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An asymptotic criterion for third-order dynamic equations with positive and negative coefficients

Nadide Utku* , Tongxing Li† and M. Tamer Şenel‡

Abstract

We establish a criterion for the asymptotic properties of all bounded solutions to a class of third-order linear dynamic equations with positive and negative coefficients. New theorem improves and complements the related results reported in the literature. An example is provided to illustrate the main results.

Keywords: asymptotic behavior, third-order dynamic equation, linear equation, positive and negative coefficients, time scale.

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1. Introduction

In this paper, we deal with the asymptotic behavior of all bounded solutions to a class of third-order linear dynamic equations with positive and negative coefficients

$$\left(rx^{\Delta\Delta}\right)^{\Delta}(t) + B(t)x(\beta(t)) - C(t)x(\gamma(t)) = 0, \tag{1.1}$$

where $t_0 \in \mathbb{T}$ and $t \in [t_0, \infty)_{\mathbb{T}}$. Throughout the paper, we always assume that the following hypotheses are satisfied:

(h1) $r \in \mathrm{C}^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}}, (0,\infty))$, $B, C \in \mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}}, [0,\infty))$, and

$$\int_{t_0}^{\infty} \frac{\Delta t}{r(t)} = \infty; \tag{1.2}$$

(h2) $\beta, \gamma \in \mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}}, \mathbb{T})$ are strictly increasing functions such that $\lim_{t\to\infty} \beta(t) = \lim_{t\to\infty} \gamma(t) = \infty$;

(h3) $\delta := \gamma^{-1} \circ \beta \in \mathrm{C}^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}}, \mathbb{T})$ is strictly increasing with $\delta([t_0,\infty)_{\mathbb{T}}) = [\delta(t_0), \infty)_{\mathbb{T}}$ and $\delta(t) < t$, the notation γ^{-1} stands for the inverse of the function γ;

*Institute of Sciences, Erciyes University, 38039, Kayseri, Turkey
Email: nadideutku@gmail.com
†Department of Mathematics, Linyi University, Linyi, Shandong 276005, P. R. China
Email: litongx2007@163.com
‡Department of Mathematics, Faculty of Sciences, Erciyes University, 38039, Kayseri, Turkey
Email: senel@erciyes.edu.tr

(h4) $D \in \mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},(0,\infty))$, where $D(t) := B(t) - \delta^{\Delta}(t)C(\delta(t))$.

A solution of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is called nonoscillatory. Hilger [11] initiated the theory of time scales (which unifies continuous and discrete analysis). Agarwal et al. [3] and Bohner and Peterson [5] summarize and organize much of the time scale calculus and advances in dynamic equations on time scales.

In recent years, there has been an increasing interest in obtaining sufficient conditions for the oscillatory and asymptotic behavior of solutions to various classes of differential and dynamic equations on time scales. We refer the reader to [1,2,4,6–10,12–27] and the references cited therein. For the study of asymptotic properties of third-order dynamic equations, Agarwal et al. [1] and Erbe et al. [8] established Hille and Nehari type criteria for third-order dynamic equations

$$(a(rx^{\Delta})^{\Delta})^{\Delta}(t) + p(t)x(\tau(t)) = 0$$

and

$$x^{\Delta^3}(t) + p(t)x(t) = 0,$$

respectively. Assuming that γ is a quotient of odd positive integers, Agarwal et al. [4], Hassan [10], and Li et al. [21] studied a third-order nonlinear delay dynamic equation

$$(a((rx^{\Delta})^{\Delta})^{\gamma})^{\Delta}(t) + f(t, x(\tau(t))) = 0.$$

Şenel [26] examined a third-order dynamic equation

$$(a(rx^{\Delta})^{\Delta})^{\Delta}(t) + p(t, x(t), x^{\Delta}(t)) + F(t, x(t)) = 0.$$

Grace et al. [9] considered a third-order neutral delay dynamic equation

$$(r(t)(x(t) - a(t)x(\tau(t)))^{\Delta\Delta})^{\Delta} + p(t)x^{\gamma}(\delta(t)) = 0.$$

So far, there are few results regarding the oscillation of dynamic equations with positive and negative coefficients. Karpuz and Öcalan [14] investigated a first-order delay dynamic equation

$$x^{\Delta}(t) + B(t)x(\beta(t)) - C(t)x(\gamma(t)) = 0.$$

Chen et al. [7] considered a second-order nonlinear dynamic equation

$$(rx^{\Delta})^{\Delta}(t) + p(t)f(x(\xi(t))) - q(t)h(x(\delta(t))) = 0.$$

Karpuz and Öcalan [16] and Karpuz et al. [17] studied the first-order neutral delay dynamic equations

$$[x(t) - A(t)x(\alpha(t))]^{\Delta} + B(t)x(\beta(t)) - C(t)x(\gamma(t)) = 0$$

and

$$[x(t) + A(t)x(\alpha(t))]^{\Delta} + B(t)F(x(\beta(t))) - C(t)G(x(\gamma(t))) = \varphi(t),$$

respectively. Karpuz et al. [19] obtained some necessary and sufficient conditions which guarantee that every solution y of a neutral differential equation

$$(y(t) - p(t)y(r(t)))^{(n)} + q(t)G(y(g(t))) - u(t)H(y(h(t))) = f(t)$$

is either oscillatory or satisfies $\lim_{t\to\infty} y(t) = 0$.

In the real world, one can predict dynamic behavior of solutions of third-order partial differential equations by using the qualitative behavior of the third-order differential equations; see, for instance, Agarwal et al. [2]. In order to develop oscillation theory of third-order dynamic equations with positive and negative coefficients, we present an asymptotic test for equation (1.1) in the next section. As usual, all functional equalities and inequalities considered in the paper are assumed to hold for all t large enough.

2. Main results

In what follows, the notation δ^{-1} stands for the inverse of the function δ and

$$z(t) := x(t) - \int_t^\infty \int_v^\infty \frac{1}{r(u)} \int_{\delta(u)}^u \frac{B(\delta^{-1}(s))}{\delta^\Delta(\delta^{-1}(s))} x(\gamma(s))\Delta s \Delta u \Delta v$$

for $t \in [t_0, \infty)_{\mathbb{T}}$.

2.1. Theorem. *Assume that conditions (h1)–(h4) are satisfied and let*

$$\lim_{t\to\infty} \int_t^\infty [\sigma(v) - t] F(v)\Delta v < \infty, \tag{2.1}$$

where

$$F(v) := \frac{1}{r(v)} \int_{\delta(v)}^v \frac{B(\delta^{-1}(s))}{\delta^\Delta(\delta^{-1}(s))} \Delta s.$$

Then every bounded solution x of (1.1) *is either oscillatory or $\lim_{t\to\infty} x(t)$ exists (finite).*

Proof. Without loss of generality, we may assume that x is a bounded eventually positive solution of (1.1). Then there exists a $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t) > 0$, $x(\beta(t)) > 0$, and $x(\gamma(t)) > 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}$. Differentiation of z yields

$$z^\Delta(t) = x^\Delta(t) + \int_t^\infty \frac{1}{r(u)} \int_{\delta(u)}^u \frac{B(\delta^{-1}(s))}{\delta^\Delta(\delta^{-1}(s))} x(\gamma(s))\Delta s \Delta u$$

and

$$z^{\Delta\Delta}(t) = x^{\Delta\Delta}(t) - \frac{1}{r(t)} \int_{\delta(t)}^t \frac{B(\delta^{-1}(s))}{\delta^\Delta(\delta^{-1}(s))} x(\gamma(s))\Delta s.$$

Writing the latter equality in the form

$$r(t)z^{\Delta\Delta}(t) = r(t)x^{\Delta\Delta}(t) - \int_{\delta(t)}^t \frac{B(\delta^{-1}(s))}{\delta^\Delta(\delta^{-1}(s))} x(\gamma(s))\Delta s.$$

Using (1.1) and [5, Theorem 1.93], we deduce that

$$\begin{aligned}
(rz^{\Delta\Delta})^\Delta(t) &= (rx^{\Delta\Delta})^\Delta(t) - \frac{B(\delta^{-1}(t))}{\delta^\Delta(\delta^{-1}(t))} x(\gamma(t)) + B(t)x(\beta(t)) \\
&= -B(t)x(\beta(t)) + C(t)x(\gamma(t)) - \frac{B(\delta^{-1}(t))}{\delta^\Delta(\delta^{-1}(t))} x(\gamma(t)) + B(t)x(\beta(t)) \\
&= C(t)x(\gamma(t)) - \frac{B(\delta^{-1}(t))}{\delta^\Delta(\delta^{-1}(t))} x(\gamma(t)) \\
&= -\left(\frac{B(\delta^{-1}(t))}{\delta^\Delta(\delta^{-1}(t))} - C(t) \right) x(\gamma(t)).
\end{aligned}$$

Then, we obtain

$$(rz^{\Delta\Delta})^\Delta(t) = -\frac{D(\delta^{-1}(t))}{\delta^\Delta(\delta^{-1}(t))} x(\gamma(t)) < 0, \tag{2.2}$$

which implies that $rz^{\Delta\Delta}$ is decreasing, and thus the sign of $z^{\Delta\Delta}$ is fixed. Next, we assert that there exists a $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that $z^{\Delta\Delta}(t) > 0$ for $t \in [t_2, \infty)_{\mathbb{T}}$. If $z^{\Delta\Delta} < 0$, then there exist a $t_3 \in [t_1, \infty)_{\mathbb{T}}$ and a constant $M > 0$ such that, for $t \in [t_3, \infty)_{\mathbb{T}}$,

$$z^{\Delta\Delta}(t) \le -\frac{M}{r(t)} < 0.$$

Integrating the latter inequality from t_3 to t, we obtain

$$z^{\Delta}(t) \le z^{\Delta}(t_3) - M \int_{t_3}^{t} \frac{\Delta s}{r(s)}.$$

Letting $t \to \infty$ and using condition (1.2), we have $\lim_{t\to\infty} z^{\Delta}(t) = -\infty$. It follows from inequalities $z^{\Delta\Delta} < 0$ and $z^{\Delta} < 0$ that

$$\lim_{t\to\infty} z(t) = -\infty,$$

which contradicts the fact that z is bounded. Hence, there exists a $t_4 \in [t_1, \infty)_{\mathbb{T}}$ such that, for $t \in [t_4, \infty)_{\mathbb{T}}$,

$$z(t) > 0, \quad z^{\Delta}(t) < 0, \quad z^{\Delta\Delta}(t) > 0, \quad (rz^{\Delta\Delta})^{\Delta}(t) < 0, \tag{2.3}$$

or

$$z(t) < 0, \quad z^{\Delta}(t) < 0, \quad z^{\Delta\Delta}(t) > 0, \quad (rz^{\Delta\Delta})^{\Delta}(t) < 0. \tag{2.4}$$

Assume first that (2.3) holds. Using condition (2.1) and the definition of z, we conclude that there exists a constant $\ell \ge 0$ such that $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} z(t) = \ell$. Assume now that (2.4) holds. Then

$$x(t) \le \int_t^{\infty} \int_v^{\infty} \frac{1}{r(u)} \int_{\delta(u)}^{u} \frac{B(\delta^{-1}(s))}{\delta^{\Delta}(\delta^{-1}(s))} x(\gamma(s)) \Delta s \Delta u \Delta v.$$

On the other hand, by virtue of [12, Lemma 2.1],

$$\int_t^{\infty} \int_v^{\infty} \frac{1}{r(u)} \int_{\delta(u)}^{u} \frac{B(\delta^{-1}(s))}{\delta^{\Delta}(\delta^{-1}(s))} \Delta s \Delta u \Delta v = \int_t^{\infty} [\sigma(v) - t] F(v) \Delta v.$$

It follows now from condition (2.1) that $\lim_{t\to\infty} x(t) = 0$. This completes the proof. ∎

2.2. Example. For $t \ge t_0$, consider a third-order differential equation

$$x'''(t) + \frac{b}{t^4} x(t) - \frac{c}{t^5} x(2t) = 0, \tag{2.5}$$

where b and c are positive constants. It is not difficult to verify that all assumptions of Theorem 2.1 are satisfied. Hence, every bounded solution x of (2.5) is either oscillatory or $\lim_{t\to\infty} x(t)$ exists (finite).

3. Conclusions

Most oscillation and asymptotic results reported in the literature for third-order dynamic equation (1.1) and its particular cases have been obtained in the case where $C(t) = 0$. In this paper, we establish an asymptotic criterion for equation (1.1) under the assumption that $C(t) \ge 0$, which, in a certain sense, improves and complements the related results in the cited papers.

We stress that the study of asymptotic behavior of equation (1.1) in the case $C(t) \ge 0$ brings additional difficulties. The main difficulty one encounters lies in how to obtain inequality such as (2.2). Since $z^{\Delta} < 0$, it is hard to establish criteria which ensure that all bounded solutions of (1.1) are just oscillatory. The question regarding the study of sufficient conditions which guarantee that all bounded solutions of (1.1) tend to zero remains open at the moment.

It is not easy to use the technique exploited in this paper for deriving similar results for the odd-order dynamic equation

$$\left(rx^{\Delta^{n-1}}\right)^{\Delta}(t) + B(t)x(\beta(t)) - C(t)x(\gamma(t)) = 0, \tag{3.1}$$

where $n \geq 3$ is an odd natural number. Therefore, an interesting problem for future research can be formulated as follows.

(P) Is it possible to establish similar asymptotic tests for equation (3.1)?

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