Hacettepe Journal of Mathematics and Statistics
Volume 45 (5) (2016), 1367 – 1378

Some starlikeness and convexity properties for two new $p-$valent integral operators

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Abstract

In this paper, we define two new general $p-$valent integral operators in the unit disc $\mathbb{U}$ and obtain the properties of $p-$valent starlikeness and $p-$valent convexity of these integral operators of $p-$valent functions on some classes of $\beta-$uniformly $p-$valent starlike and $\beta-$uniformly $p-$valent convex functions of complex order and type α $(0 \leq \alpha < p)$. As special cases, the properties of $p-$valent starlikeness and $p-$valent convexity of the operators $\int_0^z pt^{p-1}\left(\frac{f(t)}{t^p}\right)^{\delta} dt$ and $\int_0^z pt^{p-1}\left(\frac{g'(t)}{pt^{p-1}}\right)^{\delta} dt$ are given.

Keywords: Analytic functions; Integral operators; $\beta-$uniformly $p-$valent starlike and $\beta-$uniformly $p-$valent convex functions; Complex order.

2000 AMS Classification: 30C80, 30C45.

Received : 06.05.2015 *Accepted* : 19.11.2015 *Doi* : 10.15672/HJMS.20164514281

1. Introduction and Preliminaries

Let $\mathcal{A}_p$ denote the class of the form

$$(1.1) \qquad f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in \mathbb{N} = \{1, 2, ..., \}),$$

which are analytic in the open disc $\mathbb{U} = \{z \in \mathbb{C} : \ |z| < 1\}$.

A function $f \in \mathcal{S}_p^*(\gamma, \alpha)$ is $p-$valently starlike of complex order $\gamma\,(\gamma \in \mathbb{C} - \{0\})$ and type $\alpha\,(0 \leq \alpha < p)$, that is, $f \in \mathcal{S}_p^*(\gamma, \alpha)$, if it is satisfies the following condition

$$(1.2) \qquad \mathrm{Re}\left\{p + \frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)} - p\right)\right\} > \alpha \quad (z \in \mathbb{U}).$$

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Furthermore, a function $f \in \mathcal{C}_p(\gamma, \alpha)$ is $p-$valently convex of complex order $\gamma\,(\gamma \in \mathbb{C} - \{0\})$ and type $\alpha\,(0 \leq \alpha < p)$, that is, $f \in \mathcal{C}_p(\gamma, \alpha)$ if it satisfies the following condition;

$$(1.3)\qquad \operatorname{Re}\left\{p + \frac{1}{\gamma}\left(1 + \frac{zf''(z)}{f'(z)} - p\right)\right\} > \alpha \quad (z \in \mathbb{U}).$$

In particular cases, for $p = 1$ in the classes $\mathcal{S}_p^*(\gamma, \alpha)$ and $\mathcal{C}_p(\gamma, \alpha)$, we obtain the classes $\mathcal{S}^*(\gamma, \alpha)$ and $\mathcal{C}(\gamma, \alpha)$ of starlike functions of complex order $\gamma\,(\gamma \in \mathbb{C} - \{0\})$ and type $\alpha\,(0 \leq \alpha < 1)$ and convex functions of complex order $\gamma\,(\gamma \in \mathbb{C} - \{0\})$ and type $\alpha\,(0 \leq \alpha < 1)$, respectively, which were introduced and studied by Frasin [15]. Also, for $\alpha = 0$ in the classes $\mathcal{S}_p^*(\gamma, \alpha)$ and $\mathcal{C}_p(\gamma, \alpha)$, we obtain the classes $\mathcal{S}_p^*(\gamma)$ and $\mathcal{C}_p(\gamma)$, which are called $p-$valently starlike of complex order $\gamma\,(\gamma \in \mathbb{C} - \{0\})$ and $p-$valently convex of complex order $\gamma\,(\gamma \in \mathbb{C} - \{0\})$, respectively. Setting $p = 1$ and $\alpha = 0$, we obtain the classess $\mathcal{S}^*(\gamma)$ and $\mathcal{C}(\gamma)$. The class $\mathcal{S}^*(\gamma)$ of starlike functions of complex order $\gamma\,(\gamma \in \mathbb{C} - \{0\})$ was defined by Nasr and Aouf (see [21]) while the class $\mathcal{C}(\gamma)$ of convex functions of complex order $\gamma\,(\gamma \in \mathbb{C} - \{0\})$ was considered earlier by Wiatrowski (see [27]). Note that $\mathcal{S}_p^*(1, \alpha) = \mathcal{S}_p^*(\alpha)$ and $\mathcal{C}_p(1, \alpha) = \mathcal{C}_p(\alpha)$ are, respectively, the classes of $p-$valently starlike and $p-$valently convex functions of order $\alpha\,(0 \leq \alpha < p)$ in $\mathbb{U}$. In special cases, $\mathcal{S}_p^*(0) = \mathcal{S}_p^*$ and $\mathcal{C}_p(0) = \mathcal{C}_p$ are, respectively, the familiar classes of $p-$valently starlike and $p-$valently convex functions in $\mathbb{U}$. Also, we note that $\mathcal{S}_1^*(\alpha) = \mathcal{S}^*(\alpha)$ and $\mathcal{C}_1(\alpha) = \mathcal{C}(\alpha)$ are, respectively, the usual classes of starlike and convex functions of order $\alpha\,(0 \leq \alpha < 1)$ in $\mathbb{U}$. In special cases, $\mathcal{S}_1^*(0) = \mathcal{S}^*$ and $\mathcal{C}_1 = \mathcal{C}$ are, respectively, the familiar classes of starlike and convex functions in $\mathbb{U}$.

A function $f \in \beta - \mathcal{US}_p(\alpha)$ is $\beta-$uniformly $p-$valently starlike of order α $(0 \leq \alpha < p)$, that is, $f \in \beta - \mathcal{US}_p(\alpha)$ if it is satisfies the following condition

$$(1.4)\qquad \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \beta\left|\frac{zf'(z)}{f(z)} - p\right| + \alpha \quad (\beta \geq 0,\ z \in \mathbb{U}).$$

Furthermore, a function $f \in \beta - \mathcal{UC}_p(\alpha)$ is $\beta-$uniformly $p-$valently convex of order $\alpha\,(0 \leq \alpha < p)$, that is, $f \in \beta - \mathcal{UC}_p(\alpha)$ if it satisfies the following condition

$$(1.5)\qquad \operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \beta\left|1 + \frac{zf''(z)}{f'(z)} - p\right| + \alpha \quad (\beta \geq 0,\ z \in \mathbb{U}).$$

These classes generalize various other classes which are worthy to mention here. For example $p = 1$, the classes $\beta - \mathcal{US}(\alpha)$ and $\beta - \mathcal{UC}(\alpha)$ introduced by Bharti, Parvatham and Swaminathan (see [2]). Also, the class $\beta - \mathcal{UC}_1(0) = \beta - \mathcal{UCV}$ is the known class of $\beta-$uniformly convex functions [17]. Using the Alexander type relation, we can obtain the class $\beta - \mathcal{US}_p(\alpha)$ in the following way:

$$f \in \beta - \mathcal{UC}_p(\alpha) \Leftrightarrow \frac{zf'}{p} \in \beta - \mathcal{US}_p(\alpha).$$

The class $1 - \mathcal{UC}_1(0) = \mathcal{UCV}$ of uniformly convex functions was defined by Goodman [16] while the class $1 - \mathcal{US}_1(0) = \mathcal{SP}$ was considered by Rønning [26].

When the classes $\mathcal{S}_p^*(\gamma, \alpha)$ with $\beta - \mathcal{US}_p(\alpha)$ and $\mathcal{C}_p(\gamma, \alpha)$ with $\beta - \mathcal{UC}_p(\alpha)$ are thought together, we define following classes. Let $0 \leq \alpha < p$, $\beta \geq 0$ and $\gamma \in \mathbb{C} - \{0\}$. A function $f \in \mathcal{A}_p$ is in the class $\beta - \mathcal{US}_p(\gamma, \alpha)$ if and only if for all $z \in \mathbb{U}$

$$\operatorname{Re}\left\{p + \frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)} - p\right)\right\} > \beta\left|\frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)} - p\right)\right| + \alpha$$

and in the class $\beta - \mathcal{UC}_p(\gamma, \alpha)$ if and only if for all $z \in \mathbb{U}$

$$\operatorname{Re}\left\{p + \frac{1}{\gamma}\left(\frac{zf''(z)}{f'(z)} + 1 - p\right)\right\} > \beta\left|\frac{1}{\gamma}\left(\frac{zf''(z)}{f'(z)} + 1 - p\right)\right| + \alpha.$$

For $f \in \mathcal{A}_p$ given by (1.1) and $g(z)$ given by

$$g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k \tag{1.6}$$

their convolution (or Hadamard product), denoted by $(f * g)$, is defined as follows

$$(f * g)(z) = z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k = (g * f)(z) \quad (z \in \mathbb{U}).$$

For a function f in $\mathcal{A}_p$, in [13], the authors defined the *multiplier transformations* $\mathcal{D}_{p,\lambda,\mu}^m$ as follows.

1.1. Definition. Let $f \in \mathcal{A}_p$. For the parameters $\lambda, \mu \in \mathbb{R}$; $0 \leq \mu \leq \lambda$ and $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, define the multiplier transformations $\mathcal{D}_{p,\lambda,\mu}^m$ on $\mathcal{A}_p$ by the following:

$$\begin{aligned}
\mathcal{D}_{p,\lambda,\mu}^0 f(z) &= f(z) \\
\mathcal{D}_{p,\lambda,\mu}^1 f(z) &= \mathcal{D}_{p,\lambda,\mu} f(z) \\
&= \frac{1}{p}\left[\lambda\mu z^2 f''(z) + (\lambda - \mu + (1-p)\lambda\mu)\, z f'(z) + p(1 - \lambda + \mu) f(z)\right] \\
&\;\;\vdots \\
\mathcal{D}_{p,\lambda,\mu}^m f(z) &= \mathcal{D}_{p,\lambda,\mu}\left(\mathcal{D}_{p,\lambda,\mu}^{m-1}\right)
\end{aligned}$$

for $z \in \mathbb{U}$ and $p \in \mathbb{N} := \{1, 2, ...\}$.

If $f(z)$ is given by (1.1), then from the definition of the multiplier transformations $\mathcal{D}_{p,\lambda,\mu}^m f(z)$, we can easily see that

$$\mathcal{D}_{p,\lambda,\mu}^m f(z) = z^p + \sum_{k=p+1}^{\infty} \Phi_p^k(m, \lambda, \mu) a_k z^k$$

where

$$\Phi_p^k(m, \lambda, \mu) = \left[\frac{(k-p)(\lambda\mu k + \lambda - \mu) + p}{p}\right]^m.$$

By using the operator $\mathcal{D}_{p,\lambda,\mu}^m f(z)$ $(m \in \mathbb{N}_0)$, we introduce the new classes $\beta - \mathcal{US}_p(m, \lambda, \mu, \gamma, \alpha)$ and $\beta - \mathcal{UC}_p(m, \lambda, \mu, \gamma, \alpha)$ as follows:

$$\beta - \mathcal{US}_p(m, \lambda, \mu, \gamma, \alpha) = \left\{f \in \mathcal{A}_p : \ \mathcal{D}_{p,\lambda,\mu}^m f(z) \in \beta - \mathcal{US}_p(\gamma, \alpha)\right\}$$

and

$$\beta - \mathcal{UC}_p(m, \lambda, \mu, \gamma, \alpha) = \left\{f \in \mathcal{A}_p : \ \mathcal{D}_{p,\lambda,\mu}^m f(z) \in \beta - \mathcal{UC}_p(\gamma, \alpha)\right\}$$

where $f \in \mathcal{A}_p$, $0 \leq \alpha < p$, $\beta \geq 0$ and $\gamma \in \mathbb{C} - \{0\}$.

We note that by specializing the parameters m, p, γ, β and α in the classes $\beta - \mathcal{US}_p(m, \lambda, \mu, \gamma, \alpha)$ and $\beta - \mathcal{UC}_p(m, \lambda, \mu, \gamma, \alpha)$, these classes are reduced to several well-known subclasses of analytic functions. For example, for $m = 0$ the classes $\beta - \mathcal{US}_p(m, \lambda, \mu, \gamma, \alpha)$ and $\beta - \mathcal{UC}_p(m, \lambda, \mu, \gamma, \alpha)$ are reduced to the classes $\beta - \mathcal{US}_p(\gamma, \alpha)$ and $\beta - \mathcal{UC}_p(\gamma, \alpha)$, respectively. Someone can find more information about these classes in Cağlar [10], Deniz, Orhan and Sokol [11], Deniz, Cağlar and Orhan [12] and Orhan, Deniz and Raducanu [22].

1.2. Definition. Let $l = (l_1, l_2, ..., l_n) \in \mathbb{N}_0^n$, $\delta = (\delta_1, \delta_2, ..., \delta_n) \in \mathbb{R}_+^n$ for all $i = \overline{1, n}$, $n \in \mathbb{N}$. We define the following general integral operators

$$\mathcal{I}_{n,p,l}^{\delta,\lambda,\mu}(f_1, f_2, ..., f_n) : \mathcal{A}_p^n \to \mathcal{A}_p$$

$$\mathcal{I}_{n,p,l}^{\delta,\lambda,\mu}(f_1, f_2, ..., f_n) = \mathcal{F}_{n,p,l}^{\delta,\lambda,\mu}(z),$$

$$\mathcal{F}_{n,p,l}^{\delta,\lambda,\mu}(z) = \int_0^z pt^{p-1} \prod_{i=1}^{n} \left(\frac{\mathcal{D}_{p,\lambda,\mu}^{l_i} f_i(t)}{t^p} \right)^{\delta_i} dt \tag{1.7}$$

and

$$\mathcal{J}_{n,p,l}^{\delta,\lambda,\mu}(g_1, g_2, ..., g_n) : \mathcal{A}_p^n \to \mathcal{A}_p$$

$$\mathcal{J}_{n,p,l}^{\delta,\lambda,\mu}(g_1, g_2, ..., g_n) = \mathcal{G}_{n,p,l}^{\delta,\lambda,\mu}(z),$$

$$\mathcal{G}_{n,p,l}^{\delta,\lambda,\mu}(z) = \int_0^z pt^{p-1} \prod_{i=1}^{n} \left(\frac{\left(\mathcal{D}_{p,\lambda,\mu}^{l_i} g_i(t)\right)'}{pt^{p-1}} \right)^{\delta_i} dt \tag{1.8}$$

where $f_i, g_i \in \mathcal{A}_p$ for all $i = \overline{1,n}$ and $\mathcal{D}_{p,\lambda,\mu}^{l}$ is defined in Definition 1.1.

1.3. Remark. We note that if $l_1 = l_2 = ... = l_n = 0$, then the integral operator $\mathcal{F}_{n,p,l}^{\delta,\lambda,\mu}(z)$ is reduced to the operator $F_p(z)$ which was studied by Frasin (see [14]). Upon setting $p = 1$ in the operator (1.7), we can obtain the integral operator $\mathbb{F}_n(z)$ which was studied by Oros G.I. and Oros G.A. (see [23]). For $p = 1$ and $l_1 = l_2 = ... = l_n = 0$ in (1.7), the integral operator $\mathcal{F}_{n,p,l}^{\delta,\lambda,\mu}(z)$ is reduced to the operator $F_m(z)$ which was studied by Breaz D. and Breaz N. (see [6]). Observe that when $p = n = 1$, $l_1 = 0$ and $\delta_1 = \delta$, we obtain the integral operator $I_\delta(f)(z)$ which was studied by Pescar and Owa (see [24]), for $\delta_1 = \delta \in [0, 1]$ special case of the operator $I_\delta(f)(z)$ was studied by Miller, Mocanu and Reade (see [19]). For $p = n = 1$, $l_1 = 0$ and $\delta_1 = 1$ in (1.7), we have Alexander integral operator $I(f)(z)$ in [1].

1.4. Remark. For $l_1 = l_2 = ... = l_n = 0$ in (1.8) the integral operator $\mathcal{G}_{n,p,l}^{\delta,\lambda,\mu}(z)$ is reduced to the operator $G_p(z)$ which was studied by Frasin (see [14]). For $p = 1$ and $l_1 = l_2 = ... = l_n = 0$ in (1.8), the integral operator $\mathcal{G}_{n,p,l}^{\delta,\lambda,\mu}(z)$ is reduced to the operator $G_{\delta_1,\delta_2,...,\delta_m}(z)$ which was studied by Breaz D., Owa and Breaz N. (see [8]). If $p = n = 1$, $l_1 = 0$ and $\delta_1 = \delta$, we obtain the integral operator $G(z)$ which was introduced and studied by Pfaltzgraff (see [25]) and Kim and Merkes (see [18]).

In this paper, we consider the integral operators $\mathcal{F}_{n,p,l}^{\delta,\lambda,\mu}(z)$ and $\mathcal{G}_{n,p,l}^{\delta,\lambda,\mu}(z)$ defined by (1.7) and (1.8), respectively, and study their properties on the classes $\beta-\mathcal{US}_p(m, \lambda, \mu, \gamma, \alpha)$ and $\beta-\mathcal{UC}_p(m, \lambda, \mu, \gamma, \alpha)$. As special cases, the order of $p-$valently convexity and $p-$valently starlikeness of the operators $\int_0^z pt^{p-1}\left(\frac{f(t)}{t^p}\right)^{\delta} dt$ and $\int_0^z pt^{p-1}\left(\frac{g'(t)}{pt^{p-1}}\right)^{\delta} dt$ are given.

2. Convexity of the integral operators $\mathcal{F}_{n,p,l}^{\delta,\lambda,\mu}(z)$ and $\mathcal{G}_{n,p,l}^{\delta,\lambda,\mu}(z)$

First, in this section we prove a sufficient condition for the integral operator $\mathcal{F}_{n,p,l}^{\delta,\lambda,\mu}(z)$ to be $p-$valently convex of complex order.

2.1. Theorem. *Let $l = (l_1, l_2, ...l_n) \in \mathbb{N}_0^n$, $\delta = (\delta_1, \delta_2, ..., \delta_n) \in \mathbb{R}_+^n$, $0 \le \alpha_i < p$, $\gamma \in \mathbb{C} - \{0\}$ such that $0 < \sum_{i=1}^n \delta_i (p - \alpha_i) \le p$, $\beta_i \ge 0$ and $f_i \in \beta_i - \mathcal{US}_p(l_i, \lambda, \mu, \gamma, \alpha_i)$ for all $i = \overline{1,n}$. Then, the integral operator $\mathcal{F}_{n,p,l}^{\delta,\lambda,\mu}$ defined by (1.7) is $p-$valently convex of complex order γ ($\gamma \in \mathbb{C} - \{0\}$) and type $p-\sum_{i=1}^n \delta_i (p - \alpha_i)$, that is, $\mathcal{F}_{n,p,l}^{\delta,\lambda,\mu} \in \mathcal{C}_p(\gamma, p-\sum_{i=1}^n \delta_i (p - \alpha_i))$.*

Proof. From the definition (1.7), we observe that $\mathcal{F}_{n,p,l}^{\delta,\lambda,\mu}(z) \in \mathcal{A}_p$. On the other hand, it is easy to see that

$$\left[\mathcal{F}_{n,p,l}^{\delta,\lambda,\mu}(z)\right]' = pz^{p-1}\prod_{i=1}^{n}\left(\frac{\mathcal{D}_{p,\lambda,\mu}^{l_i} f_i(z)}{z^p}\right)^{\delta_i}. \tag{2.1}$$

Now we differentiate (2.1) logarithmically and we easily obtain

$$p+\frac{1}{\gamma}\left(\frac{z\left[\mathcal{F}_{n,p,l}^{\delta,\lambda,\mu}(z)\right]''}{\left[\mathcal{F}_{n,p,l}^{\delta,\lambda,\mu}(z)\right]'}+1-p\right) = p+\sum_{i=1}^{n}\delta_i\left(p+\frac{1}{\gamma}\left(\frac{z\left(\mathcal{D}_{p,\lambda,\mu}^{l_i} f_i\right)'(z)}{\left(\mathcal{D}_{p,\lambda,\mu}^{l_i} f_i\right)(z)}-p\right)\right)-p\sum_{i=1}^{n}\delta_i. \tag{2.2}$$

Then, we calculate the real part of both sides of (2.2) and obtain

$$\begin{aligned}\operatorname{Re}&\left\{p+\frac{1}{\gamma}\left(\frac{z\left[\mathcal{F}_{n,p,l}^{\delta,\lambda,\mu}(z)\right]''}{\left[\mathcal{F}_{n,p,l}^{\delta,\lambda,\mu}(z)\right]'}+1-p\right)\right\} \\ &= \sum_{i=1}^{n}\delta_i \operatorname{Re}\left\{p+\frac{1}{\gamma}\left(\frac{z\left(\mathcal{D}_{p,\lambda,\mu}^{l_i} f_i\right)'(z)}{\left(\mathcal{D}_{p,\lambda,\mu}^{l_i} f_i\right)(z)}-p\right)\right\}-p\sum_{i=1}^{n}\delta_i+p.\end{aligned} \tag{2.3}$$

Since $f_i \in \beta_i - \mathcal{US}_p(l_i, \lambda, \mu, \gamma, \alpha_i))$ for all $i = \overline{1,n}$ from (2.3), we have

$$\begin{aligned}\operatorname{Re}&\left\{p+\frac{1}{\gamma}\left(\frac{z\left[\mathcal{F}_{n,p,l}^{\delta,\lambda,\mu}(z)\right]''}{\left[\mathcal{F}_{n,p,l}^{\delta,\lambda,\mu}(z)\right]'}+1-p\right)\right\} \\ &> \sum_{i=1}^{n}\frac{\delta_i\beta_i}{|\gamma|}\left|\frac{z\left(\mathcal{D}_{p,\lambda,\mu}^{l_i} f_i\right)'(z)}{\left(\mathcal{D}_{p,\lambda,\mu}^{l_i} f_i\right)(z)}-p\right|+p-\sum_{i=1}^{n}\delta_i\left(p-\alpha_i\right).\end{aligned} \tag{2.4}$$

Because $\sum_{i=1}^{n}\frac{\delta_i\beta_i}{|\gamma|}\left|\frac{z\left(\mathcal{D}_{p,\lambda,\mu}^{l_i} f_i\right)'(z)}{\left(\mathcal{D}_{p,\lambda,\mu}^{l_i} f_i\right)(z)}-p\right| > 0$, from (2.4), we obtain

$$\operatorname{Re}\left\{p+\frac{1}{\gamma}\left(\frac{z\left[\mathcal{F}_{n,p,l}^{\delta,\lambda,\mu}(z)\right]''}{\left[\mathcal{F}_{n,p,l}^{\delta,\lambda,\mu}(z)\right]'}+1-p\right)\right\} > p-\sum_{i=1}^{n}\delta_i\left(p-\alpha_i\right).$$

Therefore, the operator $\mathcal{F}_{n,p,l}^{\delta,\lambda,\mu}(z)$ is $p-$valently convex of complex order $\gamma\,(\gamma \in \mathbb{C}-\{0\})$ and type $p-\sum_{i=1}^{n}\delta_i\left(p-\alpha_i\right)$. The proof of Theorem 2.1 is completed. ∎

2.2. Remark.

(1) Letting $\gamma = 1$ and $l_i = 0$ for all $i = \overline{1,n}$ in Theorem 2.1, we obtain Theorem 2.1 in [14].
(2) Letting $p = 1$, $\gamma = 1$ and $l_i = 0$ for all $i = \overline{1,n}$ in Theorem 2.1, we obtain Theorem 1 in [4].
(3) Letting $p = 1$, $\gamma = 1$ and $\alpha_i = l_i = 0$ for all $i = \overline{1,n}$ in Theorem 2.1, we obtain Theorem 2.5 in [7].
(4) Letting $p = 1$, $\beta = 0$ and $l_i = 0$ for all $i = \overline{1,n}$ in Theorem 2.1, we obtain Theorem 1 in [3].
(5) Letting $p = 1$, $\beta = 0$, $\alpha_i = \alpha$ and $l_i = 0$ for all $i = \overline{1,n}$ in Theorem 2.1, we obtain Theorem 1 in [9].

(6) Letting $p = 1$, $\beta = 0$, $\alpha_i = 0$ and $l_i = 0$ for all $i = \overline{1,n}$ in Theorem 2.1, we obtain Theorem 1 in [5].

Putting $n = 1$, $l_1 = 0$, $\delta_1 = \delta$, $\alpha_1 = \alpha$, $\beta_1 = \beta$ and $f_1 = f$ in Theorem 2.1, we have

2.3. Corollary. *Let $\delta > 0$, $0 \leq \alpha < p$, $\beta \geq 0$, $\gamma \in \mathbb{C} - \{0\}$ and $f \in \beta - \mathcal{US}_p(\gamma, \alpha)$. If $\delta \in (0, p \diagup (p - \alpha)]$, then $\int_0^z pt^{p-1} \left(\frac{f(t)}{t^p}\right)^\delta dt$ is convex of complex order $\gamma \left(\gamma \in \mathbb{C} - \{0\}\right)$ and type $p - \delta (p - \alpha)$ in $\mathbb{U}$.*

2.4. Theorem. *Let $l = (l_1, l_2, ..., l_n) \in \mathbb{N}_0^n$, $\delta = (\delta_1, \delta_2, ..., \delta_n) \in \mathbb{R}_+^n$, $0 \leq \alpha_i < p$, $\beta_i \geq 0$, $\gamma \in \mathbb{C} - \{0\}$ and $f_i \in \beta_i - \mathcal{US}_p(l_i, \lambda, \mu, \gamma, \alpha_i)$ for all $i = \overline{1,n}$. If*

$$\left| \frac{z \left(\mathcal{D}_{p,\lambda,\mu}^{l_i} f_i\right)'(z)}{\left(\mathcal{D}_{p,\lambda,\mu}^{l_i} f_i\right)(z)} - p \right| > -\frac{p + \sum_{i=1}^n \delta_i (\alpha_i - p)}{\sum_{i=1}^n \frac{\delta_i \beta_i}{|\gamma|}} \tag{2.5}$$

for all $i = \overline{1,n}$, then the integral operator $\mathcal{F}_{n,p,l}^{\delta,\lambda,\mu}(z)$ defined by (1.7) is $p-$valently convex of complex order γ $(\gamma \in \mathbb{C} - \{0\})$.

Proof. From (2.4) and (2.5), we easily get $\mathcal{F}_{n,p,l}^{\delta,\lambda,\mu}(z)$ is $p-$valently convex of complex order γ. ∎

From Theorem 2.4, we easily get

2.5. Corollary. *Let $l = (l_1, l_2, ..., l_n) \in \mathbb{N}_0^n$, $\delta = (\delta_1, \delta_2, ..., \delta_n) \in \mathbb{R}_+^n$, $0 \leq \alpha_i < p$, $\beta_i \geq 0$, $\gamma \in \mathbb{C} - \{0\}$ and $f_i \in \beta_i - \mathcal{US}_p(l_i, \lambda, \mu, \gamma, \alpha_i)$ for all $i = \overline{1,n}$. If $\mathcal{D}_{p,\lambda,\mu}^{l_i} f_i \in \mathcal{S}_p^*(\sigma)$, where $\sigma = p - \left(p - \sum_{i=1}^n \delta_i (p - \alpha_i)\right) \diagup \sum_{i=1}^n \frac{\delta_i \beta_i}{|\gamma|}$; $0 \leq \sigma < p$ for all $i = \overline{1,n}$, then the integral operator $\mathcal{F}_{n,p,l}^{\delta,\lambda,\mu}(z)$ is $p-$valently convex of complex order $\gamma \left(\gamma \in \mathbb{C} - \{0\}\right)$.*

Putting $n = 1$, $l_1 = 0$, $\delta_1 = \delta$, $\alpha_1 = \alpha$, $\beta_1 = \beta$ and $f_1 = f$ in Corollary 2.5, we have

2.6. Corollary. *Let $\delta > 0$, $0 \leq \alpha < p$, $\beta > 0$, $\gamma \in \mathbb{C} - \{0\}$ and $f \in \mathcal{S}_p^*(\rho)$ where $\rho = [\delta(p\beta + (p - \alpha)|\gamma|) - p|\gamma|] \diagup \delta\beta$; $0 \leq \rho < p$, then the integral operator $\int_0^z pt^{p-1} \left(\frac{f(t)}{t^p}\right)^\delta dt$ is $p-$valently convex of complex order $\gamma \left(\gamma \in \mathbb{C} - \{0\}\right)$ in $\mathbb{U}$.*

Next, we give a sufficient condition for the integral operator $\mathcal{G}_{n,p,l}^{\delta,\lambda,\mu}(z)$ to be $p-$valently convex of complex order.

2.7. Theorem. *Let $l = (l_1, l_2, ...l_n) \in \mathbb{N}_0^n$, $\delta = (\delta_1, \delta_2, ..., \delta_n) \in \mathbb{R}_+^n$, $0 \leq \alpha_i < p$, $\gamma \in \mathbb{C} - \{0\}$ such that $0 < \sum_{i=1}^n \delta_i (p - \alpha_i) \leq p$, $\beta_i \geq 0$ and $g_i \in \beta_i - \mathcal{UC}_p(l_i, \lambda, \mu, \gamma, \alpha_i)$ for all $i = \overline{1,n}$. Then, the integral operator $\mathcal{G}_{n,p,l}^{\delta,\lambda,\mu}$ defined by (1.8) is $p-$valently convex of complex order $\gamma \left(\gamma \in \mathbb{C} - \{0\}\right)$ and type $p - \sum_{i=1}^n \delta_i (p - \alpha_i)$, that is, $\mathcal{G}_{n,p,l}^{\delta,\lambda,\mu} \in \mathcal{C}_p(\gamma, p - \sum_{i=1}^n \delta_i (p - \alpha_i))$.*

Proof. From the definition (1.8), we observe that $\mathcal{G}_{n,p,l}^{\delta,\lambda,\mu}(z) \in \mathcal{A}_p$. On the other hand, it is easy to see that

$$\left[\mathcal{G}_{n,p,l}^{\delta,\lambda,\mu}(z)\right]' = pz^{p-1} \prod_{i=1}^n \left(\frac{\left(\mathcal{D}_{p,\lambda,\mu}^{l_i} g_i(z)\right)'}{pz^{p-1}} \right)^{\delta_i}. \tag{2.6}$$

Now, we differentiate (2.6) logarithmically and then do some simple calculations, we have

$$\text{(2.7)}\qquad \operatorname{Re}\left\{p+\frac{1}{\gamma}\left(\frac{z\left[\mathcal{G}_{n,p,l}^{\delta,\lambda,\mu}(z)\right]''}{\left[\mathcal{G}_{n,p,l}^{\delta,\lambda,\mu}(z)\right]'}+1-p\right)\right\}$$

$$=\sum_{i=1}^{n}\delta_i \operatorname{Re}\left\{p+\frac{1}{\gamma}\left(1+\frac{z\left(\mathcal{D}_{p,\lambda,\mu}^{l_i}g_i\right)''(z)}{\left(\mathcal{D}_{p,\lambda,\mu}^{l_i}g_i\right)'(z)}-p\right)\right\}-p\sum_{i=1}^{n}\delta_i+p.$$

Since $g_i \in \beta_i - \mathcal{UC}_p(l_i,\lambda,\mu,\gamma,\alpha_i)$ for all $i=\overline{1,n}$ from (2.7), we have

$$\text{(2.8)}\qquad \operatorname{Re}\left\{p+\frac{1}{\gamma}\left(\frac{z\left[\mathcal{G}_{n,p,l}^{\delta,\lambda,\mu}(z)\right]''}{\left[\mathcal{G}_{n,p,l}^{\delta,\lambda,\mu}(z)\right]'}+1-p\right)\right\}$$

$$> p-p\sum_{i=1}^{n}\delta_i+\sum_{i=1}^{n}\delta_i\left\{\beta_i\left|\frac{1}{\gamma}\left(\frac{z\left(\mathcal{D}_{p,\lambda,\mu}^{l_i}g_i\right)''(z)}{\left(\mathcal{D}_{p,\lambda,\mu}^{l_i}g_i\right)'(z)}+1-p\right)\right|+\alpha_i\right\}$$

$$= p-\sum_{i=1}^{n}\delta_i\left(p-\alpha_i\right)+\sum_{i=1}^{n}\frac{\delta_i\beta_i}{|\gamma|}\left|\frac{z\left(\mathcal{D}_{p,\lambda,\mu}^{l_i}g_i\right)''(z)}{\left(\mathcal{D}_{p,\lambda,\mu}^{l_i}g_i\right)'(z)}+1-p\right|$$

$$> p-\sum_{i=1}^{n}\delta_i\left(p-\alpha_i\right).$$

Therefore, the operator $\mathcal{G}_{n,p,l}^{\delta,\lambda,\mu}(z)$ is $p-$valently convex of complex order $\gamma\,(\gamma\in\mathbb{C}-\{0\})$ and type $p-\sum_{i=1}^{n}\delta_i\left(p-\alpha_i\right)$. This evidently completes the proof of Theorem 2.7. ∎

2.8. Remark.

(1) Letting $\gamma=1$ and $l_i=0$ for all $i=\overline{1,n}$ in Theorem 2.7, we obtain Theorem 3.1 in [14].

(2) Letting $p=1$, $\beta=0$ and $l_i=0$ for all $i=\overline{1,n}$ in Theorem 2.7, we obtain Theorem 3 in [3].

(3) Letting $p=1$, $\beta=0$, $\alpha_i=\mu$ and $l_i=0$ for all $i=\overline{1,n}$ in Theorem 2.7, we obtain Theorem 3 in [9].

(4) Letting $p=1$, $\beta=0$, $\alpha_i=0$ and $l_i=0$ for all $i=\overline{1,n}$ in Theorem 2.7, we obtain Theorem 2 in [5].

Putting $n=1$, $l_1=0$, $\delta_1=\delta$, $\alpha_1=\alpha$, $\beta_1=\beta$ and $g_1=g$ in Theorem 2.7, we have

2.9. Corollary. *Let $\delta>0$, $0\le\alpha<p$, $\beta\ge 0$, $\gamma\in\mathbb{C}-\{0\}$ and $g\in\beta-\mathcal{UC}_p(\gamma,\alpha)$. If $\delta\in(0,p/(p-\alpha)]$, then $\int_0^z pt^{p-1}\left(\frac{g't)}{pt^{p-1}}\right)^{\delta}dt$ is $p-$valently convex of complex order $\gamma\,(\gamma\in\mathbb{C}-\{0\})$ and type $p-\delta\,(p-\alpha)$ in $\mathbb{U}$.*

2.10. Theorem. *Let $l=(l_1,l_2,...,l_n)\in\mathbb{N}_0^n$, $\delta=(\delta_1,\delta_2,...,\delta_n)\in\mathbb{R}_+^n$, $0\le\alpha_i<p$, $\beta_i\ge 0$, $\gamma\in\mathbb{C}-\{0\}$ and $g_i\in\beta_i-\mathcal{UC}_p(l_i,\lambda,\mu,\gamma,\alpha_i)$ for all $i=\overline{1,n}$. If*

$$\text{(2.9)}\qquad \left|\frac{z\left(\mathcal{D}_{p,\lambda,\mu}^{l_i}g_i\right)''(z)}{\left(\mathcal{D}_{p,\lambda,\mu}^{l_i}g_i\right)'(z)}+1-p\right|>-\frac{p+\sum_{i=1}^{n}\delta_i\left(\alpha_i-p\right)}{\sum_{i=1}^{n}\frac{\delta_i\beta_i}{|\gamma|}}$$

for all $i=\overline{1,n}$, then the integral operator $\mathcal{G}_{n,p,l}^{\delta,\lambda,\mu}(z)$ defined by (1.8) is $p-$valently convex of complex order $\gamma\,(\gamma\in\mathbb{C}-\{0\})$.

Proof. From (2.8) and (2.9), we easily get $\mathcal{G}_{n,p,l}^{\delta,\lambda,\mu}(z)$ is $p-$valently convex of complex order γ. ∎

From Theorem 2.10, we easily get

2.11. Corollary. *Let $l = (l_1, l_2, ..., l_n) \in \mathbb{N}_0^n$, $\delta = (\delta_1, \delta_2, ..., \delta_n) \in \mathbb{R}_+^n$, $0 \leq \alpha_i < p$, $\beta_i \geq 0$, $\gamma \in \mathbb{C} - \{0\}$ and $g_i \in \beta_i - \mathcal{UC}_p(l_i, \lambda, \mu, \gamma, \alpha_i)$ for all $i = \overline{1,n}$. If $\mathcal{D}_{p,\lambda,\mu}^{l_i} g_i \in \mathcal{C}_p(\sigma)$, where $\sigma = p - \left(p - \sum_{i=1}^n \delta_i \left(p - \alpha_i\right)\right) \diagup \sum_{i=1}^n \frac{\delta_i \beta_i}{|\gamma|}$; $0 \leq \sigma < p$ for all $i = \overline{1,n}$, then the integral operator $\mathcal{G}_{n,p,l}^{\delta,\lambda,\mu}(z)$ is $p-$valently convex of complex order $\gamma\,(\gamma \in \mathbb{C} - \{0\})$.*

Putting $n = 1$, $l_1 = 0$, $\delta_1 = \delta$, $\alpha_1 = \alpha$, $\beta_1 = \beta$ and $g_1 = g$ in Corollary 2.11, we have

2.12. Corollary. *Let $\delta > 0$, $0 \leq \alpha < p$, $\beta > 0$, $\gamma \in \mathbb{C} - \{0\}$ and $g \in \mathcal{C}(\rho)$ where $\rho = [\delta(p\beta + (p - \alpha)\,|\gamma|) - p\,|\gamma|] \diagup \delta\beta$; $0 \leq \rho < p$, then the integral operator $\int_0^z pt^{p-1} \left(\frac{g' t)}{pt^{p-1}}\right)^\delta dt$ is convex of complex order $\gamma\,(\gamma \in \mathbb{C} - \{0\})$ in $\mathbb{U}$.*

3. Starlikeness of the integral operators $\mathcal{F}_{n,p,l}^{\delta,\lambda,\mu}(z)$ and $\mathcal{G}_{n,p,l}^{\delta,\lambda,\mu}(z)$

In this section, we will give the sufficient conditions for the integral operators $\mathcal{F}_{n,p,l}^{\delta,\lambda,\mu}$ and $\mathcal{G}_{n,p,l}^{\delta,\lambda,\mu}(z)$ to be $p-$valently starlike of complex order.
Let

$$H(\mathbb{U}) = \{f : \mathbb{U} \to \mathbb{C} : \ f \text{ analytic}\}$$
$$H\,[a, n] = \left\{f \in H(\mathbb{U}) : \ f(z) = a + a_n z^n + a_{n+1} z^{n+1} + ..., \ \ z \in \mathbb{U}, \ a \in \mathbb{C}, \ n \in \mathbb{N}_0\right\}.$$

In order to prove our main results, we shall need the following lemma due to S. S. Miller and P. T. Mocanu [20].

3.1. Lemma. *Let the function $\psi : \mathbb{C}^2 \times \mathbb{U} \to \mathbb{U}$ satisfy*

$$\text{Re } \psi(i\rho, \sigma; z) \leq 0$$

for all $\rho, \sigma \in \mathbb{R}$, $n \geq 1$ with $\sigma \leq -\frac{n}{2}(1+\rho^2)$. If $P \in H[1, n]$ and $\text{Re } \psi(P(z), zP'(z); z) > 0$ for every $z \in \mathbb{U}$, then

$$\text{Re } P(z) > 0.$$

3.2. Lemma. *Let $n \in \mathbb{N}$, $\kappa \in \mathbb{R}$, $u, v \in \mathbb{C}$ such that $\text{Im } v \leq 0$, $\text{Re}\,(u - \kappa v) \geq 0$. Assume the following condition*

$$\text{Re}\left\{P(z) + \frac{zP'(z)}{u - vP(z)}\right\} > \kappa, \quad (z \in \mathbb{U})$$

is satisfy such that $P \in H\,[P(0), n]$, $P(0) \in \mathbb{R}$ and $P(0) > \kappa$. Then,

$$\text{Re } P(z) > \kappa, \quad (z \in \mathbb{U}\,).$$

Proof. Firstly, we consider the function $R : \mathbb{U} \to \mathbb{C}$,

$$R(z) = \frac{P(z) - \kappa}{P(0) - \kappa}.$$

Then, $R(z) \in H\,[1, n]$. Furthermore, since $P(0) - \kappa > 0$ and

$$\text{Re}\left\{P(z) + \frac{zP'(z)}{u - vP(z)}\right\} > \kappa, \quad (z \in \mathbb{U})\,,$$

we have

$$\text{Re}\left\{R(z) + \frac{zR'(z)}{u - v\kappa - v\,(P(0) - \kappa)\,R(z)}\right\} > 0, \quad (z \in \mathbb{U})\,.$$

Now, we define the function ψ as follows

$$\psi(R(z), zR'(z); z) = R(z) + \frac{zR'(z)}{u - v\kappa - v\,(P(0) - \kappa)\,R(z)}.$$

Thus,

$$\operatorname{Re}\psi(R(z), zR'(z); z) > 0.$$

Now, so then we can use Lemma 3.1, we must show that the following condition

$$\operatorname{Re}\ \psi(i\rho, \sigma; z) \leq 0$$

is satisfied for $\rho \leq 0$, $\sigma \leq -\frac{1+\rho^2}{2}$ and $z \in \mathbb{U}$. Indeed, from hypothesis, we obtain

$$\begin{aligned}
\operatorname{Re}\ \psi(i\rho, \sigma; z) &= \operatorname{Re}\ \frac{\sigma}{u - v\kappa - v\,(P(0) - \kappa)\,\rho i} \\
&= \operatorname{Re}\ \frac{\sigma}{u_1 + iu_2 - (v_1 + iv_2)\,\kappa - (v_1 + iv_2)\,(P(0) - \kappa)\,\rho i} \\
&= \frac{\sigma\,[u_1 - v_1\kappa + v_2\rho\,(P(0) - \kappa)]}{[u_1 - v_1\kappa + v_2\rho\,(P(0) - \kappa)]^2 + [u_2 - v_2\kappa + v_1\rho\,(P(0) - \kappa)]^2} \leq 0.
\end{aligned}$$

Hence, from Lemma 3.1, we get Re $R(z) > 0$. Moreover, from the definition of $R(z)$, we obtain

$$\operatorname{Re} P(z) > \kappa, \quad (z \in \mathbb{U})\,.$$

∎

Now, we prove the following theorem using Lemma 3.2

3.3. Theorem. *Let* $l = (l_1, l_2, ...l_n) \in \mathbb{N}_0^n$, $\delta = (\delta_1, \delta_2, ..., \delta_n) \in \mathbb{R}_+^n$, $0 \leq \alpha_i < p$, $\gamma \in \mathbb{C} - \{0\}$ *such that* $0 < \sum_{i=1}^n \delta_i\,(p - \alpha_i) \leq p$, $\operatorname{Im}\gamma \geq 0$, $\operatorname{Re}\gamma \leq \frac{p}{\sum_{i=1}^n \delta_i (p-\alpha_i)}$, $\beta_i \geq 0$ *and* $f_i \in \beta_i - \mathcal{US}_p(l_i, \lambda, \mu, \gamma, \alpha_i)$ *for all* $i = \overline{1, n}$. *Then, the integral operator* $\mathcal{F}_{n,p,l}^{\delta,\lambda,\mu}$ *defined by (1.7) is* $p-$*valently starlike of complex order* $\gamma\,(\gamma \in \mathbb{C} - \{0\})$ *and type* $p - \sum_{i=1}^n \delta_i\,(p - \alpha_i)$, *that is,* $\mathcal{F}_{n,p,l}^{\delta,\lambda,\mu} \in \mathcal{S}_p^*(\gamma, p - \sum_{i=1}^n \delta_i\,(p - \alpha_i))$.

Proof. We define the analytic function $q:\ \mathbb{U} \to \mathbb{C},\ q(0) = p$ as follows

$$q(z) = p + \frac{1}{\gamma}\left(\frac{z\left[\mathcal{F}_{n,p,l}^{\delta,\lambda,\mu}(z)\right]'}{\left[\mathcal{F}_{n,p,l}^{\delta,\lambda,\mu}(z)\right]} - p\right).$$

Thus, we obtain

$$p + \gamma\,(q(z) - p) = \frac{z\left[\mathcal{F}_{n,p,l}^{\delta,\lambda,\mu}(z)\right]'}{\left[\mathcal{F}_{n,p,l}^{\delta,\lambda,\mu}(z)\right]}$$

$$\Rightarrow \quad \frac{\gamma z q'(z)}{p(1-\gamma) + \gamma q(z)} = 1 + \frac{z\left[\mathcal{F}_{n,p,l}^{\delta,\lambda,\mu}(z)\right]''}{\left[\mathcal{F}_{n,p,l}^{\delta,\lambda,\mu}(z)\right]'} - \frac{z\left[\mathcal{F}_{n,p,l}^{\delta,\lambda,\mu}(z)\right]'}{\left[\mathcal{F}_{n,p,l}^{\delta,\lambda,\mu}(z)\right]}$$

$$\Rightarrow \quad p + \gamma\,(q(z) - p) + \frac{\gamma z q'(z)}{p(1-\gamma) + \gamma q(z)} = 1 + \frac{z\left[\mathcal{F}_{n,p,l}^{\delta,\lambda,\mu}(z)\right]''}{\left[\mathcal{F}_{n,p,l}^{\delta,\lambda,\mu}(z)\right]'}$$

$$\Rightarrow \quad q(z) + \frac{zq'(z)}{p(1-b) + bq(z)} = p + \frac{1}{\gamma}\left[1 - p + \frac{z\left[\mathcal{F}_{n,p,l}^{\delta,\lambda,\mu}(z)\right]''}{\left[\mathcal{F}_{n,p,l}^{\delta,\lambda,\mu}(z)\right]'}\right].$$

When we consider this last equality and the inequality (2.2), we can write

$$q(z)+\frac{zq'(z)}{p(1-\gamma)+\gamma q(z)}=p+\sum_{i=1}^{n}\delta_i\left(p+\frac{1}{\gamma}\left(\frac{z\left(D^{l_i}_{p,\lambda,\mu}f_i\right)'(z)}{D^{l_i}_{p,\lambda,\mu}f_i(z)}-p\right)\right)-p\sum_{i=1}^{n}\delta_i.$$

Similarly to the proof of Theorem 2.1, it can be easly seen that

$$\mathrm{Re}\left\{q(z)+\frac{zq'(z)}{p(1-\gamma)+\gamma q(z)}\right\}>p-\sum_{i=1}^{n}\delta_i\left(p-\alpha_i\right).$$

Here, $q(0)=p>p-\sum_{i=1}^{n}\delta_i\left(p-\alpha_i\right)$ and the function q is analytic on $\mathbb{U}$. Also, when we write $\kappa=p-\sum_{i=1}^{n}\delta_i\left(p-\alpha_i\right)$, $u=p(1-\gamma)$ and $v=-\gamma$, we find $\mathrm{Im}\ v\leq 0$ and $\mathrm{Re}\left(u-\kappa v\right)\geq 0$. Hence, all the conditions of Lemma 3.1 are satisfied and so

$$\mathrm{Re}\, q(z)=\mathrm{Re}\left\{p+\frac{1}{\gamma}\left(\frac{z\left[\mathcal{F}^{\delta,\lambda,\mu}_{n,p,l}(z)\right]'}{\left[\mathcal{F}^{\delta,\lambda,\mu}_{n,p,l}(z)\right]}-p\right)\right\}>p-\sum_{i=1}^{n}\delta_i\left(p-\alpha_i\right).$$

Thus, the proof of the theorem is completed. ∎

Putting $n=1$, $l_1=0$, $\delta_1=\delta$, $\alpha_1=\alpha$, $\beta_1=\beta$ and $f_1=f$ in Theorem 3.3, we have

3.4. Corollary. *Let* $\delta>0$, $0\leq\alpha<p$, $\beta\geq 0$, $\gamma\in\mathbb{C}-\{0\}$, $\mathrm{Im}\ \gamma\geq 0$, $\mathrm{Re}\ \gamma\leq\frac{p}{\delta(p-\alpha)}$ *and* $f\in\beta-\mathcal{US}_p(\gamma,\alpha)$. *If* $\delta\in\left(0,\frac{p}{p-\alpha}\right]$ *then* $\int_0^z pt^{p-1}\left(\frac{f(t)}{t^p}\right)^{\delta}dt\in\mathcal{S}^*_p(\gamma,p-\delta\left(p-\alpha\right))$.

From Theorem 3.3, we obtain the following result.

3.5. Theorem. *Let* $l=(l_1,l_2,...l_n)\in\mathbb{N}_0^n$, $\delta=(\delta_1,\delta_2,...,\delta_n)\in\mathbb{R}_+^n$, $0\leq\alpha_i<p$, $\gamma\in\mathbb{C}-\{0\}$ *such that* $0<\sum_{i=1}^{n}\delta_i\left(p-\alpha_i\right)\leq p$, $\mathrm{Im}\ \gamma\geq 0$, $\mathrm{Re}\,\gamma\leq\frac{p}{\sum_{i=1}^{n}\delta_i(p-\alpha_i)}$, $\beta_i\geq 0$ *and* $f_i\in\beta_i-\mathcal{US}_p(l_i,\lambda,\mu,\gamma,\alpha_i)$ *for all* $i=\overline{1,n}$. *If the inequality (2.5) is satisfied for all* $i=\overline{1,n}$, *then the integral operator* $\mathcal{F}^{\delta,\lambda,\mu}_{n,p,l}(z)$ *defined by (1.7) is* $p-$*valently starlike of complex order* $\gamma\left(\gamma\in\mathbb{C}-\{0\}\right)$.

From Theorem 3.5, we get the following result.

3.6. Corollary. *Let* $l=(l_1,l_2,...l_n)\in\mathbb{N}_0^n$, $\delta=(\delta_1,\delta_2,...,\delta_n)\in\mathbb{R}_+^n$, $0\leq\alpha_i<p$, $\gamma\in\mathbb{C}-\{0\}$ *such that* $0<\sum_{i=1}^{n}\delta_i\left(p-\alpha_i\right)\leq p$, $\mathrm{Im}\ \gamma\geq 0$, $\mathrm{Re}\,\gamma\leq\frac{p}{\sum_{i=1}^{n}\delta_i(p-\alpha_i)}$, $\beta_i\geq 0$ *and* $f_i\in\beta_i-\mathcal{US}_p(l_i,\lambda,\mu,\gamma,\alpha_i)$ *for all* $i=\overline{1,n}$. *If* $\mathcal{D}^{l_i}_{p,\lambda,\mu}f_i\in\mathcal{S}^*_p(\sigma)$, *where* $\sigma=p-\left(p-\sum_{i=1}^{n}\delta_i\left(p-\alpha_i\right)\right)\big/\sum_{i=1}^{n}\frac{\delta_i\beta_i}{|\gamma|}$; $0\leq\sigma<p$ *for all* $i=\overline{1,n}$, *then the integral operator* $\mathcal{F}^{\delta,\lambda,\mu}_{n,p,l}(z)$ *is* $p-$*valently starlike of complex order* $\gamma\left(\gamma\in\mathbb{C}-\{0\}\right)$.

Next, we give a sufficient condition for the integral operator $\mathcal{G}^{\delta,\lambda,\mu}_{n,p,l}(z)$ to be $p-$valently starlike of complex order.

3.7. Theorem. *Let* $l=(l_1,l_2,...l_n)\in\mathbb{N}_0^n$, $\delta=(\delta_1,\delta_2,...,\delta_n)\in\mathbb{R}_+^n$, $0\leq\alpha_i<p$, $\gamma\in\mathbb{C}-\{0\}$ *such that* $0<\sum_{i=1}^{n}\delta_i\left(p-\alpha_i\right)\leq p$, $\mathrm{Im}\ \gamma\geq 0$, $\mathrm{Re}\,\gamma\leq\frac{p}{\sum_{i=1}^{n}\delta_i(p-\alpha_i)}$, $\beta_i\geq 0$ *and* $f_i\in\beta_i-\mathcal{UC}_p(l_i,\lambda,\mu,\gamma,\alpha_i)$ *for all* $i=\overline{1,n}$. *Then, the integral operator* $\mathcal{G}^{\delta,\lambda,\mu}_{n,p,l}$ *defined by (1.8) is* $p-$*valently starlike of complex order* $\gamma\left(\gamma\in\mathbb{C}-\{0\}\right)$ *and type* $p-\sum_{i=1}^{n}\delta_i\left(p-\alpha_i\right)$, *that is,* $\mathcal{G}^{\delta,\lambda,\mu}_{n,p,l}\in\mathcal{S}^*_p(\gamma,p-\sum_{i=1}^{n}\delta_i\left(p-\alpha_i\right))$.

Proof. Let us define the analytic function $q:\mathbb{U}\to\mathbb{C}$ given by

$$q(z)=p+\frac{1}{\gamma}\left(\frac{z\left(\mathcal{G}^{\delta,\lambda,\mu}_{n,p,l}(z)\right)'}{\left(\mathcal{G}^{\delta,\lambda,\mu}_{n,p,l}(z)\right)}-p\right).$$

Then, we follow the same steps as in the proof of Theorem 3.3, so we omit the details involved in this case. ∎

Putting $n=1$, $l_1=0$, $\delta_1=\delta$, $\alpha_1=\alpha$, $\beta_1=\beta$ and $g_1=g$ in Theorem 3.7, we have

3.8. Corollary. *Let* $\delta>0$, $0\le\alpha<p$, $\beta\ge 0$, $\gamma\in\mathbb{C}-\{0\}$, $\operatorname{Im}\gamma\ge 0$, $\operatorname{Re}\gamma\le\frac{p}{\delta(p-\alpha)}$ *and* $f\in\beta-\mathcal{UC}_p(\gamma,\alpha)$. *If* $\delta\in\left(0,\frac{p}{p-\alpha}\right]$, *then* $\int_0^z pt^{p-1}\left(\frac{g'(t)}{pt^{p-1}}\right)^{\delta}dt\in\mathcal{S}_p^*(\gamma,p-\delta(p-\alpha))$.

From Theorem 3.7, we obtain the following result.

3.9. Theorem. *Let* $l=(l_1,l_2,...l_n)\in\mathbb{N}_0^n$, $\delta=(\delta_1,\delta_2,...,\delta_n)\in\mathbb{R}_+^n$, $0\le\alpha_i<p$, $\gamma\in\mathbb{C}-\{0\}$ *such that* $0<\sum_{i=1}^n\delta_i(p-\alpha_i)\le p$, $\operatorname{Im}\gamma\ge 0$, $\operatorname{Re}\gamma\le\frac{p}{\sum_{i=1}^n\delta_i(p-\alpha_i)}$, $\beta_i\ge 0$ *and* $f_i\in\beta_i-\mathcal{US}_p(l_i,\lambda,\mu,\gamma,\alpha_i)$ *for all* $i=\overline{1,n}$. *If the inequality (2.9) is satisfied for all* $i=\overline{1,n}$, *then the integral operator* $\mathcal{G}_{n,p,l}^{\delta,\lambda,\mu}(z)$ *defined by (1.8) is* $p-$*valently starlike of complex order* $\gamma\,(\gamma\in\mathbb{C}-\{0\})$.

We obtain the following corollary using Theorem 3.9.

3.10. Corollary. *Let* $l=(l_1,l_2,...l_n)\in\mathbb{N}_0^n$, $\delta=(\delta_1,\delta_2,...,\delta_n)\in\mathbb{R}_+^n$, $0\le\alpha_i<p$, $\gamma\in\mathbb{C}-\{0\}$ *such that* $0<\sum_{i=1}^n\delta_i(p-\alpha_i)\le p$, $\operatorname{Im}\gamma\ge 0$, $\operatorname{Re}\gamma\le\frac{p}{\sum_{i=1}^n\delta_i(p-\alpha_i)}$, $\beta_i\ge 0$ *and* $f_i\in\beta_i-\mathcal{US}_p(l_i,\lambda,\mu,\gamma,\alpha_i)$ *for all* $i=\overline{1,n}$. *If* $\mathcal{D}_{p,\lambda,\mu}^{l_i}f_i\in\mathcal{C}_p(\sigma)$, *where* $\sigma=p-\left(p-\sum_{i=1}^n\delta_i(p-\alpha_i)\right)\big/\sum_{i=1}^n\frac{\delta_i\beta_i}{|\gamma|}$; $0\le\sigma<p$ *for all* $i=\overline{1,n}$, *then the integral operator* $\mathcal{G}_{n,p,l}^{\delta,\lambda,\mu}(z)$ *is* $p-$*valently starlike of complex order* $\gamma\,(\gamma\in\mathbb{C}-\{0\})$.

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