

$$(f_1-f_3-f)^2=(1-k)(f_4-f_6)^2. \tag{43}$$

Then from (35), we have $f=f_1-f_3$ implies $f_4=f_6$.
Thus we can state the following:

Theorem 3. *A generalized (k,μ)-space form $M(f_1.....,f_6)$ with $f_1\neq f_3$ is Ricci pseudo-symmetric, then the space form is an Einstein manifold.*

Suppose the generalized (k,μ)-space form satisfying the curvature condition $Q(g,S)=0$. Then we have

$$(X\wedge_g Y\cdot S)(U,V)=0. \tag{44}$$

Using (3) and (6) in (44), we get

$$-g(Y,U)S(X,V)+g(X,U)S(Y,V)-g(Y,V)S(U,X)+g(X,V)S(U,Y)=0. \tag{45}$$

Taking $X=U=\xi$ in (45) and using (19) and (9), we obtain

$$S(Y,V)=2n(f_1-f_3)g(Y,V). \tag{46}$$

Thus we can state the following:

Theorem 4. *If a $(2n+1)$-dimensional generalized (k,μ)-space form $M(f_1.....,f_6)$ satisfying the condition $Q(g,S)=0$, then the space form is either Ricci flat or an Einstein manifold.*

Definition 5. *A vector field V is said to be confirmal Killing vector field if it satisfies $L_V g = \rho g$, for some function ρ.*

If the manifold admitting a Ricci solitons (g,V,λ) is an Einstein manifold then the vector field V is confirmal Killing. Now by substituting (46) in (8), we get

$$(L_V g)(X,Y) = \rho g(X,Y). \tag{47}$$

Where $\rho = -2\{2n(f_1-f_3)+\lambda\}$. i.e. V is confirmal Killing.
This leads to the following:

Theorem 6. *Let (g,V,λ) be a Ricci soliton in generalized (k,μ)-space form $M(f_1.....,f_6)$. The potential vector field V is confirmal Killing if and only if $Q(g,S)=0$, holds in M.*

Proposition 7. *Let (g,V,λ) be a Ricci soliton in generalized (k,μ)-space form $M(f_1.....,f_6)$ with $f_1 \neq f_3$ and $f_4 \neq f_6$. The potential vector field V is confirmal Killing if and only if the space form is Ricci-semisymmetric.*

Proposition 8. *Let (g,V,λ) be a Ricci soliton in generalized (k,μ)-space form $M(f_1.....,f_6)$ with $f \neq f_1-f_3$ and $f_4 \neq f_6$. The potential vector field V is confirmal Killing if and only if the space form is Ricci pseudo-semisymmetric.*

Proposition 9. *Let (g,V,λ) be a Ricci soliton in generalized (k,μ)-space form $M(f_1.....,f_6)$ with $f_1 \neq f_3$ and $f_4 \neq f_6$. The potential vector field V is confirmal Killing if and only if $Q(S,R)=0$, holds in M.*

Suppose that a generalized (k,μ)-space form $M(f_1.....,f_6)$, admits a Ricci soliton (g,V,λ), then from (8), we have

$$g(\nabla_X V,Y)+g(X,\nabla_Y V)+2S(X,Y)+2\lambda g(X,Y)=0. \tag{48}$$

Replacing $X=Y=\xi$ in (48), we get

$$2g(\nabla_\xi V,\xi)+2S(\xi,\xi)+2\lambda=0. \tag{49}$$

If $V \perp \xi$, it provides $\eta(\nabla_X V) = g(\phi X+\phi hX,V)$. Hence $\eta(\nabla_\xi V)=0$. Therefore on using (19) in (49), we obtain

$$\lambda = -2n(f_1-f_3). \tag{50}$$

Hence we can state the following:

Theorem 10. *A generalized (k,μ)-space form $M(f_1.....,f_6)$ admitting a Ricci soliton (g,V,λ), where the potential vector field V is orthogonal to ξ is shrinking if $f_1>f_3$, expanding if $f_1<f_3$ or steady if $f_1=f_3$.*

Definition 11. *A vector field V is called torse forming vector field if it satisfies $\nabla_X V = fX+\gamma(X)V$, where f is a smooth function and γ is a $1-$form.*

From (48) and using (18), we can write

$$\nabla_X V = -\{2nf_1+3f_2-f_3+\lambda\}X-\{(2n-1)f_4-f_6\}hX+\{3f_2+(2n-1)f_3\}\eta(X)\xi. \tag{51}$$

If $(2n-1)f_4=f_6$, then the vector field $V(=b\xi)$ is torse forming, where $f=-\{2nf_1+3f_2-f_3+\lambda\}, \gamma(X)$ is $1-$form and $b=3f_2+(2n-1)f_3$.
Thus we state the following:

Theorem 12. *A generalized (k,μ)-space form $M(f_1.....,f_6)$ admitting a Ricci soliton (g,V,λ), where the vector field V is collinear with ξ. Then the the vector field V is torse forming.*

If the vector field V is torse forming vector field, then equation (48) becomes

$$2fg(X,Y)+\gamma(X)g(V,Y)+\gamma(Y)g(V,X)+2S(X,Y)+2\lambda g(X,Y)=0. \tag{52}$$

Taking $Y=\xi$ in (52), we get

$$\{2f+4n(f_1-f_3)+2\lambda\}\eta(X)+\gamma(\xi)g(V,X)+\gamma(X)\eta(V)=0. \tag{53}$$

Replacing X by ξ in (53), we obtain

$$\lambda = -\{\eta(V)\gamma(\xi) + f + 2n(f_1 - f_3)\}. \tag{54}$$

If $f = -\eta(V)\gamma(\xi)$, then from (54), we get

$$\lambda = 2n(f_3 - f_1). \tag{55}$$

Thus we can state the following:

Theorem 13. *If (g,V,λ) is a Ricci soliton in a generalized (k,μ)-space form $M(f_1,\dots,f_6)$ and V is torse forming with $f = -\eta(V)\gamma(\xi)$, then the Ricci soliton is shrinking if $f_1 > f_3$, expanding if $f_1 < f_3$ or steady if $f_1 = f_3$.*

4. Conclusions

Generalized (k,μ)-Space forms generalize the notion of (k,μ)-Space forms and generalized Sasakian space forms. Some semi-symmetry, Ricci pseudo symmetry on generalized (k,μ)-Space form leads to the Einstein condition. Further the potential vector field of a Ricci soliton in a generalized (k,μ)-Space form reduces to torse forming or conformal Killing under certain conditions.

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