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# A classification of biharmonic hypersurfaces in the Minkowski spaces of arbitrary dimension

Nurettin Cenk Turgay *

**Abstract**

In this paper we study hypersurfaces with the mean curvature function $H$ satisfying $\langle \nabla H, \nabla H \rangle = 0$ in a Minkowski space of arbitrary dimension. First, we obtain some conditions satisfied by connection forms of biconservative hypersurfaces with the mean curvature function whose gradient is light-like. Then, we use these results to get a classification of biharmonic hypersurfaces. In particular, we prove that if a hypersurface is biharmonic, then it must have at least 6 distinct principal curvatures under the hypothesis of having mean curvature function satisfying the condition above.

**Keywords:** biharmonic submanifolds, Lorentzian hypersurfaces, biconservative hypersurfaces, finite type submanifolds.

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## 1. Introduction

After Bang-Yen Chen conjectured that every biharmonic submanifold of a Euclidean space is minimal, biharmonic and biconservative submanifolds in semi-Euclidean spaces have been studied by many geometers (cf. [4, 5, 7, 8]). In particular, many results on biharmonic submanifolds in the Minkowski 4-space $\mathbb{E}^4_1$ and the semi-Euclidean space $\mathbb{E}^4_2$ have appeared since the middle of 1990s, [1, 2, 6, 9, 18].

On the other hand, several geometrical properties of biconservative submanifolds in Euclidean spaces have been obtained and some classification results of biconservative hypersurfaces have been given so far, [3, 12, 15, 17]. For example in [12], Hasanis and Vlachos obtained the complete classification of biconservative hypersurfaces in $\mathbb{E}^3$ and $\mathbb{E}^4$. Furthermore, Yu Fu have recently proved that the only biconservative surfaces in $\mathbb{E}^3_1$ are surfaces of revolution and null scrolls, [10]. Most recently, the complete classification of biconservative surfaces in 4-dimensional Lorentzian space forms is obtained in [11]

*Istanbul Technical University, Faculty of Science and Letters, Department of Mathematics, 34469 Maslak, Istanbul, Turkey, Email: turgayn@itu.edu.tr

Let $M$ be a hypersurface in $\mathbb{E}^{n+1}_s,\ s=0,1$ with the shape operator $S$, mean curvature $H$ and $x : M \to \mathbb{E}^m$ an isometric immersion. $M$ is said to be biharmonic if the equation $\Delta^2 x = 0$ is satisfied or, equivalently, the system of differential equations

(BC) $\quad S(\nabla H) + \varepsilon \dfrac{nH}{2}\nabla H = 0,$

(BH1) $\quad \Delta H + H\mathrm{tr}S^2 = 0$

is satisfied, where $N$ is the unit normal vector field (see [6, 13]) and $\varepsilon = \langle N, N\rangle$.

On the other hand, a hypersurface satisfying (BC) is said to be a biconservative hypersurface. From (BC), one can see that if a hypersurface $M$ with non-constant mean curvature is biconservative, then $\nabla H$ is an eigenvector of its shape operator. Note that along with the increase of index, the difference between Euclidean space and Minkowski space is the appearance of light-like vectors. Thus, the hypersurfaces with light-like $\nabla H$ has no counterparts in Euclidean spaces and they are worth to be studied separately in terms of being biconservative or biharmonic.

**1.1. Remark.** For ease of elaboration, we want to abbreviate a hypersurface with mean curvature whose gradient is light-like to a MCGL-hypersurface.

In this work we study MCGL-hypersurfaces in the Minkowski space of arbitrary dimension. In Sect. 2, after we describe our notations, we give a summary of the basic facts and formulas that we will use. In Sect. 3, we focus on biconservative MCGL-hypersurfaces and obtain some necessary conditions. In Sect. 4, we prove the non-existence of biharmonic MCGL-hypersurfaces under some conditions.

## 2. Prelimineries

Let $\mathbb{E}^m_s$ denote the pseudo-Euclidean $m$-space with the canonical pseudo-Euclidean metric tensor $g$ of index $s$ given by

$$g = -\sum_{i=1}^{s} dx_i^2 + \sum_{j=s+1}^{m} dx_j^2,$$

where $(x_1, x_2, \ldots, x_m)$ is a rectangular coordinate system in $\mathbb{E}^m_s$. A non-zero vector $v \in \mathbb{E}^m_s$ is called space-like, time-like or light-like if $\langle v, v\rangle > 0$, $\langle v, v\rangle < 0$ or $\langle v, v\rangle = 0$, respectively.

Consider an oriented hypersurface $M$ of the Minkowski space $\mathbb{E}^{n+1}_1$. We denote the Levi-Civita connections of $\mathbb{E}^{n+1}_1$ and $M$ by $\widetilde{\nabla}$ and $\nabla$, respectively. Then, the Gauss and Weingarten formulas are given, respectively, by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y)N, \tag{2.1}$$

$$\widetilde{\nabla}_X N = -S(X) \tag{2.2}$$

for all tangent vectors fields $X,\ Y$, where $h$, $\nabla^\perp$ and $S$ are the second fundamental form, the normal connection and the shape operator of $M$, respectively, and $N$ is the unit normal vector field associated with the orientation of $M$.

The Gauss and Codazzi equations are given, respectively, by

$$R(X,Y,Z,W) = \langle h(Y,Z), h(X,W)\rangle - \langle h(X,Z), h(Y,W)\rangle, \tag{2.3}$$

$$(\bar{\nabla}_X h)(Y,Z) = (\bar{\nabla}_Y h)(X,Z), \tag{2.4}$$

where $R$ is the curvature tensor associated with the connection $\nabla$ and $\bar{\nabla}h$ is defined by

$$(\bar{\nabla}_X h)(Y,Z) = \nabla^\perp_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

$M$ is said to be Lorentzian if its tangent space $T_mM$ at every point $m \in M$ has two linearly independent null vectors. In this case, there exists a pseudo-orthonormal frame field $\{e_1, e_2, \ldots, e_n\}$ of the tangent bundle of $M$ satisfying

$$\langle e_A, e_B\rangle = 1 - \delta_{AB}, \quad \langle e_A, e_a\rangle = 0, \quad \langle e_a, e_b\rangle = \delta_{ab}$$

for all $A, B = 1, 2$, $a, b = 3, 4, \ldots, n$. Then, the Levi-Civita connection $\nabla$ of $M$ becomes

$$\nabla_{e_i} e_1 = \phi_i e_1 + \sum_{b=3}^{n} \omega_{1b}(e_i) e_b, \tag{2.5a}$$

$$\nabla_{e_i} e_2 = -\phi_i e_2 + \sum_{b=3}^{n} \omega_{2b}(e_i) e_b, \tag{2.5b}$$

$$\nabla_{e_i} e_a = \omega_{2a}(e_i) e_1 + \omega_{1a}(e_i) e_2 + \sum_{b=3}^{n} \omega_{ab}(e_i) e_b, \tag{2.5c}$$

where $\phi_i = \phi(e_i) = \langle \nabla_{e_i} e_2, e_1\rangle$ and $\omega_{jk}(e_i) = \langle \nabla_{e_i} e_j, e_k\rangle$, i.e., $\phi = -\omega_{12}$.

On the other hand, the shape operator $S$ of an oriented Lorentzian hypersurface in $\mathbb{E}^{n+1}_1$ can be non-diagonalizable. If $S$ is non-diagonalizable, then its characteristic polynomial may also have complex roots. However, in this case all eigenvectors of $S$ are space-like.

Now, assume that $M$ has non-diagonalizable shape operator $S$ and consider the case that all of the eigenvalues of $S$ are real at any point of $M$. In this case, locally, we may assume that the multiplicities of eigenvalues are constant at every point of $M$. Therefore, [14, Lemma 2.3 and Lemma 2.5] imply that there exists an appropriate pseudo-orthonormal frame field $\{e_1, e_2, \ldots, e_n\}$ of smooth vector fields such that the matric representation of $S$ is in one of the following two forms.

$$\text{Case I. } S = \begin{pmatrix} k_1 & 1 & 0 & 0 & \ldots & 0 \\ 0 & k_1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & k_3 & 0 & \ldots & 0 \\ 0 & 0 & 0 & k_4 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & k_n \end{pmatrix}, \tag{2.6}$$

$$\text{Case II. } S = \begin{pmatrix} k_1 & 0 & 1 & 0 & \ldots & 0 \\ 0 & k_1 & 0 & 0 & \ldots & 0 \\ 0 & -1 & k_1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & k_4 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & k_n \end{pmatrix}$$

for some **smooth** functions $k_1, k_3, k_4, \ldots, k_n$, where the eigenvector $e_1$ of $S$ is light-like, (see also [13, 16]). With the abuse of terminology, we call these vector fields $e_1, e_2, \ldots, e_n$ as principal directions and the functions $k_1, k_3, k_4, \ldots, k_n$ as principal curvatures. Moreover, we put

$$s_1 = 2k_1 + k_3 + \cdots + k_n = nH,$$

where $H$ is the mean curvature of $M$.

## 3. Biconservative MCGL-hypersurfaces

In this section we focus on biconservative MCGL-hypersurfaces in the Minkowski space $\mathbb{E}_1^{n+1}$. As we described in the previous section, the shape operator $S$ of a MCGL-hypersurfaces in the Minkowski space $\mathbb{E}_1^{n+1}$ is one of two forms given in (2.6). We study these two cases separately.

**3.1. Case I.** Consider a hypersurface $M$ in the Minkowski space $\mathbb{E}_1^{n+1}$ with the shape operator $S$ given by case I of (2.6). Then, we have

$$\begin{aligned} & h(e_1, e_2) = -k_1, \quad h(e_2, e_2) = -1, \\ (3.1) \qquad & h(e_A, e_B) = \delta_{AB} k_A, \\ & h(e_1, e_1) = h(e_1, e_A) = h(e_2, e_A) = 0, \qquad A, B = 3, 4, \dots, n. \end{aligned}$$

Now, assume that $M$ is a biconservative MCGL-hypersurface, i.e., $\nabla s_1$ is light-like and (BC) is satisfied. Then, $e_1$ is proportional to $\nabla s_1$ and we have

$$k_1 = -\frac{s_1}{2}, \quad k_3 + k_4 + \cdots + k_n = 2s_1, \tag{3.2a}$$

$$e_1(k_1) = e_3(k_1) = e_4(k_1) = \cdots = e_n(k_1) = 0, \quad e_2(k_1) \neq 0. \tag{3.2b}$$

Let the distinct principal curvatures of $M$ be $K_1, K_2, \dots, K_p$ with the multiplicities $\nu_1, \nu_2, \dots, \nu_p$, respectively, i.e., the characteristic polynomial of $S$ is

$$\rho_S(t) = (t - K_1)^{\nu_1} (t - K_2)^{\nu_2} \dots (t - K_p)^{\nu_p} \tag{3.3}$$

with $K_1 = k_1$ and $\nu_1 \geq 2$. We also suppose that the functions $K_\alpha - K_\beta$ does not vanish on $M$, for all $\alpha \neq \beta \in \{1, 2, \dots, p\}$. Then, (3.2a) becomes

$$K_1 = -\frac{s_1}{2}, \quad \nu_2 K_2 + \nu_3 K_3 + \cdots + \nu_p K_p = (-2 - \nu_1) K_1. \tag{3.4}$$

On the other hand, from the Codazzi equation (2.4) for $X = e_1$, $Y = Z = e_A$ we get

$$\psi_\alpha = \omega_{1A}(e_A) = \frac{e_1(K_A)}{K_1 - K_A} \quad \text{if } k_A = K_\alpha, \ \alpha = 2, 3, \dots, p. \tag{3.5}$$

By rearranging the indices if necessary, we may assume that $\psi_2, \psi_3, \dots, \psi_r \neq 0$ and $\psi_{r+1} = \psi_{r+2} = \cdots = \psi_p = 0$ for some $r \leq p$. Thus, from (3.5) we have

$$e_1(K_A) = 0 \quad \text{if } k_A = K_\alpha, \ \alpha > r. \tag{3.6}$$

From Codazzi equation (2.4) for $X = e_1$, $Y = e_A$, $Z = e_B$ and $X = e_A$, $Y = e_B$, $Z = e_1$ we obtain

$$\omega_{1A}(e_B)(k_1 - k_A) = \omega_{1B}(e_A)(k_1 - k_B) = \omega_{AB}(e_1)(k_A - k_B), \quad A, B = 2, 3, \dots, n. \tag{3.7}$$

Moreover, from the equation $[e_A, e_B](k_1) = 0$ we have

$$\omega_{1A}(e_B) = \omega_{1B}(e_A).$$

By combining the above equation with (3.7) one may obtain

$$\omega_{1B}(e_A) = 0 \quad \text{if } k_A, k_B \neq K_1. \tag{3.8}$$

On the other hand, from the Codazzi equation $X = e_1$, $Y = e_1$, $Z = e_j$ and $X = e_2$, $Y = e_1$, $Z = e_j$ we have

$$\omega_{1j}(e_1) = 0, \quad j = 3, 4, \dots, n. \tag{3.9}$$

In addition, by combining the Codazzi equation (2.4) for $X = e_A$, $Y = e_1$, $Z = e_a$ and $[e_a, e_A]\,(k_1) = 0$, we obtain

$$\omega_{aA}(e_1) = \omega_{1A}(e_a) = \omega_{1a}(e_A) = 0 \tag{3.10}$$

for all $a, A = 3, 4, \dots, n$ such that $k_a = K_1 \neq k_A$. By summing up (3.8)-(3.10) we obtain

$$\begin{aligned} \nabla_{e_1} e_1 = \phi_1 e_1, \quad \nabla_{e_A} e_1 = \phi_A e_1 + \omega_{1A}(e_A) e_A, \\ \omega_{1A}(e_x) = 0, \quad x \neq 2, x \neq A \end{aligned} \tag{3.11}$$

for all $A = 3, 4, \dots, n$ such that $K_1 \neq k_A$.

Hence, by combaining (3.11) and the Gauss equation $R(e_A, e_1, e_1, e_A) = 0$ we obtain

$$e_1(\omega_{1A}(e_A)) = \omega_{1A}(e_A)(\phi_1 - \omega_{1A}(e_A)) \quad \text{if } k_A \neq K_1$$

from which we get

$$e_1(\psi_\alpha) = \psi_\alpha(\phi_1 - \psi_\alpha), \quad \psi_\alpha = 2, 3, \dots, r. \tag{3.12}$$

Next, we obtain the following lemma which we will use later.

**3.1. Lemma.** Let $M$ be a biconservative MCGL-hypersurface in the Minkowski space $\mathbb{E}_1^{n+1}$ with the shape operator given by (3.1). Then the functions $\psi_3, \psi_4, \dots, \psi_r$ defined above satisfy

$$W(\psi_2, \psi_3, \dots, \psi_r) \begin{pmatrix} \nu_2(K_1 - K_2) \\ \nu_3(K_1 - K_3) \\ \vdots \\ \nu_r(K_1 - K_r) \end{pmatrix} = 0, \tag{3.13a}$$

where $W(\psi_2, \psi_3, \dots, \psi_r)$ is an $r \times r$ matrix given by

$$W(\psi_2, \psi_3, \dots, \psi_r) = \begin{pmatrix} \psi_2 & \psi_3 & \dots & \psi_r \\ \psi_2^2 & \psi_3^2 & \dots & \psi_r^2 \\ \vdots & \vdots & \ddots & \vdots \\ \psi_2^r & \psi_3^r & \dots & \psi_r^r \end{pmatrix}. \tag{3.13b}$$

*Proof.* By applying $e_1$ to the second equation in (3.4) and using (3.2b), we obtain

$$\nu_2 e_1(K_2) + \nu_3 e_1(K_3) + \dots + \nu_p e_1(K_p) = 0. \tag{3.14}$$

Now, by induction we would like to show

$$\sum_{\alpha=2}^{r} (\psi_\alpha)^t \nu_\alpha (K_1 - K_\alpha) = 0, \quad t = 1, 2, \dots. \tag{3.15}$$

Note that by combining (3.5) and (3.14) one can obtain (3.15) for $t = 1$. Suppose that (3.15) is satisfied for $t = l - 1$, i.e.,

$$\sum_{\alpha=2}^{r} (\psi_\alpha)^{l-1} \nu_\alpha (K_1 - K_\alpha) = 0, \quad n = 1, 2, \dots. \tag{3.16}$$

By applying $e_1$ to this equation and using (3.2b), (3.5) and (3.12) we obtain

$$\sum_{\alpha=2}^{r} (l-1)(\psi_\alpha)^{l-1} \nu_\alpha (\phi_1 - \psi_\alpha)(K_1 - K_\alpha) = \sum_{\alpha=2}^{r} (\psi_\alpha)^l \nu_\alpha (K_1 - K_\alpha).$$

By combining this equation and (3.16) we obtain (3.15) for $t = l$. Thus, we have (3.15) for all $t$ which implies (3.13). $\square$

**3.2. Case II.** In this subsection, we consider the hypersurfaces with the shape operator given by case II of (2.6) in the Minkowski space $\mathbb{E}^{n+1}$. Now, assume that $M$ is a biconservative MCGL-hypersurface. In this case, we have

$$\begin{aligned} &h(e_1,e_2)=-k_1, \quad h(e_1,e_1)=h(e_1,e_3)=h(e_2,e_2)=0,\\ &h(e_3,e_3)=k_1, \quad h(e_A,e_B)=\delta_{AB}k_A, \\ &h(e_1,e_1)=h(e_1,e_A)=h(e_2,e_A)=h(e_3,e_A)=0, \qquad A,B=4,5,\dots,n. \end{aligned} \tag{3.17}$$

Assume that the characteristic polynomial of $S$ is as given by (3.3) with $K_1=k_1=-s_1/2$ and $\nu_1\geq 3$. Then, we have (3.4) and

$$e_1(K_1)=e_3(K_1)=e_4(K_1)=\cdots=e_n(K_1)=0, \quad e_2(K_1)\neq 0. \tag{3.18}$$

We again suppose that the functions $K_\alpha-K_\beta$ does not vanish on $M$.

Note that the Codazzi equation (2.4) for $X=e_1$, $Y=e_A$, $Z=e_A$ gives $e_1(k_A)=\omega_{1A}(e_A)(k_1-k_A)$ if $k_1\neq k_A$. Let $\psi_2,\psi_3,\dots,\psi_p$ be the functions defined by (3.5) such that $\psi_2,\psi_3,\dots,\psi_r\neq 0$ and $\psi_{r+1}=\psi_{r+2}=\cdots=\psi_p=0$ for some $r\leq p$.

(3.18) implies $[e_1,e_A](k_1)=0$. By computing the left-hand side of this equation we get $\omega_{1A}(e_1)=0, A=3,4,\dots,n$. In addition, the Codazzi equation (2.4) for $X=e_1$, $Y=e_2$, $Z=e_3$ gives $\phi_1=0$. Thus, we have $\nabla_{e_1}e_1=0$. Next, similar to previous subsection, we apply the Codazzi equation (2.4) for $X=e_i, Y=e_j, Z=e_k$ for each triplet (i, j, k) in the set $\{(1,2,a), (1,3,A), (3,A,1), (1,A,B), (A,B,1), (1,a,A)\}$ and combine equations obtained with $[e_A,e_B](k_1)=[e_A,e_a](k_1)=0$ to get $\nabla_{e_A}(e_1)\in \mathrm{span}\{e_1,e_A\}$ and $\omega_{1A}(e_x)=0,\ x\neq 2,A,$ where $A,B,a=4,5,\dots,n$ with $A\neq B$, $k_A,k_B\neq K_1$, $k_a=K_1$. By combaining these equations with the Gauss equation $R(e_3,e_1,e_1,e_3)=0$ we obtain

$$e_1(\psi_\alpha)=-\psi_\alpha^2, \quad \alpha=1,2,\dots,r.$$

Therefore, similar to Lemma 3.1 we have

**3.2. Lemma.** Let $M$ be a biconservative MCGL-hypersurface in the Minkowski space $\mathbb{E}_1^{n+1}$ with the shape operator given by (3.17). Then the functions $\psi_3,\psi_4,\dots,\psi_r$ defined above satisfy (3.13).

**3.3. Biconservative hypersurfaces.** In this subsection, we would like to obtain conditions satisfied by connection forms of biconservative MCGL-hypersurfaces (See [17, 10, 11] for implicit examples of biconservative hypersurfaces that have recently obtained).

Now we would like to obtain some necessary conditions for being biconservative of an MCGL-hypersurface by using Lemma 3.1 and Lemma 3.2.

**3.3. Proposition.** Let $M$ be an MCGL-hypersurface in the Minkowski space $\mathbb{E}_1^{n+1}$ and $e_1,e_2,\dots,e_n$ its principal directions with corresponding principal curvatures $k_1,k_1,k_3,k_4,\dots,k_n$ such that $e_1$ is proportional to gradient of its mean curvature. If $M$ is biconservative, then

(i) For any $3\leq i\leq n$ such that $k_i\neq k_1, \omega_{1i}(e_i)\neq 0$, there exists a $j\neq i$ such that $\omega_{1j}(e_j)=\omega_{1i}(e_i),\ k_j\neq k_i$.

(ii) Let $I_i=\{3\leq j\leq n|\omega_{1j}(e_j)=\omega_{1i}(e_i)\}$. Then,

$$\sum_{j\in I_i}(k_1-k_j)=0. \tag{3.19}$$

(iii) There exists a $j\in\{3,4,\dots,n\}$ such that $e_1(k_j)=\omega_{1l}(e_j)=0,\ k_1\neq k_l$.

*Proof.* Let $K_1,\dots,K_n$ and $\psi_2,\dots,\psi_r$ be the functions defined on the beginning of this section.

Assume that $\psi_2 \neq 0$ and $\psi_2 \neq \psi_j$, $2 < j \leq r$. Then, we have $\det W(\psi_2, \psi_3, \ldots, \psi_r) = 0$ from (3.13) since the functions $K_1 - K_2$ is non-vanishing by the assumptions. Therefore, $\psi_3, \ldots, \psi_r$ are not distinct and we may assume $\psi_{r-1} = \psi_r$. Thus (3.13) gives

$$W(\psi_2, \psi_3, \ldots, \psi_{r-1}) \begin{pmatrix} \nu_2(K_1 - K_2) \\ \nu_3(K_1 - K_3) \\ \vdots \\ \nu_r(K_1 - K_r) + \nu_{r-1}(K_1 - K_{r-1}) \end{pmatrix} = 0.$$

Since $(K_1 - K_2)$ is non-vanishing, the above equation implies that $\psi_3, \ldots, \psi_{r-1}$ are not distinct and we may assume either $\psi_{r-2} = \psi_{r-1}$ or $\psi_3 = \psi_4$. By repeating this procedure, one can get $\psi_3 = \cdots = \psi_{r-1}$ and

$$\begin{aligned} \psi_2(K_1 - K_2) + \psi_3 \left( \sum_{\alpha=3}^{r} \nu_\alpha (K_1 - K_\alpha) \right) &= 0, \\ \psi_2^2(K_1 - K_2) + \psi_3^2 \left( \sum_{\alpha=3}^{r} \nu_\alpha (K_1 - K_\alpha) \right) &= 0 \end{aligned}$$

which gives $\psi_2 = \psi_3$ or $K_1 - K_2 = 0$ which yields a contradiction. Hence we have (i) of the proposition.

Let $l - 1$ of $\psi_2, \psi_3, \ldots \psi_r$ be distinct and by rearranging indices if necessary, assume that they are $\psi_2, \psi_3, \ldots \psi_l$. Note that we have $l - 1 \leq (r-1)/2$ because of (i) of the proposition. Moreover, we have $\det W(\psi_2, \psi_3, \ldots, \psi_l) \neq 0$. Thus, (3.13) implies

$$W(\psi_2, \psi_3, \ldots, \psi_l) \begin{pmatrix} \sum\limits_{j \in I_2} \nu_j (K_1 - K_j) \\ \sum\limits_{j \in I_3} \nu_j (K_1 - K_j) \\ \vdots \\ \sum\limits_{j \in I_l} \nu_j (K_1 - K_j) \end{pmatrix} = 0$$

which gives (ii) of the proposition.

Now, assume that all of the functions $\omega_{1j}(e_j)$ are non-zero, i.e., $r = p$ and $\psi_2, \psi_3, \ldots \psi_l$ are distinct. Note that we have $\bigcup\limits_{j=2}^{l} I_j = \{2, 3, \ldots, p\}$ and (ii) of the proposition implies $\sum\limits_{j \in I_\alpha} \nu_j (K_1 - K_j) = 0$ or, equivalently,

$$\sum_{j \in I_\alpha} \nu_j K_j = \left( \sum_{j \in I_\alpha} \nu_j \right) K_1, \quad \alpha = 2, 3, \ldots, l.$$

By summing these equations over $\alpha$ we get

$$\nu_2 K_2 + \nu_3 K_3 + \cdots + \nu_p K_p = (\nu_2 + \nu_3 + \cdots + \nu_p) K_1.$$

However, this equation and (3.4) give $K_1 \equiv 0$ on $M$ which implies $\nabla s_1 = 0$. This is a contradiction because we have assummed that $\nabla s_1$ is light-like. Hence, we have (iii) of the proposition. $\square$

## 4. Biharmonic MCGL-Hypersurfaces

In this section we study biharmonic MCGL-hypersurfaces with the shape operator given by (3.1) in the Minkowski space $\mathbb{E}^{n+1}_1$ and obtain some classification results.

Let $M$ be a biharmonic MCGL- hypersurface with the shape operator given by (3.1). Then, we have (3.2a)-(3.13) obtained in the Sect. 3.1. In addition, from the Codazzi equation $X = e_2$, $Y = e_1$, $Z = e_2$ and $X = e_A$, $Y = e_2$, $Z = e_A$ we have

(4.1) $\quad e_2(k_1) = 2\phi_1 = \omega_{1A}(e_A), \quad \text{if } k_A = K_1,\ A > 2.$

Moreover, since $e_1e_2(k_1) = [e_1, e_2](k_1)$, by using (3.2b) we get

(4.2) $\quad e_1e_2(k_1) = -\phi_1 e_2(k_1).$

This equation and (4.1) imply

(4.3) $\quad e_1(\phi_1) = -\phi_1^2.$

Now we would like to consider the biharmonic equation (BH1). By a direct calculation using (3.2b) and (4.2) we get

$$\left(e_1e_2 + e_2e_1 - \sum_{j=3}^{n} e_je_j - \nabla_{e_1}e_2 - \nabla_{e_2}e_1\right)(k_1) = 0$$

which gives

$$\begin{aligned}\Delta k_1 &= \sum_{j=3}^{n} \omega_{1j}(e_j)e_2(k_1) = \sum_{\alpha=1}^{r}\left(\sum_{k_A=K_\alpha} \omega_{1A}(e_A)e_2(k_1)\right)\\ &= (2\nu_1\phi_1 + \nu_2\psi_2 + \nu_3\psi_3 + \cdots + \nu_r\psi_r)\, e_2(k_1).\end{aligned}$$

By combaining the above equation and (4.1), we see that the biharmonic equation (BH1) becomes

(BH2) $\quad (4\nu_1\phi_1 + 2\nu_2\psi_2 + 2\nu_3\psi_3 + \cdots + 2\nu_r\psi_r)\phi_1 = -k_1(\nu_1K_1^2 + \nu_2K_2^2 + \cdots + \nu_pK_p^2).$

**4.1. Theorem.** *There exists no biharmonic MCGL-hypersurface with at most 5 distinct principal curvatures and the shape operator given by* (3.1) *in the Minkowski space* $\mathbb{E}_1^{n+1}$.

*Proof.* Let the distinct principal curvatures of $M$ be $K_1, K_2, K_3, K_4, K_5$ with the multiplicities $\nu_1, \nu_2, \nu_3, \nu_4, \nu_5$, respectively, and consider the functions $\psi_2, \psi_3, \psi_4, \psi_5$ defined by (3.5). Now, toward contradiction we assume that $M$ is a biharmonic MCGL-hypersurface, i.e., (BC) and (BH1) are satisfied.

*Case I.* $p < 4$. If the number of distinct principal curvatures is less then 4, the proof directly follows from Proposition 3.3.

*Case II.* $p = 4$. Next, we consider the case that $M$ has exactly 4 distinct principal curvatures, i.e., $K_4 = K_5$. Then, because of (iii) of Proposition 3.3, we may assume $\psi_2 = 0$. Note that if $\psi_3 = 0$, then (i) of Proposition 3.3 implies $\psi_4 = 0$. In this subcase, we have $r = 1$ and (3.6) implies $e_1(K_\alpha) = 0$, $\alpha = 1, 2, 3, 4$. Thus (BH2) becomes

(4.4) $\quad 4\nu_1\phi_1^2 = -k_1(\nu_1K_1^2 + \nu_2K_2^2 + \nu_3K_3^2 + \nu_4K_4^2).$

By applying $e_1$ to this equation and using (4.3) one can find $\nu_1\phi_1^3 = 0$. However, this equation and (4.4) implies $k_1 \equiv 0$. Thus, we have $\nabla s_1 = 0$ which contradicts with being light-like of $\nabla s_1$. Hence, $\psi_3$ and $\psi_4$ are non-zero.

Therefore, (i) and (ii) of Proposition 3.3 imply

(4.5) $\quad \psi_3 = \psi_4, \quad \nu_3(K_1 - K_3) + \nu_4(K_1 - K_4) = 0.$

Thus, (BH2) becomes

(4.6) $\quad (a\phi_1 + b\psi_3)\phi_1 = -k_1(\nu_1K_1^2 + \nu_2K_2^2 + \nu_3K_3^2 + \nu_4K_4^2),$

where $a = 4\nu_1$ and $b = 2(\nu_3 + \nu_4)$ are some non-negative constants. Note that $\psi_2 = 0$ and (3.5) imply $e_1(K_2) = 0$.

Next, we apply $e_1$ to (4.6) and use (3.2b), (4.3), (3.12) to get

$$-(2a\phi_1^2 + b\psi_3^2)\phi_1 = -k_1 e_1(\nu_3 K_3^2 + \nu_4 K_4^2). \tag{4.7}$$

Then we use, (3.5) and (4.5) to compute the right-hand side of (4.7) and get

$$-(2a\phi_1^2 + b\psi_3^2)\phi_1 = -2k_1\psi_3(bK_1^2 - \nu_3 K_3^2 - \nu_4 K_4^2). \tag{4.8}$$

By applying $e_1$ to (4.8) again and using (3.2b), (4.3), (3.12) we get

$$\begin{aligned}(6a\phi_1^3 - b\phi_1\psi_3^2 + 2b\psi_3^3)\phi_1 = &- 2k_1\psi_3(\phi_1 - \psi_3)(bK_1^2 - \nu_3 K_3^2 - \nu_4 K_4^2)\\ &+ 2k_1\psi_3 e_1(\nu_3 K_3^2 + \nu_4 K_4^2)\end{aligned} \tag{4.9}$$

By combining (4.7), (4.8) and (4.9) we get

$$\left(6a\phi_1^3 - b\phi_1\psi_3^2 + 2b\psi_3^3 + (\phi_1 - 3\psi_3)(2a\phi_1^2 + b\psi_3^2)\right)\phi_1 = 0. \tag{4.10}$$

Thus, we have $\psi_3 = c\phi_1$ for a constant $c$. However, in this case, from (4.3) and (3.12) we get $c = 2$, i.e., $\psi_3 = 2\phi_1$. However, this equation and (4.10) give $(a + 2b)\phi_1^4 = 0$ which is impossible to be satisfied because $a$, $b$ are non-negative constants. Thus, the proof for this case is completed.

*Case III.* $p = 5$. Then, because of (iii) of Proposition 3.3, we may assume $\psi_2 = 0$. Note that, if $\psi_3 = 0$, then we have either $\psi_4 = \psi_5 \neq 0$ or $\psi_3 = \psi_4 = \psi_5 = 0$. However, these subcases and the other possible subcase $\psi_3 = \psi_4 = \psi_5$ are similar to case II. □

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