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One step iterative scheme for a pair of nonexpansive mappings in a convex metric space

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Abstract

We propose and analyse a one step explicit iteration scheme for a pair of nonexpansive mappings in a uniformly convex metric space. Our results refine and generalize several recent and comparable results in uniformly convex Banach spaces and $CAT(0)$ spaces, simultaneously.

Keywords: Convex metric space, one step iterative scheme, nonexpansive mapping, common fixed point, strong convergence and $\triangle-$convergence.

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1. Introduction and preliminaries

The fixed point theory of nonexpansive mappings proposed in the setting of Banach spaces extremly depends on the linear structure of the underlying space. A nonlinear framework for theory of iterative construction of fixed points of nonexpansive mappings is a metric space embedded with a "convex structure". In the literature, different notions of convexity in metric spaces are provided (see, for example, Kirk [10, 11], Penot [15] and Takahashi [20]).

Takahashi [20] introduced the notion of a convex structure in a metric space X as a mapping $W : X^2 \times I \to X$ satisfying

$$d\left(u, W\left(x, y, \alpha\right)\right) \leq \alpha d(u, x) + (1 - \alpha)d(u, y) \qquad (1.1)$$

for all $x, y, u \in X$ and $\alpha \in I = [0, 1]$. A metric space X together with a convex structure W is known as a convex metric space. For the sake of simplicity, we also denote a convex metric space by X. A nonempty subset C of X is convex if $W(x, y, \alpha) \in C$ for all $x, y \in C$ and $\alpha \in I$.There are many examples of convex metric spaces which cannot be imbedded in any Banach space (see [20]). Some other examples of convex metric spaces are Hadamard manifolds [3] and $CAT\left(0\right)$ spaces [2, 9].

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A convex metric space X is uniformly convex [5, 19] if for any $\varepsilon > 0$, there exists $\alpha > 0$ such that $d\left(z, W\left(x, y, \frac{1}{2}\right)\right) \leq r\left(1-\alpha\right) < r$ for all $r > 0$ and $x, y, z \in X$ with $d\left(z, x\right) \leq r, d\left(z, y\right) \leq r$ and $d\left(x, y\right) \geq r\varepsilon$.

A closed subset X of the unit ball $S_1\left(0\right) = \{x \in H : \|x\| \leq 1\}$ in a Hilbert space H with diameter $\delta\left(X\right) \leq \sqrt{2}$, turns out to be a uniformly convex metric space with $d\left(x, y\right) = \cos^{-1}\langle x, y\rangle$ for all $x, y \in X$ and $W\left(x, y, \alpha\right) = \frac{\alpha x+(1-\alpha)y}{\|\alpha x+(1-\alpha)y\|}$ for all $x, y \in X$ and $\alpha \in I$.

A mapping T on a subset C of X is nonexpansive if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in C$. A point $x \in C$ is a fixed point of T if $Tx = x$. Denote by $F(T)$, the set of all fixed points of T.

Ishikawa iterative scheme [6] is a two step iterative scheme and has been extensively used to approximate common fixed points of nonexpansive mappings by a number of researchers (see, for example, [7, 13, 21, 22]).

In order to reduce the computational cost of a two step iterative scheme, we propose a one step iterative scheme for a pair of nonexpansive mappings $S, T : C \to C$ in a convex metric space as follows:

$$x_{n+1} = W\left(Tx_n, W\left(Sx_n, x_n, \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right) \tag{1.2}$$

where $0 < a \leq \alpha_n,\ \beta_n \leq b < 1$ and satisfy $\alpha_n + \beta_n < 1$(see also [1]).

In Banach space setting, (1.2) becomes one step iterative scheme [23]:

$$x_{n+1} = \alpha_n Tx_n + \beta_n Sx_n + \left(1 - \alpha_n - \beta_n\right) x_n. \tag{1.3}$$

When $S = I$ in (1.2), it reduces to Mann iterative scheme [14]:

$$x_{n+1} = W\left(Tx_n, x_n, \alpha_n\right). \tag{1.4}$$

One of the interesting and important aspect of approximation theory of fixed points is to consider an iterative scheme with bounded error term and therefore such an iterative scheme has been widely studied by a number of researchers in various frames of work; see, for instance, [7] and references therein. It is remarked that the scheme (1.2) can be reshaped as Mann iteration scheme with errors by replacing $\{Sx_n\}$ or $\{Tx_n\}$ with $\{u_n\}$(*i.e.*, the error term).

Let $\{x_n\}$ be a bounded sequence in a metric space X. For $x \in X$, define $r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n)$. Then (i) $r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in C\}$ is called the asymptotic radius of $\{x_n\}$ with respect to $C \subseteq X$, (ii) For any $y \in C$, the set $A(\{x_n\}) = \{x \in X : r(x, \{x_n\} \leq r(y, \{x_n\})\}$ is called the asymptotic center of $\{x_n\}$ with respect to $C \subseteq X$.

A subset C of a metric space X is Chebyshev if for every $x \in X$, there exists $z \in C$ such that $d\left(z, x\right) < d\left(c, x\right)$ for all $c \in C$ and $c \neq z$. If C is a Chebyshev subset of a metric space X, then we define the nearest point projection $P : X \to C$ by sending x to z. This is consistent with the notion of orthogonal projection onto a subspace of a Euclidean space. It has been shown in [4] that every closed convex subset of a uniformly convex metric space is Chebyshev.

A sequence $\{x_n\}$ in X is said to $\triangle-$converge to $x \in X$ [12] if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\triangle-\lim_n x_n = x$.

It has been shown in the literature that the notion of $\triangle-$convergence and weak convergence in Banach spaces share many useful properties.

In this manuscript, we approximate the common fixed points of two nonexpansive mappings by one step iterative scheme (1.2) in a convex metric space.

For the development of our main results, some key results are listed in the form of lemmas:

1.1. Lemma. *([4]). Let C be a nonempty closed convex subset of a uniformly convex metric space and $\{x_n\}$ a bounded sequence in C such that $A(\{x_n\}) = \{y\}$ and $r(\{x_n\}) = \rho$. If $\{y_m\}$ is another sequence in C such that $\lim_{m\to\infty} r(y_m, \{x_n\}) = \rho$, then $\lim_{m\to\infty} y_m = y$.*

1.2. Lemma. *([18]). Let X be a uniformly convex metric space with continuous convex structure W. Then for any $\varepsilon > 0$ and $r > 0$, there exists $\delta > 0$ such that*

$$d(z, W(x, y, \alpha)) \le r(1 - 2\min\{\alpha, 1-\alpha\}\delta)$$

for all $x, y, z \in X, d(z, x) \le r, d(z, y) \le r, d(x, y) \ge r\varepsilon$ and $\alpha \in I$.

From now onwards, for a pair of nonexpansive mappings $S, T : C \to C$, we set $F = F(T) \cap F(S)$.

2. Main Results

We start with the following lemma.

2.1. Lemma. *Let C be a closed and convex subset of a convex metric space X and let S, T be nonexpansive mappings on C such that $F \neq \phi$. Then for the sequence $\{x_n\}$ defined in (1.2), $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F$.*

Proof. Let $p \in F$. Applying (1.1) to (1.2), we have

$$\begin{aligned}
d(x_{n+1}, p) &= d\left(W\left(Tx_n, W\left(Sx_n, x_n, \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right), p\right) \\
&\le \alpha_n d(Tx_n, p) + (1-\alpha_n)\, d\left(W\left(Sx_n, x_n, \frac{\beta_n}{1-\alpha_n}\right), p\right) \\
&\le \alpha_n d(x_n, p) + (1-\alpha_n)\left[\frac{\beta_n}{1-\alpha_n} d(Sx_n, p) + \left(1 - \frac{\beta_n}{1-\alpha_n}\right) d(Sx_n, p)\right] \\
&\le \alpha_n d(x_n, p) + (1-\alpha_n)\left[\frac{\beta_n}{1-\alpha_n} d(x_n, p) + \left(1 - \frac{\beta_n}{1-\alpha_n}\right) d(x_n, p)\right] \\
&= \alpha_n d(x_n, p) + \beta_n d(x_n, p) + (1-\alpha_n-\beta_n)\, d(x_n, p) \\
&= d(x_n, p).
\end{aligned}$$

That is,

$$d(x_{n+1}, p) \le d(x_n, p) \text{ for all } p \in F. \tag{2.1}$$

This gives that $\{x_n\}$ is a decreasing and bounded below sequence of nonnegative real numbers, therefore $\lim_{n\to\infty} d(x_n, p)$ exists. ∎

The following lemma provides an analogue of Schu Lemma [16] in the setting of convex metric spaces and is needed in the next lemma.

2.2. Lemma. *Let X be a uniformly convex metric space with continuous convex structure W. Let $x \in X$ and $\{a_n\}$ be a sequence in $[b, c]$ for some $b, c \in (0, 1)$. If $\{u_n\}$ and $\{v_n\}$ are sequences in X such that $\limsup_{n\longrightarrow\infty} d(u_n, x) \le r$, $\limsup_{n\longrightarrow\infty} d(v_n, x) \le r$ and $\lim_{n\longrightarrow\infty} d(W(u_n, v_n, a_n), x) = r$ for some $r \ge 0$, then $\lim_{n\to\infty} d(u_n, v_n) = 0$.*

Proof. The case $r = 0$ is trivial. Suppose $r > 0$ and assume $\lim_{n\to\infty} d(u_n, v_n) \neq 0$. If $n_0 \ge 1$, then $d(u_{n_i}, v_{n_i}) \ge \frac{\alpha}{2} > 0$ for some $\alpha \in (0, r]$ and for $n_i \ge n_0$. Since $\limsup_{i\longrightarrow\infty} d(u_{n_i}, x) \le r$ and $\limsup_{n\longrightarrow\infty} d(v_{n_i}, x) \le r$, so $\max\{d(u_{n_i}, x), d(v_{n_i}, x)\} \le$

$r+\frac{1}{n_i}$ for $n_i \geq n_0$ and $d(u_{n_i}, v_{n_i}) \geq \frac{\alpha}{2} = \left(r+\frac{1}{n_i}\right)\frac{\alpha_{n_i}}{2(n_i r+1)} \geq \left(r+\frac{1}{n_i}\right)\frac{\alpha}{2(r+1)}$. Therefore Lemma 1.2 gives that

$$\begin{aligned} d(W(u_{n_i}, v_{n_i}, a_{n_i}), x) &\leq \left(r+\frac{1}{n_i}\right)(1-2\min\{a_{n_i}, 1-a_{n_i}\}\delta) \\ &\leq \left(r+\frac{1}{n_i}\right)(1-2a_{n_i}(1-a_{n_i})\delta) \\ &\leq \left(r+\frac{1}{n_i}\right)(1-2b(1-c)\delta). \end{aligned}$$

Thus, by letting $i \to \infty$, we obtain

$$\lim_{i\to\infty} d(W(u_{n_i}, v_{n_i}, a_{n_i}), x) \leq (1-2b(1-c)\delta) r < r,$$

a contradiction. ∎

2.3. Lemma. *Let C be a nonempty, closed and convex subset of a uniformly convex metric space X with continuous convex structure W and let S, T be nonexpansive mappings on C such that $F \neq \phi$. Then for the sequence $\{x_n\}$ in (1.2), we have*

$$\lim_{n\to\infty} d(x_n, Sx_n) = 0 = \lim_{n\to\infty} d(x_n, Tx_n).$$

Proof. It follows from Lemma 2.1 that $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F$. Assume that $\lim_{n\to\infty} d(x_n, p) = c$. If $c = 0$, the result is trivial. For $c > 0, \lim_{n\to\infty} d(x_{n+1}, p) = c$ gives that

$$\lim_{n\to\infty} d\left(W\left(Tx_n, W\left(Sx_n, x_n, \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right), p\right) = c. \tag{2.2}$$

Nonexpansiveness of T gives that

$$\limsup_{n\to\infty} d(Tx_n, p) \leq \limsup_{n\to\infty} d(x_n, p) = c. \tag{2.3}$$

Since

$$\begin{aligned} d\left(W\left(Sx_n, x_n, \frac{\beta_n}{1-\alpha_n}\right), p\right) &\leq \frac{\beta_n}{1-\alpha_n} d(Sx_n, p) + \left(1-\frac{\beta_n}{1-\alpha_n}\right) d(x_n, p) \\ &\leq d(x_n, p), \end{aligned}$$

therefore

$$\limsup_{n\to\infty} d\left(W\left(Sx_n, x_n, \frac{\beta_n}{1-\alpha_n}\right), p\right) \leq c. \tag{2.4}$$

Using Lemma 2.2 $\left(\text{with } x = p, r = c, a_n = \alpha_n, u_n = Tx_n, v_n = W\left(Sx_n, x_n, \frac{\beta_n}{1-\alpha_n}\right)\right)$ together with (2.2-2.4), we get

$$\lim_{n\to\infty} d\left(Tx_n, W\left(Sx_n, x_n, \frac{\beta_n}{1-\alpha_n}\right)\right) = 0. \tag{2.5}$$

Now the estimate

$$\begin{aligned} d(x_{n+1}, Tx_n) &\leq d\left(W\left(Tx_n, W\left(Sx_n, x_n, \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right), Tx_n\right) \\ &\leq (1-\alpha_n) d\left(W\left(Sx_n, x_n, \frac{\beta_n}{1-\alpha_n}\right), Tx_n\right) \\ &\leq (1-b) d\left(W\left(Sx_n, x_n, \frac{\beta_n}{1-\alpha_n}\right), Tx_n\right), \end{aligned}$$

together with (2.5) implies that

$$\lim_{n\to\infty} d(x_{n+1}, Tx_n) = 0. \tag{2.6}$$

Since S is nonexpansive, $\limsup_{n\to\infty} d(Sx_n, p) \leq c$.
By triangle inequality, we have

$$\begin{aligned} d(x_{n+1}, p) &\leq d(x_{n+1}, Tx_n) + d\left(Tx_n, W\left(Sx_n, x_n, \frac{\beta_n}{1-\alpha_n}\right)\right) \\ &\quad + d\left(W\left(Sx_n, x_n, \frac{\beta_n}{1-\alpha_n}\right), p\right). \end{aligned}$$

Taking $\liminf_{n\to\infty}$ on both sides in the above inequality, we have

$$c \leq \liminf_{n\to\infty} d\left(W\left(Sx_n, x_n, \frac{\beta_n}{1-\alpha_n}\right), p\right).$$

Therefore

$$\lim_{n\to\infty} d\left(W\left(Sx_n, x_n, \frac{\alpha_n}{1-\beta_n}\right), p\right) = c. \tag{2.7}$$

Again by Lemma 2.2 (with $x = p, r = c, a_n = \frac{\alpha_n}{1-\beta_n}, u_n = S_n x_n, v_n = x_n$), we get

$$\lim_{n\to\infty} d(x_n, Sx_n) = 0. \tag{2.8}$$

Further note that

$$\begin{aligned} d(x_{n+1}, x_n) &\leq d(x_{n+1}, Tx_n) + d\left(Tx_n, W\left(Sx_n, x_n, \frac{\alpha_n}{1-\beta_n}\right)\right) \\ &\quad + d\left(W\left(Sx_n, x_n, \frac{\alpha_n}{1-\beta_n}\right), x_n\right) \\ &\leq d(x_{n+1}, Tx_n) + d\left(Tx_n, W\left(Sx_n, x_n, \frac{\alpha_n}{1-\beta_n}\right)\right) \\ &\quad + (1 - \frac{\alpha_n}{1-\beta_n}) d(x_n, Sx_n) \\ &\leq d(x_{n+1}, Tx_n) + d\left(Tx_n, W\left(Sx_n, x_n, \frac{\alpha_n}{1-\beta_n}\right)\right) \\ &\quad + \left(\frac{1-2a}{1-b}\right) d(x_n, Sx_n). \end{aligned}$$

Letting $n \to \infty$ in the above estimate, we have

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{2.9}$$

As a direct consequence of (2.6) and (2.9), the inequality

$$d(x_n, Tx_n) \leq d(x_n, x_{n+1}) + d(x_{n+1}, Tx_n)$$

provides that

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

Hence

$$\lim_{n\to\infty} d(x_n, Sx_n) = 0 = \lim_{n\to\infty} d(x_n, Tx_n).$$

∎

The conclusion of Lemma 2.3 is interesting because the sequence generated by (1.2) gives an approximate fixed point sequence for both S and T without assuming that these mappings commute.

Now we state a result concerning $\triangle-$convergence of the iterative scheme (1.2). The method of proof is closely related to Theorem 3.1 in [8].

2.4. Theorem. *Let C be a nonempty, closed and convex subset of a uniformly convex complete metric space X with continuous convex structure W and $S,T : C \to C$ be nonexpansive mappings with $F \neq \phi$. Then the sequence $\{x_n\}$ in (1.2), $\triangle-$converges to an element of F.*

Proof. In the proof of Lemma 2.1, it has been shown that $\{x_n\}$ is bounded. Therefore $\{x_n\}$ has a unique asymptotic centre, that is, $A(\{x_n\}) = \{x\}$. Let $\{u_n\}$ be any subsequence of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. First, we show that $u \in C$. Suppose $u \notin C$. As C is a Chebyshev set, we can define a nearest point projection $P : X \to C$.Therefore $d\left(Pu, u_n\right) < d\left(u, u_n\right) \Longrightarrow r\left(Pu, \{u_n\}\right) < r\left(u, \{u_n\}\right) \Longrightarrow u$ is not the asymptotic center of $\{u_n\}$, a contradiction. Hence $u \in C$. Also by Lemma 2.2, we have $\lim_{n\to\infty} d(u_n, Tu_n) = 0 = \lim_{n\to\infty} d(u_n, Su_n)$. Define $\{z_m\}$ in C by $z_m = T^m u$.
Observe that

$$\begin{aligned} d(z_m, u_n) &\leq d(T^m u, T^m u_n) + \sum_{j=1}^{m} d(T^j u_n, T^{j-1} u_n) \\ &\leq d(u, u_n) + m d(Tu_n, u_n). \end{aligned}$$

Therefore, we have

$$r(z_m, \{u_n\}) = \limsup_{n\to\infty} d(z_m, u_n) \leq \limsup_{n\to\infty} d(u, u_n) = r(u, \{u_n\}).$$

This implies that $|r(z_m, \{u_n\}) - r(u, \{u_n\})| \to 0$ as $m \to \infty$. It follows from Lemma 1.1 that $\lim_{m\to\infty} T^m u = u$. Since C is closed, so $\lim_{m\to\infty} T^m u = u \in C$ and $\lim_{m\to\infty} T^{m+1} u = Tu$.That is, $Tu = u$. Similarly we have $Su = u$. Therefore $\lim_{n\to\infty} d(x_n, u)$ exists by Lemma 2.1. If $x \neq u$, then by the uniqueness of asymptotic centres, we have

$$\begin{aligned} \limsup_{n\to\infty} d(u_n, u) &< \limsup_{n\to\infty} d(u_n, x) \\ &\leq \limsup_{n\to\infty} d(x_n, x) \\ &< \limsup_{n\to\infty} d(x_n, u) \\ &= \limsup_{n\to\infty} d(u_n, u), \end{aligned}$$

a contradiction. Hence $x = u$.
Therefore, $A(\{u_n\}) = \{u\}$ for all subsequences $\{u_n\}$ of $\{x_n\}$. This proves that $\{x_n\}$ $\triangle-$converges to x. ∎

Using the concept of near point projection, we establish the following theorem.

2.5. Theorem. *Let C be a nonempty, closed and convex subset of a complete uniformly convex metric space X and $S,T : C \to C$ be nonexpansive mappings. Let P be the nearest point projection of C onto F. For an initial value x_1, define $\{x_n\}$ as given in (1.2) where $\alpha_n, \beta_n \in [a,b]$ for some $a, b \in \mathbb{R}$ with $0 < a \leq b < 1$. Then $\{Px_n\}$ converges strongly to a point of F.*

Proof. It follows from (2.1) that, for any $n \geq 1, m \geq 1$, we have

$$d\left(Px_n, x_{n+m}\right) \leq d\left(Px_n, x_{n+m-1}\right) \leq d\left(Px_n, x_{n+m-2}\right) \leq ... \leq d\left(Px_n, x_{n+1}\right).$$

That is,

(2.10) $\quad d\left(Px_{n},x_{n+m}\right)\leq d\left(Px_{n},x_{n}\right)$ for $n\geq1,m\geq1.$

In order to prove the result, we show that $\{Px_n\}$ is a Cauchy sequence. By definition of nearest point projection and (2.10), we have

$$d\left(Px_{n+1},x_{n+1}\right)\leq d\left(Px_{n},x_{n+1}\right)\leq d\left(Px_{n},x_{n}\right).$$

Hence $d\left(Px_{n},x_{n}\right)\rightarrow c$(say). If $c=0$, then for an arbitrary $\varepsilon>0$, there exists an integer $n_0\geq1$ such that

(2.11) $\quad d\left(Px_{n},x_{n}\right)<\varepsilon$ for all $n\geq n_0$.

By (2.11), for $m>n\geq n_0$,we have

$$\begin{aligned}
d\left(Px_{n},Px_{m}\right) &\leq d\left(Px_{n},Px_{n_0}\right)+d\left(Px_{n_0},Px_{m}\right)\\
&\leq d\left(Px_{n},x_{n}\right)+d\left(x_{n},Px_{n_0}\right)+d\left(Px_{n_0},x_{m}\right)+d\left(x_{m},Px_{m}\right)\\
&< 4\varepsilon.
\end{aligned}$$

This proves that $\{Px_n\}$ is a Cauchy sequence. Assume that $c>0$ and $\{Px_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon>0$ and two subsequences $\{Px_{n_i}\}$ and $\{Px_{m_i}\}$ of $\{Px_n\}$ such that $d\left(Px_{n_i},Px_{m_i}\right)\geq\varepsilon$ for all $i\geq1$. Since $\{d\left(Px_{n},x_{n}\right)\}$ is a decreasing sequence and $d\left(Px_{n},x_{n}\right)\rightarrow c$, therefore we have

$$c\leq d\left(Px_{n},x_{n}\right)\leq c+\frac{1}{n}\text{ for }n\geq n_0.$$

Let $n_0\leq n_i,m_i\leq l$. By (2.10), we have

$$d\left(Px_{n_i},x_{l}\right)\leq d\left(Px_{n_i},x_{n_i}\right)<c+\frac{1}{n}\text{ and }d\left(Px_{m_i},x_{l}\right)\leq d\left(Px_{m_i},x_{m_i}\right)<c+\frac{1}{n}.$$

Moreover,

$$d\left(Px_{n_i},Px_{m_i}\right)\geq\left(\frac{\varepsilon}{c+\frac{1}{n}}\right)\left(c+\frac{1}{n}\right)\geq\left(\frac{\varepsilon}{c+1}\right)\left(c+\frac{1}{n}\right).$$

By uniform convexity of X, there exists $\delta\left(\frac{\varepsilon}{c+1}\right)>0$ such that

$$d\left(x_{l},\frac{1}{2}Px_{n_i}\oplus\frac{1}{2}Px_{m_i}\right)\leq\left(c+\frac{1}{n}\right)\left(1-\delta\left(\frac{\varepsilon}{c+1}\right)\right).$$

Let $n\rightarrow\infty$ in the above inequality, we have

$$c\leq d\left(Px_{l},x_{l}\right)\leq d\left(x_{l},\frac{1}{2}Px_{n_i}\oplus\frac{1}{2}Px_{m_i}\right)\leq c\left(1-\delta\left(\frac{\varepsilon}{c+1}\right)\right)<c,$$

a contradiction.

This proves that $\{Px_n\}$ is a Cauchy sequence in F. As F is closed, therefore it converges to a point of F. ∎

Recall that a mapping $T:C\rightarrow C$ is semi-compact if every bounded sequence $\{x_n\}$ has a convergent subsequence whenever $d(x_n,Tx_n)\rightarrow0$.

Let $f:[0,\infty)\rightarrow[0,\infty)$ be a nondecreasing function with $f(0)=0$ and $f(t)>0$ for all $t\in(0,\infty)$.The mappings $S,T:C\rightarrow C$ with $F\neq\phi$, satisfy Condition (I) [7] (see also [17]) if

$$\frac{1}{2}\left[d\left(x,Tx\right)+d\left(x,Sx\right)\right]\geq f(d(x,F))\text{ for }x\in C,$$

where $d(x,F)=\inf_{p\in F}d\left(x,p\right)$.

Using Lemma 2.3, we obtain the following strong convergence theorem.

2.6. Theorem. *Let C be a nonempty, closed and convex subset of a uniformly convex complete metric space with continuous convex structure W and let $S,T : C \to C$ be nonexpansive mappings with $F \neq \phi$. If S and T satisfy Condition (I), then the sequence $\{x_n\}$ defined in (1.2), converges strongly to an element of F.*

Proof. By Lemma 2.3, we have

$$\lim_{n\to\infty} d(x_n, Sx_n) = 0 = \lim_{n\to\infty} d(x_n, Tx_n).$$

Using Condition (I), we get that that $\lim_{n\to\infty} d(x_n, F) = 0$. For a given $\varepsilon > 0$, there exists $N_\varepsilon \geq 1$ and $y_\varepsilon \in F$ such that $d(x_n, y_\varepsilon) < \varepsilon$ for all $n \geq N_\varepsilon$.Thus, if $\varepsilon_k = 2^{-k}$ for $k \geq 1$, then corresponding to each ε_k, there exist $N_k \geq 1$ and $y_k \in F$ such that $d(x_n, y_k) \leq \frac{\varepsilon_k}{4}$ for all $n \geq N_k$. On choosing $N_{k+1} \geq N_k$ for any $k \geq 1$,we have that

$$\begin{aligned} d(y_k, y_{k+1}) &\leq d\left(y_k, x_{N_{k+1}}\right) + d\left(x_{N_{k+1}}, y_{k+1}\right) \\ &< \frac{\varepsilon_k}{4} + \frac{\varepsilon_{k+1}}{4} = \frac{3}{4}\varepsilon_{k+1}. \end{aligned}$$

If $x \in S[y_{k+1}, \varepsilon_{k+1}]$, then

$$\begin{aligned} d(x, y_k) &\leq d(x, y_{k+1}) + d(y_{k+1}, y_k) \\ &< \varepsilon_{k+1} + \frac{3}{4}\varepsilon_{k+1} = \frac{7}{4}\varepsilon_{k+1} < 2\varepsilon_{k+1} = \varepsilon_k. \end{aligned}$$

That is, $x \in S[y_k, \varepsilon_k]$.Hence $\{S[y_k, \varepsilon_k] : k \geq 1\}$ is a decreasing sequence of nonempty, bounded, closed and convex subsets in a uniformly convex complete metric space and so $\cap_{k=1}^{\infty} S[y_k, \varepsilon_k] \neq \emptyset$ by Theorem 1([19], p. 200). Now there exists a $p \in X$ such that

$$d(y_k, p) \leq \frac{1}{2^k} \to 0 \text{ as } k \to \infty.$$

That is, $y_k \to p$. Since F is closed, therefore $p \in F$.

In view of the inequality

$$d(x_n, y_k) \leq \frac{\varepsilon_k}{4} \text{ for all } n \geq N_k,$$

we get that $x_n \to p$. ∎

We can also prove the following strong convergence theorem.

2.7. Theorem. *Let C be a closed and convex subset of a uniformly convex complete metric space X and let $S,T : C \to C$ be nonexpansive mappings with $F \neq \phi$. If, either S or T is semi-compact, then the sequence $\{x_n\}$ defined in (1.2), converges strongly to an element of F.*

2.8. Remark. (1) Our results can be extended for two finite families of nonexpansive mappings (2) Our results are valid in uniformly convex Banach spaces and $CAT(0)$ spaces, simultaneously.

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