Hacettepe Journal of Mathematics and Statistics
Volume 44 (5) (2015), 1147 – 1156

Some results on σ-ideal of σ-prime ring

Selin Türkmen* and Neşet Aydın†

Abstract

Let R be a σ-prime ring with characteristic not 2, $Z(R)$ be the center of R, I be a nonzero σ-ideal of R, $\alpha, \beta : R \to R$ be two automorphisms, d be a nonzero (α, β)-derivation of R and h be a nonzero derivation of R. In the present paper, it is shown that (i) If $d(I) \subset C_{\alpha,\beta}$ and β commutes with σ then R is commutative. (ii) Let α and β commute with σ. If $a \in I \cap S_\sigma(R)$ and $[d(I), a]_{\alpha,\beta} \subset C_{\alpha,\beta}$ then $a \in Z(R)$. (iii) Let α, β and h commute with σ. If $dh(I) \subset C_{\alpha,\beta}$ and $h(I) \subset I$ then R is commutative.

Keywords: σ-prime ring, σ-ideal, (α, β)-derivation

2000 AMS Classification: Primary 16N60; Secondary 16W25, 16U80

Received 27/03/2014 : Accepted 28/07/2014 Doi : 10.15672/HJMS.2015449665

1. Introduction

Let R be an associative ring with center $Z(R)$. R is said to be 2-torsion free if whenever $2x = 0$ with $x \in R$, then $x = 0$. Recall that a ring R is prime if $aRb = 0$ implies $a = 0$ or $b = 0$. An involution σ of a ring R is an additive mapping satisfying $\sigma(xy) = \sigma(y)\sigma(x)$ and $\sigma^2(x) = x$ for all $x, y \in R$. A ring R equipped with an involution σ is said to be σ-prime if $aRb = aR\sigma(b) = 0$ implies $a = 0$ or $b = 0$. Note that every prime ring which has an involution σ is a σ-prime but the converse is in generally not true. An example, due to Shuliang [8], if R^0 denotes the opposite ring of a prime ring R, then $R \times R^0$ equipped with the exchange involution σ_{ex}, defined by $\sigma_{ex}(x, y) = (y, x)$, is σ_{ex}-prime but not prime. An additive subgroup I of R is said to be an ideal of R if $xr, rx \in I$ for all $x \in I$ and $r \in R$. An ideal I which satisfies $\sigma(I) = I$ is called a σ- ideal of R. An example, due to Rehman [8], Set $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\}$. We define a map $\sigma : R \to R$ as follows: $\sigma \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} c & -b \\ 0 & a \end{pmatrix}$. It is easy to check that $I = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in \mathbb{Z} \right\}$ is a

*Çanakkale Onsekiz Mart University, Dept. Math. Çanakkale - Turkey
Email: selinvurkac@gmail.com
†Çanakkale Onsekiz Mart University, Dept. Math. Çanakkale - Turkey
Email: neseta@comu.edu.tr

σ-ideal of R. Note that an ideal I of a ring R may be not a σ-ideal. Let $R = \mathbb{Z} \times \mathbb{Z}$. Consider a map $\sigma : R \to R$ defined by $\sigma((a, b)) = (b, a)$ for all $(a, b) \in R$. For an ideal $I = \mathbb{Z} \times \{0\}$ of R, I is not a σ-ideal of R since $\sigma(I) = \{0\} \times \mathbb{Z} \neq I$. $S_\sigma(R)$ will denote the set of symmetric and skew symmetric elements of R. i.e. $S_\sigma(R) = \{x \in R \mid \sigma(x) = \pm x\}$. As usual the commutator $xy - yx$ will be denoted by $[x, y] = xy - yx$. An additive mapping $h : R \to R$ is called a derivation if $h(xy) = h(x)y + xh(y)$ holds for all $x, y \in R$. For a fixed $a \in R$, the mapping $I_a : R \to R$ is given by $I_a(x) = [a, x]$ is a derivation which is said to be an inner derivation which is determined by a. Let α and β be two maps of R. Set $C_{\alpha,\beta} = \{c \in R \mid c\alpha(r) = \beta(r)c \text{ for all } r \in R\}$ and known as (α, β)-center of R. In particular, $C_{1,1} = Z(R)$ is the center of R, where $1 : R \to R$ is identity map. As usual the (α, β)-commutator $a\alpha(b) - \beta(b)a$ will be denoted by $[a, b]_{\alpha,\beta} = a\alpha(b) - \beta(b)a$. An additive mapping $d : R \to R$ is called an (α, β)-derivation if $d(xy) = d(x)\alpha(y) + \beta(x)d(y)$ holds for all $x, y \in R$. For a fixed $a \in R$, the mapping $I_a : R \to R$ is given by $I_a(x) = [a, x]_{\alpha,\beta}$ is an (α, β)-inner derivation which is determined by a.

Many studies have been objected the relationship between commutativity of a ring and the act of derivations defined on this ring. These results have been generalized by many authors in several ways. Herstein [2] proved that if R is a prime ring of characteristic not 2, d is a nonzero derivation of R and $a \in R$ such that $[a, d(R)] = 0$ then $a \in Z(R)$. N. Aydın and K. Kaya [1] proved that if R is a prime ring of characteristic not 2, I is a nonzero right ideal of R, σ and τ are two automorphisms of R, $d : R \to R$ is a nonzero (σ, τ)-derivations of R and $a \in R$ such that (i) $d(I) \subset Z(R)$ then R is commutative. (ii) $[d(R), a]_{\sigma,\tau} \subset C_{\alpha,\beta}$ then $a \in Z(R)$. In [5], this result was extended to on a σ-ideal of a σ-prime ring by L. Oukhtite and S. Salhi. On the other hand, Posner [7] proved that if R is a prime ring of characteristic not 2 and d_1, d_2 are derivations of R such that the composition $d_1 d_2$ is also a derivation; then one at least of d_1, d_2 is zero. K. Kaya [3] proved that if R is a prime ring of characteristic not 2, I is a nonzero ideal of R, σ and τ are two automorphisms of R, $d_1 : R \to R$ is a nonzero (σ, τ)-derivations of R and d_2 is a nonzero derivation of R such that $d_1 d_2(I) \subset C_{\sigma,\tau}$ then R is commutative. In [4], Posner's result was extended to a nonzero σ-ideal of a σ-prime ring by L. Oukhtite and S. Salhi. Motivated by these results, we follow this line of investigation.

In this paper, our main goal is to extend these results on a σ-ideal of a σ-prime ring.

Throughout the present paper, R is a σ-prime ring, $Z(R)$ is the center of R and α, β are two automorphisms of R. We use the following basic commutator identities:

$$[x, yz] = y[x, z] + [x, y]z$$
$$[xy, z] = x[y, z] + [x, z]y$$
$$[xy, z]_{\alpha,\beta} = x[y, z]_{\alpha,\beta} + [x, \beta(z)]y = x[y, \alpha(z)] + [x, z]_{\alpha,\beta}y$$
$$[x, yz]_{\alpha,\beta} = \beta(y)[x, z]_{\alpha,\beta} + [x, y]_{\alpha,\beta}\alpha(z)$$
$$\left[[x, y]_{\alpha,\beta}, z\right]_{\alpha,\beta} = \left[[x, z]_{\alpha,\beta}, y\right]_{\alpha,\beta} + [x, [y, z]]_{\alpha,\beta}$$

The material in this work is a part of first author's PH. Dissertation which is supervised by Prof. Dr. Neşet Aydın.

2. Results

For the proof of our theorems, we give the following known Lemmas.

2.1. Lemma. [6, Theorem 2.2] *Let I be a nonzero σ-ideal of σ-prime ring R. If a, b in R are such that $aIb = 0 = aI\sigma(b)$ then $a = 0$ or $b = 0$.*

2.2. Lemma. [5, Lemma 4] *Let R be a σ-prime ring with characteristic not two, d be a derivation of R satisfying $d\sigma = \pm\sigma d$ and I be a nonzero σ-ideal of R. If $d^2(I) = 0$ then $d = 0$.*

2.3. Lemma. *Let I be a nonzero σ-ideal of R and $a \in R$. If $Ia = 0$ (or $aI = 0$) then $a = 0$.*

Proof. Since I is a σ-ideal, we know that $IR \subset I$. By hypothesis, we have $IRa \subset Ia = 0$. Thus, we get $IRa = 0$. Moreover, since I is invariant under σ, we have $\sigma(I) Ra = 0$. It follows that

$$IRa = \sigma(I) Ra = 0$$

Using σ-primeness of R, we get

$$a = 0$$

Similarly, using $RI \subset I$, one can show that if $aI = 0$ then $a = 0$. □

2.4. Lemma. *Let $a, b \in R$.*

i) *If $b, ab \in C_{\alpha,\beta}$ and a (or b) $\in S_\sigma(R)$ then $a \in Z(R)$ or $b = 0$.*
ii) *If $a, ab \in C_{\alpha,\beta}$ and a (or b) $\in S_\sigma(R)$ then $a = 0$ or $b \in Z(R)$.*

Proof. *i*) By the hypothesis, we have $[ab, r]_{\alpha,\beta} = 0$ for all $r \in R$. Expanding this equation by using $b \in C_{\alpha,\beta}$, holding for all $r \in R$

$$\begin{aligned} 0 = [ab, r]_{\alpha,\beta} &= a[b, r]_{\alpha,\beta} + [a, \beta(r)] b \\ &= [a, \beta(r)] b \end{aligned}$$

Since $b \in C_{\alpha,\beta}$, we get

$$[a, R] Rb = 0 \tag{2.1}$$

In the event of $a \in S_\sigma(R)$, we derive $\sigma([a, R]) Rb = 0$. Using the last obtained equation together with (2.1), we yield

$$[a, R] Rb = \sigma([a, R]) Rb = 0$$

Applying the σ-primeness of R, we have

$$a \in Z(R) \text{ or } b = 0$$

In case of $b \in S_\sigma(R)$, from (2.1), we get $[a, R] R\sigma(b) = 0$. Using the last obtained equation together with (2.1), we find

$$[a, R] Rb = [a, R] R\sigma(b) = 0$$

Applying the σ-primeness of R,

$$a \in Z(R) \text{ or } b = 0$$

is obtained.

ii) Since $ab \in C_{\alpha,\beta}$, we have $[ab, r]_{\alpha,\beta} = 0$ for all $r \in R$. Expanding this equation by using $a \in C_{\alpha,\beta}$, holding for all $r \in R$

$$\begin{aligned} 0 = [ab, r]_{\alpha,\beta} &= a[b, \alpha(r)] + [a, r]_{\alpha,\beta} b \\ &= a[b, \alpha(r)] \end{aligned}$$

Since $a \in C_{\alpha,\beta}$,

$$aR[b, R] = 0$$

is obtained. After here, it is similar as above. □

2.5. Lemma. *Let I be a nonzero σ-ideal of R and h be a nonzero derivation of R. If $h(I) \subset Z(R)$ then R is commutative.*

Proof. For any $x, y \in I$ and $r \in R$, using hypothesis,

$$\begin{aligned} 0 &= [r, h(xy)] = [r, h(x) y + xh(y)] \\ &= h(x)[r, y] + [r, h(x)] y + x [r, h(y)] + [r, x] h(y) \\ &= h(x)[r, y] + [r, x] h(y) \end{aligned}$$

And so,

$$h(x)[r, y] + [r, x] h(y) = 0, \ \forall x, y \in I, r \in R$$

is obtained. In the last equality, x is taken instead of r and we obtain $h(x)[x, y] = 0$ for all $x, y \in I$. Substituting y by zy where $z \in I$, it holds that

$$(2.2) \qquad h(x) I [x, y] = 0, \ \forall x, y \in I$$

It is supposed that $x \in I \cap S_\sigma(R)$. In (2.2), replacing y with $\sigma(y)$, we get $h(x) I \sigma([x, y]) = 0$ for all $y \in I$. According to Lemma 2.1, it is derived that

$$(2.3) \qquad h(x) = 0 \text{ or } x \in Z(R), \ \forall x \in I \cap S_\sigma(R)$$

Assume that $x \in I$. In this case, $x - \sigma(x) \in I \cap S_\sigma(R)$. So, from (2.3), we have $h(x - \sigma(x)) = 0$ or $x - \sigma(x) \in Z(R)$ for all $x \in I$. We set $A = \{x \in I \mid h(x - \sigma(x)) = 0\}$ and $B = \{x \in I \mid x - \sigma(x) \in Z(R)\}$. It is clear that A and B are additive subgroups of I such that $I = A \cup B$. But, a group can not be an union of two of its proper subgroups. Therefore, it is implied $I = A$ or $I = B$. In the former case, $h(x) = h(\sigma(x))$ for all $x \in I$. In (2.2), replacing y by $\sigma(y)$ and x by $\sigma(x)$, we have $h(x) I \sigma([x, y]) = 0$ for all $x, y \in I$. And so,

$$h(x) I [x, y] = h(x) I \sigma([x, y]) = 0, \ \forall x, y \in I$$

is obtained. By Lemma 2.1, get $h(x) = 0$ or $x \in Z(R)$ for all $x \in I$. In the latter case, $x - \sigma(x) \in Z(R)$ for all $x \in I$. This means $[x, r] = [\sigma(x), r]$ for all $x \in I, r \in R$. In (2.2), taking $\sigma(y)$ instead of y, we get $h(x) I \sigma([x, y]) = 0$ for all $x, y \in I$. And so,

$$h(x) I [x, y] = h(x) I \sigma([x, y]) = 0, \ \forall x, y \in I$$

is derived. According to Lemma 2.1, we have $h(x) = 0$ or $x \in Z(R)$ for all $x \in I$. So, both the cases yield either

$$h(x) = 0 \text{ or } x \in Z(R), \ \forall x \in I$$

Now, we set $K = \{x \in I \mid h(x) = 0\}$ and $L = \{x \in I \mid x \in Z(R)\}$. Each of K and L is an additive subgroup of I. Moreover, I is the set-theoretic union of K and L. But a group can not be the set-theoretic union of two proper subgroups, hence $I = K$ or $I = L$. In the former case, $h(I) = 0$. So, we have $h = 0$. But, h is a nonzero derivation of R. So, from the latter case, we get $I \subseteq Z(R)$. Therefore, R is commutative. □

2.6. Lemma. *Let I be a nonzero σ-ideal of R, d be a (α, β)-derivation of R and $a \in R$. If $ad(I) = \sigma(a) d(I) = 0$ and β commutes with σ (or $d(I) a = d(I) \sigma(a) = 0$ and α commutes with σ) then $a = 0$ or $d = 0$.*

Proof. For any $x \in I$ and $r \in R$, using $ad(I) = 0$, we get

$$\begin{aligned} 0 &= ad(xr) = ad(x) \alpha(r) + a\beta(x) d(r) \\ &= a\beta(x) d(r) \end{aligned}$$

It becomes

$$a\beta(I) d(r) = 0, \ \forall r \in R$$

Similarly, using $\sigma(a) d(I) = 0$, we derive

$$\sigma(a) \beta(I) d(r) = 0, \ \forall r \in R$$

And so,

$$a\beta(I) d(r) = \sigma(a) \beta(I) d(r) = 0, \ \forall r \in R$$

is obtained. Since β commutes with σ, $\beta(I)$ is a nonzero σ-ideal of R. Therefore, according to Lemma 2.1, we have

$$a = 0 \text{ or } d = 0$$

Let us consider $d(I) a = d(I) \sigma(a) = 0$ and α commutes with σ. Since $\alpha(I)$ is a nonzero σ-ideal of R, one can show that $a = 0$ or $d = 0$ similarly as above. □

2.7. Lemma. *Let I be a nonzero σ-ideal of R and d be a (α, β)-derivation of R. If $d(I) = 0$ and α (or β) commutes with σ then $d = 0$.*

Proof. By hypothesis, it holds that for all $x \in I$ and $r \in R$

$$\begin{aligned} 0 &= d(rx) = d(r) \alpha(x) + \beta(r) d(x) \\ &= d(r) \alpha(x) \end{aligned}$$

Thus, we get

$$d(r) \alpha(I) = 0, \ \forall r \in R$$

Since α commutes with σ, $\alpha(I)$ is a nonzero σ-ideal of R. Therefore, by Lemma 2.3, we have $d = 0$.

Suppose that β commutes with σ. For any $x \in I$ and $r \in R$, from the hypothesis, we get

$$\begin{aligned} 0 &= d(xr) = d(x) \alpha(r) + \beta(x) d(r) \\ &= \beta(x) d(r) \end{aligned}$$

So, it yields that

$$\beta(I) d(r) = 0, \ \forall r \in R$$

Since β commutes with σ, $\beta(I)$ is a nonzero σ-ideal of R. Therefore, by Lemma 2.3, we have $d = 0$. □

2.8. Theorem. *Let R be a σ-prime ring with characteristic not 2, I be a nonzero σ-ideal of R and d be a nonzero (α, β)-derivation of R such that β commutes with σ. If $d(I) \subset C_{\alpha,\beta}$ then R is commutative.*

Proof. By hypothesis, $d(x^2) = d(x) \alpha(x) + \beta(x) d(x) \in C_{\alpha,\beta}$ for all $x \in I$. Using $d(x) \in C_{\alpha,\beta}$, we get $2\beta(x) d(x) \in C_{\alpha,\beta}$. Since $char R \neq 2$, we obtain $\beta(x) d(x) \in C_{\alpha,\beta}$ which means $[\beta(x) d(x), r]_{\alpha,\beta} = 0$ for all $r \in R, x \in I$. Expanding this equation by using $d(x) \in C_{\alpha,\beta}$, we arrive

$$\begin{aligned} 0 &= [\beta(x) d(x), r]_{\alpha,\beta} = \beta(x) [d(x), r]_{\alpha,\beta} + \beta([x, r]) d(x) \\ &= \beta([x, r]) d(x) \end{aligned}$$

Since $d(x) \in C_{\alpha,\beta}$, it follows that

$$\beta([x, r]) Rd(x) = 0, \ \forall x \in I, r \in R \tag{2.4}$$

Assume that $x \in I \cap S_\sigma(R)$. In (2.4) taking $\sigma(r)$ instead of r and using the fact that β commutes with σ, we have $\sigma(\beta([x, r])) Rd(x) = 0$ for all $x \in I, r \in R$. Since R is σ-prime, we derive

$$x \in Z(R) \text{ or } d(x) = 0, \ \forall x \in I \cap S_\sigma(R)$$

Assume that $x \in I$. In this case, $x - \sigma(x) \in I \cap S_\sigma(R)$. Therefore, we have $x - \sigma(x) \in Z(R)$ or $d(x - \sigma(x)) = 0$ for all $x \in I$. Set $A = \{x \in I \mid d(x - \sigma(x)) = 0\}$ and $B = \{x \in I \mid x - \sigma(x) \in Z(R)\}$. It is clear that A and B are additive subgroups of I such that $I = A \cup B$. But, a group can not be an union of two of its proper subgroups. Therefore, we yield either $I = A$ or $I = B$. In the former case, $d(x) = d(\sigma(x))$ for all $x \in I$. In (2.4) substituting x by $\sigma(x)$ and r by $\sigma(r)$ and using the fact that β commutes with σ, we have $\sigma(\beta([x, r]))Rd(x) = 0$ for all $x \in I, r \in R$. Since R is σ-prime, we arrive $x \in Z(R)$ or $d(x) = 0$ for all $x \in I$. In the latter case, $x - \sigma(x) \in Z(R)$ for all $x \in I$. This means, $[x, r] = [\sigma(x), r]$ for all $r \in R$. In (2.4), replacing r by $\sigma(r)$ and using the fact that β commutes with σ, we get $\sigma(\beta([x, r]))Rd(x) = 0$ for all $x \in I, r \in R$. Since R is σ-prime, we have $x \in Z(R)$ or $d(x) = 0$ for all $x \in I$. As a result, both the cases yield either

$$x \in Z(R) \text{ or } d(x) = 0, \ \forall x \in I$$

Now, we set $K = \{x \in I \mid d(x) = 0\}$ and $L = \{x \in I \mid x \in Z(R)\}$. Each of K and L is an additive subgroup of I. Moreover, I is the set-theoretic union of K and L. But a group can not be the set-theoretic union of two of its proper subgroups, hence $I = K$ or $I = L$. In the former case, $d(I) = 0$. Since β commutes with σ, by Lemma 2.7, we obtain $d = 0$. But, d is a nonzero (α, β)-derivation of R, then I must be contained in $Z(R)$. So, R is commutative. □

2.9. Lemma. *Let R be a σ-prime ring with characteristic not 2, I be a nonzero σ-ideal of R, d be a (α, β)-derivation of R such that β commutes with σ and h be a derivation of R satisfying $h\sigma = \pm\sigma h$. If $dh(I) = 0$ and $h(I) \subset I$ then $d = 0$ or $h = 0$.*

Proof. By hypothesis, it holds that for all $x, y \in I$

$$\begin{aligned} 0 &= dh(xy) \\ &= dh(x)\alpha(y) + \beta(h(x))d(y) + d(x)\alpha(h(y)) + \beta(x)dh(y) \\ &= \beta(h(x))d(y) + d(x)\alpha(h(y)) \end{aligned}$$

And so,

$$\beta(h(x))d(y) + d(x)\alpha(h(y)) = 0, \ \forall x, y \in I$$

Since $h(I) \subset I$, we take $h(x)$ instead of x. Using the hypothesis, we get

$$\beta\left(h^2(x)\right)d(I) = 0, \ \forall x \in I$$

Moreover, replacing x by $\sigma(x)$ in the above obtained relation and using the fact that β commute with σ and $h\sigma = \pm\sigma h$, we derive

$$\sigma\left(\beta\left(h^2(x)\right)\right)d(I) = 0, \ \forall x \in I$$

And so,

$$\beta\left(h^2(x)\right)d(I) = \sigma\left(\beta\left(h^2(x)\right)\right)d(I) = 0, \ \forall x \in I$$

Since β commutes with σ, by Lemma 2.6, we yield either $h^2(I) = 0$ or $d = 0$. Since $h\sigma = \pm\sigma h$, by Lemma 2.2, we have $h = 0$ or $d = 0$. □

2.10. Lemma. *Let R be a σ-prime ring with characteristic not 2, I be a nonzero σ-ideal of R, d be a nonzero (α, β)-derivation of R such that β commutes with σ. If $a \in I \cap S_\sigma(R)$ and $[d(I), a]_{\alpha,\beta} = 0$ then $a \in Z(R)$.*

Proof. For any $x, y \in I$, from the hypothesis, we have $[d([x,y]),a]_{\alpha,\beta} = 0$. Since $d([x,y]) = [d(x),y]_{\alpha,\beta} - [d(y),x]_{\alpha,\beta}$, we get

$$\left[[d(y),x]_{\alpha,\beta},a\right]_{\alpha,\beta} = \left[[d(x),y]_{\alpha,\beta},a\right]_{\alpha,\beta}, \ \forall x,y \in I$$

In the above obtained relation, applying $\left[[a,b]_{\alpha,\beta},c\right]_{\alpha,\beta} = \left[[a,c]_{\alpha,\beta},b\right]_{\alpha,\beta} + [a,[b,c]]_{\alpha,\beta}$ for all $a,b,c \in R$ and using the hypothesis, it becomes

$$\begin{aligned}\left[[d(y),x]_{\alpha,\beta},a\right]_{\alpha,\beta} &= \left[[d(x),y]_{\alpha,\beta},a\right]_{\alpha,\beta} \\ &= \left[[d(x),a]_{\alpha,\beta},y\right]_{\alpha,\beta} + [d(x),[y,a]]_{\alpha,\beta} \\ &= [d(x),[y,a]]_{\alpha,\beta}\end{aligned}$$

And so,

$$\left[[d(y),x]_{\alpha,\beta},a\right]_{\alpha,\beta} = [d(x),[y,a]]_{\alpha,\beta}, \ \forall x,y \in I$$

is obtained. In the last equation, substituting x by a and using the hypothesis, we yield

$$[d(a),[y,a]]_{\alpha,\beta} = 0, \ \forall y \in I$$

The mapping $I_{d(a)} : R \to R$ is given by $I_{d(a)}(r) = [d(a),r]_{\alpha,\beta}$ is a (α,β)-derivation which is determined by $d(a)$ and $I_a : R \to R$ is given by $I_a(r) = [r,a]$ is a derivation which is determined by a. So, we derive

$$\left(I_{d(a)}I_a\right)(I) = 0$$

Since $a \in I \cap S_\sigma(R)$, we have $I_a\sigma = \pm\sigma I_a$. Therefore, by Lemma 2.9, we have

$$d(a) \in C_{\alpha,\beta} \text{ or } a \in Z(R)$$

Assume that $a \notin Z(R)$ which means that $d(a) \in C_{\alpha,\beta}$. From the hypothesis, we get $d([x,a]) = [d(x),a]_{\alpha,\beta} - [d(a),x]_{\alpha,\beta} = 0$ for all $x \in I$. That is,

$$d([I,a]) = 0 \tag{2.5}$$

On the other hand, by hypothesis, we have $[d(xy),a]_{\alpha,\beta} = 0$ for $x,y \in I$. Expanding this equation, it becomes $d(x)\alpha([y,a]) + \beta([x,a])d(y) = 0$ for all $x,y \in I$. Taking $[x,a]$ instead of x and using (2.5), we derive $\beta([[x,a],a])d(I) = 0$ for all $x \in I$. In this equation, replacing x by $\sigma(x)$ and using the fact that β commutes with σ, we obtain $\sigma(\beta[[x,a],a])d(I) = 0$ for all $x \in I$. And so, we yield

$$\beta([[x,a],a])d(I) = \sigma(\beta([[x,a],a]))d(I) = 0, \ \forall x \in I$$

Since β commutes with σ, by Lemma 2.6, it implies that $d = 0$ or $[[x,a],a] = 0$ for all $x \in I$. That is, $d = 0$ or $I_a^2(I) = 0$. Since $a \in I \cap S_\sigma(R)$, we have $I_a\sigma = \pm\sigma I_a$. So, by Lemma 2.9, we have $d = 0$. This is a contradiction which completes the proof. □

2.11. Theorem. *Let R be a σ-prime ring with characteristic not 2, I be a nonzero σ-ideal of R, d be a nonzero (α,β)-derivation of R such that α,β commute with σ. If $a \in I \cap S_\sigma(R)$ and $[d(I),a]_{\alpha,\beta} \subset C_{\alpha,\beta}$ then $a \in Z(R)$.*

Proof. By hypothesis, $\left[d(a^2),a\right]_{\alpha,\beta} \in C_{\alpha,\beta}$. Expanding this, it becomes

$$\begin{aligned}\left[d(a^2),a\right]_{\alpha,\beta} &= [d(a)\alpha(a) + \beta(a)d(a),a]_{\alpha,\beta} \\ &= d(a)\alpha[a,a] + [d(a),a]_{\alpha,\beta}\alpha(a) + \beta(a)[d(a),a]_{\alpha,\beta} \\ &\quad + \beta([a,a])d(a) \\ &= [d(a),a]_{\alpha,\beta}\alpha(a) + \beta(a)[d(a),a]_{\alpha,\beta} \in C_{\alpha,\beta}\end{aligned}$$

And so,

$$[d(a),a]_{\alpha,\beta}\,\alpha(a)+\beta(a)\,[d(a),a]_{\alpha,\beta}\in C_{\alpha,\beta}$$

is obtained. In the above obtained relation, using $[d(a),a]_{\alpha,\beta}\in C_{\alpha,\beta}$, we have $2\beta(a)\,[d(a),a]_{\alpha,\beta}\in C_{\alpha,\beta}$. Since $char R\neq 2$, we get

$$\beta(a)\,[d(a),a]_{\alpha,\beta}\in C_{\alpha,\beta} \tag{2.6}$$

Since $a\in I\cap S_\sigma(R)$, it is clear that $\beta(a)\in S_\sigma(R)$. Using the hypothesis together with (2.6), according to Lemma 2.4 (i), we yield either

$$a\in Z(R)\ \text{ or }\ [d(a),a]_{\alpha,\beta}=0$$

Assume that $a\notin Z(R)$ which means $[d(a),a]_{\alpha,\beta}=0$. On the other hand, by hypothesis, it holds that $[d([a,x]),a]_{\alpha,\beta}\in C_{\alpha,\beta}$. So,

$$[d([a,x]),a]_{\alpha,\beta}=\left[[d(a),x]_{\alpha,\beta},a\right]_{\alpha,\beta}-\left[[d(x),a]_{\alpha,\beta},a\right]_{\alpha,\beta}\in C_{\alpha,\beta}$$

is obtained. Using the hypothesis, we have

$$\left[[d(a),x]_{\alpha,\beta},a\right]_{\alpha,\beta}\in C_{\alpha,\beta},\ \forall x\in I$$

Replacing x by ax and using $[d(a),a]_{\alpha,\beta}=0$, it becomes

$$\beta(a)\left[[d(a),x]_{\alpha,\beta},a\right]_{\alpha,\beta}\in C_{\alpha,\beta},\ \forall x\in I$$

We know that $\beta(a)\in S_\sigma(R)$ and $\left[[d(a),x]_{\alpha,\beta},a\right]_{\alpha,\beta}\in C_{\alpha,\beta}$. Therefore, by Lemma 2.4 (i), we derive $\left[[d(a),x]_{\alpha,\beta},a\right]_{\alpha,\beta}=0$ for all $x\in I$. Applying the identity $\left[[a,b]_{\alpha,\beta},c\right]_{\alpha,\beta}=\left[[a,c]_{\alpha,\beta},b\right]_{\alpha,\beta}+[a,[b,c]]_{\alpha,\beta}$ for all $a,b,c\in R$ and using the assumption, we arrive

$$[d(a),[x,a]]_{\alpha,\beta}=0,\ \forall x\in I$$

The mapping $I_{d(a)}:R\to R$ is given by $I_{d(a)}(r)=[d(a),r]_{\alpha,\beta}$ is a (α,β)-derivation which is determined by $d(a)$ and $I_a:R\to R$ is given by $I_a(r)=[r,a]$ is a derivation which is determined by a. So,

$$\left(I_{d(a)}I_a\right)(I)=0$$

is obtained. Since $a\in I\cap S_\sigma(R)$, we have $I_a\sigma=\pm\sigma I_a$. According to Lemma 2.9, we yield either

$$I_{d(a)}=0 \text{ or } I_a=0$$

which means $d(a)\in C_{\alpha,\beta}$. On the other hand, by hypothesis, we have $[d(ax),a]_{\alpha,\beta}\in C_{\alpha,\beta}$ for all $x\in I$. So, we get

$$d(a)\,\alpha([x,a])+\beta(a)\,[d(x),a]_{\alpha,\beta}\in C_{\alpha,\beta},\forall x\in I \tag{2.7}$$

Commuting (2.7) with a, it follows that

$$\begin{aligned}0&=\left[d(a)\,\alpha([x,a])+\beta(a)\,[d(x),a]_{\alpha,\beta},a\right]_{\alpha,\beta}\\&=[d(a)\,\alpha([x,a]),a]_{\alpha,\beta}+\left[\beta(a)\,[d(x),a]_{\alpha,\beta},a\right]_{\alpha,\beta}\\&=d(a)\,\alpha([[x,a],a])+[d(a),a]_{\alpha,\beta}\,\alpha([x,a])\\&\quad+\beta(a)\left[[d(x),a]_{\alpha,\beta},a\right]_{\alpha,\beta}+\beta([a,a])\,[d(x),a]_{\alpha,\beta}\\&=d(a)\,\alpha([[x,a],a])+\beta(a)\left[[d(x),a]_{\alpha,\beta},a\right]_{\alpha,\beta}\end{aligned}$$

And so, it becomes

$$d(a)\alpha([[x,a],a])+\beta(a)\left[[d(x),a]_{\alpha,\beta},a\right]_{\alpha,\beta}=0,\ \forall x\in I$$

Using $[d(x),a]_{\alpha,\beta}\in C_{\alpha,\beta}$, we have $d(a)\alpha([[x,a],a])=0$ for all $x\in I$. Since $d(a)\in C_{\alpha,\beta}$,

$$d(a)R\alpha([[x,a],a])=0,\ \forall x\in I$$

is obtained. In the above obtained relation, taking $\sigma(x)$ instead of x and using the fact that α commutes with σ, we derive

$$d(a)R\sigma(\alpha([[x,a],a]))=0,\ \forall x\in I$$

And so, we yield

$$d(a)R\alpha([[x,a],a])=d(a)R\sigma(\alpha([[x,a],a]))=0,\ \forall x\in I$$

Since R is σ-prime, we get $d(a)=0$ or $[[x,a],a]=0$ for all $x\in I$. That is, $d(a)=0$ or $I_a^2(I)=0$. Since $I_a\sigma=\pm\sigma I_a$, by Lemma 2.9, we have $d(a)=0$. In (2.7), using $d(a)=0$, it becomes

$$\beta(a)[d(x),a]_{\alpha,\beta}\in C_{\alpha,\beta},\forall x\in I$$

We know that $\beta(a)\in S_\sigma(R)$ and $[d(x),a]_{\alpha,\beta}\in C_{\alpha,\beta}$ from the hypothesis. Therefore, according to Lemma 2.4 (i), we have $[d(x),a]_{\alpha,\beta}=0$ for all $x\in I$. Since $a\in I\cap S_\sigma(R)$ and β commutes with σ, by Lemma 2.10, we derive $a\in Z(R)$. This is a contradiction which completes the proof. □

2.12. Theorem. *Let R be a σ-prime ring with characteristic not 2, I be a nonzero σ-ideal of R, d be a nonzero (α,β)-derivation of R such that α and β commute with σ and h be a nonzero derivation of R which commutes with σ. If $dh(I)\subset C_{\alpha,\beta}$ and $h(I)\subset I$ then R is commutative.*

Proof. For any $x,y\in I$, from the hypothesis, we have $dh([x,y])\in C_{\alpha,\beta}$. Expanding this identity, it follows that

$$\begin{aligned}dh([x,y])&=d([h(x),y]+[x,h(y)])\\&=[(dh)(x),y]_{\alpha,\beta}-[d(y),h(x)]_{\alpha,\beta}+[d(x),h(y)]_{\alpha,\beta}\\&-[(dh)(y),x]_{\alpha,\beta}\\&=[d(x),h(y)]_{\alpha,\beta}-[d(y),h(x)]_{\alpha,\beta}\in C_{\alpha,\beta}\end{aligned}$$

And it becomes

$$[d(x),h(y)]_{\alpha,\beta}-[d(y),h(x)]_{\alpha,\beta}\in C_{\alpha,\beta},\ \forall x,y\in I$$

Since $h(I)\subset I$, we replace y by $h(y)$. So, we arrive $\left[d(x),h^2(y)\right]_{\alpha,\beta}\in C_{\alpha,\beta}$ for all $x,y\in I$. That is,

$$\left[d(I),h^2(I)\right]_{\alpha,\beta}\subset C_{\alpha,\beta}$$

Using the fact that $h(I)\subset I$ and h commutes with σ, we assure $h^2(I)\subset I\cap S_\sigma(R)$. In additional, we know that from the hypothesis α and β commute with σ. Thereby, according to Theorem 2.11, it yields $h^2(I)\subset Z(R)$. So, for all $x,y\in I$

$$\begin{aligned}h^2([x,y])&=h([h(x),y]+[x,h(y)])\\&=\left[h^2(x),y\right]+2[h(x),h(y)]+\left[x,h^2(y)\right]\\&=2[h(x),h(y)]\in Z(R)\end{aligned}$$

is obtained. Since $charR\neq 2$, we have $[h(x),h(y)]\in Z(R)$ for all $x,y\in I$. Thus,

$$[h(I),h(I)]\subset Z(R)$$

Using $h(I) \subset I \cap S_\sigma(R)$, by Theorem 2.11, we derive $h(I) \subset Z(R)$. According to Lemma 2.5, it implies that R is commutative. $\square$

Acknowledgement. The first author thanks to TÜBİTAK (The Scientific and Technical Research Council of Turkey) for financial support of BİDEB 2211 National Research Fellowship.

References

[1] Aydın N., Kaya K.: Some Generalizations in Prime Rings with (σ,τ)-Derivation, *Doğa-Tr. J. Mathematics*, vol 16 (1992) $169-176$.

[2] Herstein I. N. : A Note on Derivation II, Canad. Math. Bull., 22, 4, (1979) $509-511$.

[3] Kaya K. : (σ,τ)-Türevli Asal Halkalar Üzerine, Doğa-Tr. J. Mathematics, (1988) $42-45$.

[4] Oukhtite L., Salhi S.: On Commutativity of σ-Prime Rings, Glasnik Matematicki, vol. 41 no. 1 (2006) $57-64$.

[5] Oukhtite L., Salhi S.: Derivations and Commutativity of σ-Prime Rings, Int. J. Contemp. Sci., vol. 1 no. 9 (2006) $439-448$.

[6] Oukhtite L., Salhi S.: σ-Prime Rings with a special kind of automorphism, Int. J. Contemp. Math. Sci., vol. 2 no. 3 (2007) $127-133$.

[7] Posner E. : Derivations in Prime Rings, Proc. Amer. Math. Soc., 8, (1957).

[8] Shuliang H.: Some Generalizations in Certain Classes of Rings with Involution, Bol. Soc. Paran. Mat., 29, 1, (2011) $9-16$.