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# The Borel property for 4-dimensional matrices

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**Abstract**

In 1909 Borel has proved that "Almost all of the sequences of 0's and 1's are Cesàro summable to $\frac{1}{2}$". Then Hill has generalized Borel's result to two dimensional matrices. In this paper we investigate the Borel property for 4-dimensional matrices.

**Keywords:** Double sequences, Pringsheim convergence, the Borel Property, double sequences of 0's and 1's.

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## 1. Introduction

The summability of sequences of 0's and 1's has been studied by various authors ([1], [3], [6], [7], [8], [10]). In 1909 Borel proved that "Almost all of the sequences of 0's and 1's are Cesàro summable to $\frac{1}{2}$". Then Hill [6] has generalized Borel's result to general matrices. We say that the matrix has the Borel property, if a matrix sums almost all of the sequences of 0's and 1's to $\frac{1}{2}$. Establishing a one-to-one correspondence between the interval $(0, 1]$ and the collection of all sequences of 0's and 1's, Hill has given some necessary conditions and also some sufficient conditions for matrices to have the Borel property in [6], [7]. This property has also been examined in [5], [8].

In the present paper we investigate the Borel property for 4-dimensional matrices. In particular we exhibit some necessary and some sufficient conditions for 4-dimensional matrices to have the Borel property.

We first recall some basic notations and results related to double sequences.

A double sequence $s = (s_{ij})$ is said to be Pringsheim convergent (i.e., it is convergent in Pringsheim's sense) to $L$ if for every $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|s_{ij} - L| < \varepsilon$ whenever $i, j \geq N$ ([2], [11]). In this case $L$ is called the Pringsheim limit of $s$.

Throughout the paper when there is no confusion, convergence means the Pringsheim convergence.

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Let $X$ denote the set of all double sequences of 0's and 1's, that is

$$X = \{x = (x_{jk}) : x_{jk} \in \{0,1\} \text{ for each } j,k \in \mathbb{N}\}.$$

Let $\Re$ be the smallest $\sigma$-algebra of subsets of the set $X$ which contains all sets of the form

$$\{x = (x_{jk}) \in X : \ x_{j_1k_1} = a_1, ..., x_{j_nk_n} = a_n\}$$

where each $a_i \in \{0,1\}$ and the pairs $\{(j_ik_i)\}_{i=1}^n$ are pairwise distinct.

There exists a unique probability measure $P$ on the set $\Re$, such that

$$P\left(\{x = (x_{jk}) \in X : \ x_{j_1k_1} = a_1, ..., x_{j_nk_n} = a_n\}\right) = \frac{1}{2^n}$$

for all choices of $n$ and all pairwise disjoint pairs $\{(j_ik_i)\}_{i=1}^n$, and all choices of $a_1, ..., a_n$.

Recall that the functions $r_{jk}(x) = 2x_{jk} - 1$, for $x \in X$, are the Rademacher functions (see [4]).

Four dimensional Cesàro matrix $(C,1,1) = \left(c_{jk}^{nm}\right)$ is defined by

$$c_{jk}^{nm} = \begin{cases} \frac{1}{nm} & , \quad 1 \le j \le n \text{ and } 1 \le k \le m \\ 0 & , \quad \textit{otherwise.} \end{cases}$$

It is known that the $(C,1,1)$ matrix is an $RH$ regular, i.e., it sums every bounded convergent sequence to the same limit.

An element $x$ of $X$ is said to be normal ([4]) if for each $\varepsilon > 0$ there is a natural number $N_\varepsilon$ such that for $n,m \ge N_\varepsilon$ we have $\left| \frac{1}{nm} \sum_{\substack{j \le n \\ k \le m}} x_{jk} - \frac{1}{2} \right| < \varepsilon$. Let $\eta$ denote the set of all elements $x$ in $X$ that are normal. This means that normal elements are $(C,1,1)$-summable to $\frac{1}{2}$. It is also proved in [4] that $P(\eta) = 1$. So $(C,1,1)$ method has the Borel property.

It would be appropriate to recall the definition of bounded regularity.

**1.1. Definition.** Let $\mathcal{A} = \left(a_{jk}^{nm}\right)$ be a 4-dimensional matrix. If the limit

$$\lim_{n,m\to\infty} \sum_{j,k=1,1}^{\infty,\infty} a_{jk}^{nm} s_{jk} = L$$

exists, the double sequence $(s_{jk})$ is called $\mathcal{A}$-summable to $L$ and denoted by $s_{jk} \to L$ $(\mathcal{A})$. A matrix $\mathcal{A} = \left(a_{jk}^{nm}\right)$ is bounded regular if every bounded and convergent sequence $s = (s_{jk})$ is $\mathcal{A}$-summable to the same limit and $\mathcal{A}$-means are also bounded [9]. The next corollary characterizes bounded regular matrices.

**1.2. Proposition.** *$\mathcal{A} = \left(a_{jk}^{nm}\right)$ is bounded regular if and only if*

*(i)* $\lim_{n,m\to\infty} a_{jk}^{nm} = 0,\ (j,k = 1,2,...)$

*(ii)* $\lim_{n,m\to\infty} \sum_{j,k=1,1}^{\infty,\infty} a_{jk}^{nm} = 1,$

*(iii)* $\lim_{n,m\to\infty} \sum_{k=1}^{\infty} \left|a_{jk}^{nm}\right| = 0,\ (j = 1,2,...)$

*(iv)* $\lim_{n,m\to\infty} \sum_{j=1}^{\infty} \left|a_{jk}^{nm}\right| = 0,\ (k = 1,2,...)$

*(v)* $\sum_{j,k=1,1}^{\infty,\infty} \left|a_{jk}^{nm}\right| \le C < \infty,\ (m,n = 1,2,...).$

These conditions were first established by Robison [12].

## 2. The Borel Property

This section is devoted to the Borel property for 4-dimensional matrices.

**2.1. Theorem.** *If* $\mathcal{A} = \left(a_{jk}^{nm}\right)$ *has the Borel property, then the* $\sum_{j,k=1,1}^{\infty,\infty} a_{jk}^{nm}$ *series converges for each* $n, m$ *and tends to* $1$ *as* $n, m \to \infty$.

*Proof.* Since $\mathcal{A}$ has the Borel property, for almost all $x \in X$, we obtain $\lim_{n,m\to\infty} \sum_{j,k=1,1}^{\infty,\infty} a_{jk}^{nm} x_{jk} = \frac{1}{2}$. Indeed $P(E) = 1$ where

$$E = \left\{ x = (x_{jk}) \in X : \ (Ax)_{nm} \to \frac{1}{2} \right\}.$$

Let us define $\overline{x} = (\bar{x}_{jk})$ by

$$\bar{x}_{jk} = \begin{cases} 0 & , \quad x_{jk} = 1 \\ 1 & , \quad x_{jk} = 0 \end{cases} .$$

Let $Y = E \cap \eta$ and $\overline{Y} = \{(\overline{x}_{jk}) : \ x_{jk} \in Y\}$. We get $\overline{Y} = \overline{E} \cap \eta$. Since the mapping $(x_{jk}) \to (\overline{x}_{jk})$ preserves $P$ measure, we obtain $P\left(\overline{Y}\right) = 1$. So $Y \cap \overline{Y} \neq \emptyset$. If $x = (x_{jk}) \in Y \cap \overline{Y}$, then $x \in E$, $x \in \eta$ and $\overline{x} \in E$, $\overline{x} \in \eta$. Since $x$, $\overline{x} \in E$, it follows that

$$\sum_{j,k=1,1}^{\infty,\infty} a_{jk}^{nm} x_{jk} + \sum_{j,k=1,1}^{\infty,\infty} a_{jk}^{nm} \overline{x}_{jk} = \sum_{j,k=1,1}^{\infty,\infty} a_{jk}^{nm} \to 1 \quad (n, m \to \infty) .$$

This completes the proof. □

**2.2. Theorem.** *If* $\mathcal{A} = \left(a_{jk}^{nm}\right)$ *has the Borel property, then we have*

$$\sum_{j,k=1,1}^{\infty,\infty} \left(a_{jk}^{nm}\right)^2 < \infty$$

*for each* $n, m \in \mathbb{N}$.

*Proof.* Let $r_{jk}(x) = 2x_{jk} - 1$ be the Rademacher functions for double sequences. We have

$$\sum_{j,k=1,1}^{\infty,\infty} a_{jk}^{nm} x_{jk} = \frac{1}{2} \sum_{j,k=1,1}^{\infty,\infty} a_{jk}^{nm} + \frac{1}{2} \sum_{j,k=1,1}^{\infty,\infty} a_{jk}^{nm} r_{jk}(x) .$$

Since $\mathcal{A}$ has the Borel property and it follows from Teorem 2.1 that the series $\sum_{j,k=1,1}^{\infty,\infty} a_{jk}^{nm} r_{jk}(x)$ converges for each $n, m \in \mathbb{N}$ and almost all $x \in X$. Furthermore we obtain $\lim_{n,m} \sum_{j,k=1,1}^{\infty,\infty} a_{jk}^{nm} r_{jk}(x) = 0$ for almost all $x \in X$. So $\left( \sum_{j,k=1,1}^{\infty,\infty} a_{jk}^{nm} r_{jk}(x) \right)$ is convergent uniformly on a set $D$ with positive measure for each $n,m \in \mathbb{N}$ with respect to $x$. Hence for each $n,m \in \mathbb{N}$ and for every $\varepsilon > 0$, there exists $N_1, N_2 \in \mathbb{N}$ such that for $p, \mu > N_1$ and $q, \nu > N_2$

$$\left| \sum_{j,k=1,1}^{p,q} a_{jk}^{nm} r_{jk}(x) - \sum_{j,k=1,1}^{\mu,\nu} a_{jk}^{nm} r_{jk}(x) \right| < \varepsilon.$$

From the last inequality we immediately get

$$\begin{aligned} \varepsilon^2 P(D) &> \int_D \left( \sum_{E[\mu,p;\nu,q]} a_{jk}^{nm} r_{jk}(x) \right)^2 dP(x) \\ &= P(D) \sum_{E[\mu,p;\nu,q]} \left(a_{jk}^{nm}\right)^2 + R \end{aligned} \tag{2.1}$$

where

$$E[\mu,p;\nu,q] = \{(j,k) : \ \mu < j \le p \text{ or } \nu < k \le q\},$$

$$R = 2 \sum_{I[\mu,p;\nu,q]} a_{j_1k_1}^{nm} a_{j_2k_2}^{nm} \int_D r_{j_1k_1}(x) r_{j_2k_2}(x) \, dP(x)$$

and $I[\mu,p;\nu,q] = E[\mu,p;\nu,q] \cap \{(j,k) : \ j_1 \ne j_2 \text{ or } k_1 \ne k_2\}$. On the other hand using the Hölder inequality, we obtain

$$|R| \le 2 \left\{ \sum_{I[\mu,p;\nu,q]} \left(a_{j_1k_1}^{nm} a_{j_2k_2}^{nm}\right)^2 \right\}^{\frac{1}{2}} \left\{ \sum_{I[\mu,p;\nu,q]} \left( \int_D r_{j_1k_1}(x) r_{j_2k_2}(x) \, dP(x) \right)^2 \right\}^{\frac{1}{2}}.$$

Let $v_{j_1k_1j_2k_2}^2 = \left( \int_D r_{j_1k_1}(x) r_{j_2k_2}(x) \, dP(x) \right)^2$. From the Bessel inequality, we get

$$\sum_{\substack{1 \le j_1 < j_2 < \infty \\ 1 \le k_1 < k_2 < \infty}} v_{j_1k_1j_2k_2}^2 \le \int_X (\chi_D(x))^2 \, dP(x) = P(D).$$

For sufficiently large $p, q, \mu$ and $\nu$, we have

$$\left\{ \sum_{I[\mu,p;\nu,q]} v_{j_1k_1j_2k_2}^2 \right\}^{\frac{1}{2}} \le \frac{P(D)}{4}.$$

Hence we obtain

$$\begin{aligned} |R| &\le \frac{P(D)}{2} \left\{ \sum_{I[\mu,p;\nu,q]} \left(a_{j_1k_1}^{nm} a_{j_2k_2}^{nm}\right)^2 \right\}^{\frac{1}{2}} \\ &\le \frac{P(D)}{2} \left\{ \sum_{E[\mu,p;\nu,q]} \left(a_{j_1k_1}^{nm} a_{j_2k_2}^{nm}\right)^2 \right\}^{\frac{1}{2}} \\ &\le \frac{P(D)}{2} \sum_{E[\mu,p;\nu,q]} \left(a_{j_1k_1}^{nm}\right)^2. \end{aligned}$$

From (2.1) and last inequality, it follows that

$$\begin{aligned}\varepsilon^2 P(D) &> P(D) \sum_{E[\mu,p;\nu,q]} \left(a_{jk}^{nm}\right)^2 - \frac{P(D)}{2} \sum_{E[\mu,p;\nu,q]} \left(a_{jk}^{nm}\right)^2 \\ &= \frac{P(D)}{2} \sum_{E[\mu,p;\nu,q]} \left(a_{jk}^{nm}\right)^2 .\end{aligned}$$

Also since $P(D) > 0$, we obtain $\sum_{E[\mu,p;\nu,q]} \left(a_{jk}^{nm}\right)^2 < 2\varepsilon^2$. So for each $n, m \in \mathbb{N}$, the series $\left\{ \sum_{j,k=1,1}^{\infty,\infty} \left(a_{jk}^{nm}\right)^2 \right\}$ is convergent. Hence we obtain the result. □

**2.3. Theorem.** *If $\mathcal{A} = \left(a_{jk}^{nm}\right)$ has the Borel property and satisfies $(v)$, we have*

$$\sum_{j,k=1,1}^{\infty,\infty} \left(a_{jk}^{nm}\right)^2 = o(1), \qquad (n, m \to \infty). \tag{2.2}$$

*Proof.* Let $\sigma_{nm}(x) = \sum_{j,k=1,1}^{\infty,\infty} a_{jk}^{nm} r_{jk}(x)$. Using the equality

$$\sigma_{nm}^2(x) = \left( \sum_{j,k=1,1}^{\infty,\infty} a_{jk}^{nm} r_{jk}(x) \right) \left( \sum_{j,k=1,1}^{\infty,\infty} a_{jk}^{nm} r_{jk}(x) \right)$$

and $(v)$, we can easily obtain

$$\left|\sigma_{nm}^2(x)\right| \le \sum_{j,k=1,1}^{\infty,\infty} \left|a_{jk}^{nm}\right| \sum_{j,k=1,1}^{\infty,\infty} \left|a_{jk}^{nm}\right| < \infty$$

and hence

$$\sigma_{nm}^2(x) = \sum_{\substack{1 \le j_1, j_2 \le \infty \\ 1 \le k_1, k_2 \le \infty}} a_{j_1 k_1}^{nm} a_{j_2 k_2}^{nm} r_{j_1 k_1}(x) r_{j_2 k_2}(x)$$

is convergent uniformly almost everywhere. So we have

$$\begin{aligned}\int_X \sigma_{nm}^2(x)\, dP(x) &= \sum_{\substack{1 \le j_1, j_2 \le \infty \\ 1 \le k_1, k_2 \le \infty}} a_{j_1 k_1}^{nm} a_{j_2 k_2}^{nm} \int_X r_{j_1 k_1}(x) r_{j_2 k_2}(x)\, dP(x) \\ &= \sum_{j,k=1,1}^{\infty,\infty} \left(a_{jk}^{nm}\right)^2 .\end{aligned} \tag{2.3}$$

Since $\mathcal{A}$ has the Borel property, the uniformly bounded sequence $(\sigma_{nm}(x))$ converges to 0 for almost all $x$. From (2.3) and the Lebesgue convergence theorem, it follows that $\lim_{n,m\to\infty} \sum_{j,k=1,1}^{\infty,\infty} \left(a_{jk}^{nm}\right)^2 = 0$. This completes the proof. □

Now let us give sufficient conditions for the Borel property. First we consider the following sets

$$D_0(\mathcal{A}) = \left\{x \in X : \ (\mathcal{A}x)_{nm} \text{ diverges}\right\},$$
$$D_1(\mathcal{A}) = \left\{x \in X : \ (\mathcal{A}x)_{nm} \ \textit{converges}\right\},$$
$$D_2(\mathcal{A}) = \left\{x \in X : \ (\mathcal{A}x)_{nm} \to \frac{1}{2} \ (n,m \to \infty)\right\}.$$

We examine the relationship between these sets in the sense of $P$-measure.

**2.4. Theorem.** *Let* $\mathcal{A} = \left(a_{jk}^{nm}\right)$ *be a* 4*-dimensional bounded regular matrix. The sets* $D_1(\mathcal{A})$ *and* $D_2(\mathcal{A})$ *have the same measure and the value is either* 0 *or* 1.

*Proof.* Choose an arbitrary $x \in D_1(\mathcal{A})$ (or $D_2(\mathcal{A})$). Let $\widehat{x}$ be a sequence obtained by altering a finite term of $x$. We have the following equality

$$\begin{aligned}\sum_{j,k=1,1}^{\infty,\infty} a_{jk}^{nm}\widehat{x}_{jk} &= \sum_{j,k=1,1}^{j_0,k_0} a_{jk}^{nm}\widehat{x}_{jk} + \sum_{j>j_0 \text{ veya } k>k_0} a_{jk}^{nm}\widehat{x}_{jk}\\ &= \sum_{j,k=1,1}^{j_0,k_0} a_{jk}^{nm}\widehat{x}_{jk} + \sum_{j>j_0 \text{ veya } k>k_0} a_{jk}^{nm}x_{jk}.\end{aligned}$$

From Proposition 1.2 $(i)$, it follows $\widehat{x} \in D_1(\mathcal{A})$ (or $D_2(\mathcal{A})$). Hence the sets $D_1(\mathcal{A})$ and $D_2(\mathcal{A})$ are homogeneous [14]. Since homogeneous sets have measure 0 or 1 and $D_2(\mathcal{A}) \subset D_1(\mathcal{A})$, the proof will be completed if $P(D_1(\mathcal{A})) = 1$ implies $P(D_2(\mathcal{A})) = 1$. On the other hand we have

$$\lim_{n,m} \sum_{j,k=1,1}^{\infty,\infty} a_{jk}^{nm}x_{jk} = \lim_{n,m} \frac{1}{2} \sum_{j,k=1,1}^{\infty,\infty} a_{jk}^{nm} + \lim_{n,m} \frac{1}{2} \sum_{j,k=1,1}^{\infty,\infty} a_{jk}^{nm} r_{jk}(x) \tag{2.4}$$

where $r_{jk}(x) = 2x_{jk} - 1$. If we choose $x \in D_1(\mathcal{A})$, we get $\lim_{n,m} \sum_{j,k=1,1}^{\infty,\infty} a_{jk}^{nm} r_{jk}(x) = h(x)$ for almost all $x \in X$. From $(v)$, interchanging integral and sum we have

$$\begin{aligned}\int_X h(x)\,dx &= \int_X \left(\lim_{n,m} \sum_{j,k=1,1}^{\infty,\infty} a_{jk}^{nm} r_{jk}(x)\right) dP(x)\\ &= \lim_{n,m} \int_X \left(\sum_{j,k=1,1}^{\infty,\infty} a_{jk}^{nm} r_{jk}(x)\right) dP(x)\\ &= \lim_{n,m} \sum_{j,k=1,1}^{\infty,\infty} a_{jk}^{nm} \left(\int_X r_{jk}(x)\,dP(x)\right) = 0.\end{aligned}$$

Hence we have $h(x) = 0$ for almost all $x \in X$. Also since first part of the right hand side of (2.4) is $\frac{1}{2}$ we get $x \in D_2(\mathcal{A})$. This completes the proof. □

**2.5. Corollary.** *Let* $\mathcal{A} = \left(a_{jk}^{nm}\right)$ *be a* 4*-dimensional bounded regular matrix. The set* $D_0(\mathcal{A})$ *has measure* 0 *or* 1.

**2.6. Corollary.** *If* $\mathcal{A} = \left(a_{jk}^{nm}\right)$ *is a* 4*-dimensional bounded regular matrix sums almost all sequences of* 0*'s and* 1*'s, then the matrix has the Borel property.*

**2.7. Theorem.** *Let* $\mathcal{A} = \left(a_{jk}^{nm}\right)$ *be a 4-dimensional matrix. If* $P\left(D_1\left(\mathcal{A}\right)\right) = 1$, *then we have*

$$p_{nm} = \sum_{j,k=1,1}^{\infty,\infty} a_{jk}^{nm} \text{ converges for each } n, m \text{ and } \lim_{n,m} p_{nm} = p \text{ exists,}$$

$$\mathcal{A}_{nm} = \sum_{j,k=1,1}^{\infty,\infty} \left(a_{jk}^{nm}\right)^2 < \infty \text{ for each } n, m.$$

The proof of the theorem is similar to those of Theorems 2.1 and 2.2, and therefore is omitted.

**2.8. Lemma.** *If* $\mathcal{A}$ *satisfies condition* $(v)$, *then we have*

$$\int_X \left|\psi_{nm}\left(x\right)\right|^{2r} dP\left(x\right) \leq \frac{(2r)!}{2^r r!} \left(\mathcal{A}_{nm}\right)^r \tag{2.5}$$

*where* $r$ *is a positive integer,* $\psi_{nm}\left(x\right) = \sum_{j,k=1,1}^{\infty,\infty} a_{jk}^{nm} r_{jk}\left(x\right)$ *and* $\mathcal{A}_{nm} = \sum_{j,k=1,1}^{\infty,\infty} \left(a_{jk}^{nm}\right)^2$.

The proof can be proved using Lemma 1 of [13].

**2.9. Theorem.** *If* $\mathcal{A} = \left(a_{jk}^{nm}\right)$ *satisfies* $(ii)$, $(v)$ *and the series*

$$\sum_{n,m=1,1}^{\infty,\infty} \left( \sum_{j,k=1,1}^{\infty,\infty} \left(a_{jk}^{nm}\right)^2 \right)^r \tag{2.6}$$

*converges for some* $r > 0$, *then* $\mathcal{A}$ *has the Borel property.*

*Proof.* To complete the proof it is sufficient to show that

$$\sum_{j,k=1,1}^{\infty,\infty} a_{jk}^{nm} x_{jk} = \frac{1}{2} \sum_{j,k=1,1}^{\infty,\infty} a_{jk}^{nm} + \frac{1}{2} \sum_{j,k=1,1}^{\infty,\infty} a_{jk}^{nm} r_{jk}\left(x\right) \tag{2.7}$$

the limit of the right hand side of (2.7) equals $\frac{1}{2}$ for almost all $x \in X$. From Lemma 2.8, the inequality (2.5) holds for every positive integer $r$. On the other hand since the series in (2.6) converges for some $r > 0$, we easily get

$$\sum_{n,m=1,1}^{\infty,\infty} \int_X \left|\psi_{nm}\left(x\right)\right|^{2r} dP\left(x\right) < \infty.$$

Using the Beppo-Levi theorem, we have $\sum_{n,m=1,1}^{\infty,\infty} \left|\psi_{nm}\left(x\right)\right|^{2r} < \infty$ for almost all $x \in X$. Hence we obtain for almost all $x \in X$ that

$$\lim_{n,m\to\infty} \psi_{nm}\left(x\right) = 0.$$

This completes the proof. □

It is shown in [4] that the 4-dimensional Cesàro matrix method $(C, 1, 1)$ has the Borel property. We can also deduce this result from Theorem 2.9. We have already observed that (2.2) is a necessary condition for the Borel property. We raise the question whether the converse of Theorem 2.3 is true. The answer is no as the following example shows.

Since a 4-dimensional matrix can be considered as a matrix of infinite matrices, we can look at every entry as a matrix.

Consider the 4-dimensional Cesàro matrix, $(C,1,1) = \left(c_{jk}^{nm}\right)$. Now we construct a 4-dimensional matrix $\mathcal{A} = \left(a_{jk}^{nm}\right)$ as follows:

Shift the every column to the right in every possible order as the number of nonzero elements.

For example since there exist two possible order, we have

$$\left(a_{jk}^{11}\right) = \begin{bmatrix} 1 & 0 & 0 & ... \\ 0 & 0 & 0 & ... \\ ... \end{bmatrix}, \quad \left(a_{jk}^{12}\right) = \begin{bmatrix} 0 & 1 & 0 & ... \\ 0 & 0 & 0 & ... \\ ... \end{bmatrix}.$$

$$\left(a_{jk}^{13}\right) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & ... \\ 0 & 0 & 0 & 0 & ... \\ ... \end{bmatrix}, \quad \left(a_{jk}^{14}\right) = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & ... \\ 0 & 0 & 0 & 0 & ... \\ ... \end{bmatrix} ...$$

$$\left(a_{jk}^{17}\right) = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & ... \\ 0 & 0 & 0 & 0 & ... \\ ... \end{bmatrix}, \quad \left(a_{jk}^{18}\right) = \begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & ... \\ 0 & 0 & 0 & 0 & 0 & ... \\ ... \end{bmatrix}$$

in the above we have six possible orders. Now let us obtain $\left(a_{jk}^{21}\right), \ ... \ , \left(a_{jk}^{26}\right)$.

$$\left(a_{jk}^{21}\right) = \begin{bmatrix} \frac{1}{2} & 0 & ... \\ \frac{1}{2} & 0 & ... \\ 0 & 0 & ... \\ ... \end{bmatrix}, \quad \left(a_{jk}^{22}\right) = \begin{bmatrix} \frac{1}{2} & 0 & 0 & ... \\ 0 & \frac{1}{2} & 0 & ... \\ 0 & 0 & 0 & ... \\ ... \end{bmatrix}, ..., \left(a_{jk}^{26}\right) = \begin{bmatrix} 0 & 0 & 0 & ... \\ \frac{1}{2} & \frac{1}{2} & 0 & ... \\ 0 & 0 & 0 & ... \\ ... \end{bmatrix}.$$

Continuing this procedure we can construct the matrix $\mathcal{A}$.

Observe that the matrix $\mathcal{A}$ constructed above satisfies the condition (2.2).

Now let us consider the sequence $\{x_{jk}\}$ having $(\eta\mu + p)$ times 1 ve $(\eta\mu - p)$ times 0 in the rectangle $(\eta, 2\mu)$.

In the case of $p = 0$, an element of the matrix $\mathcal{A}$ which consists of 0's and $\frac{1}{\eta\mu}$'s sums the sequence $\{x_{jk}\}$ to 0 and the another one sums to 1. Let these terms be $(n_0, m_0)$ and $(n_1, m_1)$ respectively.

If $\left(a_{j,k}^{n_0,m_0}\right)$ containing $\dfrac{1}{\eta\mu}$'s, such that all the 0's of the sequence in the rectangle $(\eta, 2\mu)$ correspond with $\dfrac{1}{\eta\mu}$'s, we have

$$\sum_{j,k} a_{j,k}^{n_0,m_0} x_{jk} = 0.$$

Also if $\left(a_{j,k}^{n_1,m_1}\right)$ containing $\dfrac{1}{\eta\mu}$'s, such that all the 1's of the sequence in the rectangle $(\eta, 2\mu)$ correspond with $\dfrac{1}{\eta\mu}$'s, we have

$$\sum_{j,k} a_{j,k}^{n_1,m_1} x_{jk} = 1.$$

In the case of $p > 0$ there is an entry $\left(a_{j,k}^{n_0,m_0}\right)$ containing $\dfrac{1}{\eta\mu}$'s, such that all the 1's of the sequence in the rectangle $(\eta, 2\mu)$ correspond with $\dfrac{1}{\eta\mu}$'s, we have

$$\sum_{j,k} a_{j,k}^{n_0,m_0} x_{jk} = 1.$$

Also there is another entry $\left(a_{j,k}^{n_1,m_1}\right)$ containing $\dfrac{1}{\eta\mu}$'s, such that all the 0's of the sequence in the rectangle $(\eta, 2\mu)$ correspond with $\dfrac{1}{\eta\mu}$'s, we have

$$\sum_{j,k} a_{j,k}^{n_1,m_1} x_{jk} = \frac{p}{\eta\mu}.$$

In the case of $p < 0$ there is an entry $\left(a_{j,k}^{n_0,m_0}\right)$ containing $\dfrac{1}{\eta\mu}$'s, such that all the 0's of the sequence in the rectangle $(\eta, 2\mu)$ correspond with $\dfrac{1}{\eta\mu}$'s, we have

$$\sum_{j,k} a_{j,k}^{n_0,m_0} x_{jk} = 0.$$

Also there is another entry $\left(a_{j,k}^{n_1,m_1}\right)$ containing $\dfrac{1}{\eta\mu}$'s, such that all the 1's of the sequence in the rectangle $(\eta, 2\mu)$ correspond with $\dfrac{1}{\eta\mu}$'s, we have

$$\sum_{j,k} a_{j,k}^{n_1,m_1} x_{jk} = 1 + \frac{p}{\eta\mu}.$$

In any cases above, the oscillation of the sum $\sum a_{j,k}^{n,m} x_{jk}$ in the inner matrix containing $\frac{1}{\eta\mu}$'s is at least $1 - \frac{|p|}{\eta\mu}$. In order that $\{x_{jk}\}$ is $\mathcal{A}$-summable we necessarrily have $\frac{|p|}{\eta\mu} \to 1$, as $\eta, \mu \to \infty$.

Since almost all double sequences of 0's and 1's is $(C, 1, 1)$-summable to $\frac{1}{2}$, the set of sequences which $\frac{|p|}{\eta\mu}$ tends to 1 has $P$-measure 1. From this it follows that the set of sequences for which $\frac{|p|}{\eta\mu}$ tends to 1 is of $P$-measure 0. Therefore, $\mathcal{A}$ does not have the Borel property. That is condition (2.2) can not be sufficient.

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