

## Integral type contraction and coupled coincidence fixed point theorems for two pairs in G-metric spaces

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### Abstract

In this paper, we introduces the idea of integral type contraction with respect to G-metric space and by using the notion of integral type contraction we prove some coupled coincidence fixed point results for two pairs of mapping in G-metric space. Also we give an example as an application point of view.

**Keywords:** G-metric space; couple coincidence point; common fixed point; integral type contraction.

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### 1. Introduction

The study of common fixed points of mappings which satisfies certain contractive conditions has been studied by a lot of researchers due to its applications in mathematics. For the study of coincidence point of theory in metric and cone metric spaces we recommend [1, 2, 3, 4, 7, 8, 9, 10, 11, 15, 17, 18]. In 2006 Mustafa and Sims [16], introduced the idea of G-metric space and presented some fixed point theorems in G-metric space. The concept of a coupled coincidence point of mapping was introduced by V. Lakshmikantham [5, 13], they also studied some fixed point theorems in partially ordered metric spaces. In 2010 Shatanawi [19] gave the proof of coupled coincidence fixed point theorems in generalized metric spaces. Also in 2014 Manish Kumar [14] proved a coupled coincidence fixed point theorem in the setting of two pairs of mapping in G-metric space. Moreover

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in 2002, Branciari [6] gave the idea of integral type contractive mappings in complete metric spaces and they studied the existence of fixed points for mappings which is defined on complete metric space satisfying integral type contraction. Recently F. Khojasteh et al.[12], gave the idea of integral type contraction in cone metric spaces and proved some fixed point theorems in such spaces. So by using the concept of Branciari [6] of integral type contractive mapping, we presented a coupled coincidence fixed point results of integral type contractive mappings for two pairs in the setting of G-metric spaces. Also we give suitable example that support our main result.

## 2. Preliminaries

We will need the following definitions and results in this paper.

**2.1. Definition.** [16] Let  $Y$  be a non-empty set and  $G : Y \times Y \times Y \rightarrow R^+$  is a function that satisfies the following conditions:

- (1)  $G(a, b, c) = 0$  if  $a = b = c$ ,
- (2)  $G(a, a, b) > 0$  for all  $a, b \in Y$  with  $a \neq b$ ,
- (3)  $G(a, a, b) \leq G(a, b, c)$ , for all  $a, b, c \in Y$  with  $c \neq b$
- (4)  $G(a, b, c) = G(a, c, b) = G(b, c, a) = \dots$ , symmetry in all variables,
- (5)  $G(a, b, c) \leq G(a, s, s) + G(s, b, c)$  for all  $a, b, c, s \in Y$ .

Then the function  $G$  is called a generalized metric and the pair  $(Y, G)$  is called a G-metric space.

**2.2. Example.** [16] Let  $Y = \{x, y\}$ . Define  $G$  on  $Y \times Y \times Y$  by

$$G(x, x, x) = G(y, y, y) = 0, G(x, x, y) = 1, G(x, y, y) = 2$$

and extend  $G$  to  $Y \times Y \times Y$  by using the symmetry in the variables. Then it is clear that  $(Y, G)$  is a G-metric space.

**2.3. Definition.** [16] Let  $(Y, G)$  be a G-metric space and  $(a_n)$  a sequence of points of  $Y$ . A point  $a \in Y$  is said to be the limit of the sequence  $(a_n)$ , if  $\lim_{n, m \rightarrow +\infty} G(a, a_n, a_m) = 0$  and we say that the sequence  $(a_n)$  is G-convergent to  $a$ .

**2.1. Proposition.** [16] *Let  $(Y, G)$  be a G-metric space. Then the following are equivalent:*

- (1)  $(a_n)$  is G-convergent to  $a$ .
- (2)  $G(a_n, a_n, a) \rightarrow 0$  as  $n \rightarrow +\infty$ .
- (3)  $G(a_n, a, a) \rightarrow 0$  as  $n \rightarrow +\infty$ .
- (4)  $G(a_n, a_m, a) \rightarrow 0$  as  $n, m \rightarrow +\infty$ .

**2.4. Definition.** [15] Let  $(Y, G)$  be a G-metric space. A sequence  $(a_n)$  is called G-Cauchy if for every  $\epsilon > 0$ , there is  $k \in \mathbf{N}$  such that  $G(a_n, a_m, a_l) < \epsilon$ , for all  $n, m, l \geq k$ ; that is  $G(a_n, a_m, a_l) \rightarrow 0$  as  $n, m, l \rightarrow +\infty$ .

**2.2. Proposition.** [16] *Let  $(Y, G)$  be a G-metric space. Then the following are equivalent:*

- (1) *The sequence  $(a_n)$  is G-Cauchy.*
- (2) *For every  $\epsilon > 0$ , there is  $k \in \mathbf{N}$  such that  $G(a_n, a_m, a_m) < \epsilon$ , for all  $n, m \geq k$ .*

**2.5. Definition.** [16] A G-metric space  $(Y, G)$  is called G-complete if every G-Cauchy sequence in  $(Y, G)$  is G-convergent in  $(Y, G)$ .

**2.6. Definition.** [5] An element  $(a, b) \in Y \times Y$  is called a coupled coincidence point of the mappings  $F : Y \times Y \rightarrow Y$  and  $g : Y \rightarrow Y$  if  $F(a, b) = ga$  and  $F(b, a) = gb$ .

**2.7. Definition.** [13] Let  $Y$  be a non-empty set. Then we say that the mappings  $F : Y \times Y \rightarrow Y$  and  $g : Y \rightarrow Y$  are commutative if  $gF(a, b) = F(ga, gb)$ .

**2.8. Definition.** [13] An element  $(a, b) \in Y \times Y$  is called a coupled fixed point of mapping  $F : Y \times Y \rightarrow Y$  if  $F(a, b) = a$  and  $F(b, a) = b$ .

In 2002, Branciari in [6] introduced a general contractive condition of integral type as follows.

**2.9. Theorem.** [6] Let  $(Y, d)$  be a complete metric space,  $\alpha \in (0, 1)$ , and  $f : Y \rightarrow Y$  is a mapping such that for all  $x, y \in Y$ ,

$$\int_0^{d(f(x), f(y))} \phi(t) dt \leq \alpha \int_0^{d(x, y)} \phi(t) dt$$

where  $\phi : [0, +\infty) \rightarrow [0, +\infty)$  is Lebesgue-integrable mapping which is summable (i.e., with finite integral) on each compact subset of  $[0, +\infty)$  such that for each  $\epsilon > 0$ ,  $\int_0^\epsilon \phi(t) dt > 0$ , then  $f$  has a unique fixed point  $a \in Y$ , such that for each  $x \in Y$ ,  $\lim_{n \rightarrow \infty} f^n(x) = a$ .

We use the above idea of Branciari [6] and presented some coupled coincidence fixed point results of integral type contraction in G-metric space.

### 3. Main Results

In this section we will prove some fixed point results for two pairs in G-metric space by using integral type contractive mapping. We will start our work with the following important lemma.

**3.1. Lemma.** Let  $(Y, G)$  be a G-metric space. Suppose  $F, S : Y \times Y \rightarrow Y$  and  $g, h : Y \rightarrow Y$  be two mappings such that

$$(3.1) \quad \int_0^{G(F(a,b), S(p,q), S(c,r))} \varphi(t) dt \leq k \int_0^{G(ha, gp, gc) + G(hb, gq, gr)} \varphi(t) dt$$

for all  $a, b, c, p, q, r \in Y$  and  $\varphi : [0, +\infty) \rightarrow [0, +\infty)$  is a Lebesgue integrable mapping which is summable such that for each  $\epsilon > 0$ ,  $\int_0^\epsilon \varphi(t) dt > 0$ . Assume that  $(a, b)$  is coupled coincidence point of the pairs of mappings  $\{F, h\}$  and  $\{S, g\}$  and  $ga = ha$  and  $gb = hb$ . If  $k \in [0, \frac{1}{8})$ , then  $S(a, b) = ga = gb = S(b, a)$  and  $F(a, b) = ha = hb = F(b, a)$ .

*Proof.* Since  $(a, b)$  is a coupled coincidence point of the mappings  $\{F, h\}$  and  $\{S, g\}$ , we have  $ha = F(a, b)$ ,  $hb = F(b, a)$  and  $ga = S(b, a)$ ,  $gb = S(b, a)$ . Suppose  $ga \neq gb$ . Then by (3.1), we get

$$\begin{aligned} \int_0^{G(ga, gb, gb)} \varphi(t) dt &= \int_0^{G(F(a,b), S(b,a), S(b,a))} \varphi(t) dt \\ &\leq k \int_0^{G(ha, gb, gb) + G(hb, ga, ga)} \varphi(t) dt \\ &= k \int_0^{G(ga, gb, gb) + G(gb, ga, ga)} \varphi(t) dt. \end{aligned}$$

Also we have,

$$\begin{aligned} \int_0^{G(gb,ga,ga)} \varphi(t)dt &= \int_0^{G(F(b,a),S(a,b),S(a,b))} \varphi(t)dt \\ &\leq k \int_0^{(G(hb,ga,ga)+G(ha,gb,gb))} \varphi(t)dt \\ &= k \int_0^{(G(gb,ga,ga)+G(ga,gb,gb))} \varphi(t)dt. \end{aligned}$$

Therefore

$$\int_0^{G(ga,gb,gb)} \varphi(t)dt + \int_0^{G(gb,ga,ga)} \varphi(t)dt \leq 2k \int_0^{(G(ga,gb,gb)+G(gb,ga,ga))} \varphi(t)dt.$$

Since  $2k < 1$ , we get

$$\int_0^{G(ga,gb,gb)} \varphi(t)dt + \int_0^{G(gb,ga,ga)} \varphi(t)dt < \int_0^{G(ga,gb,gb)} \varphi(t)dt + \int_0^{G(gb,ga,ga)} \varphi(t)dt$$

which is a contradiction. So  $ga = gb$ , and hence

$$S(a, b) = ga = gb = S(b, a) \text{ and } F(a, b) = ha = hb = F(b, a).$$

**3.1. Theorem.** Let  $(Y, G)$  be a G-metric space. Let  $F, S : Y \times Y \rightarrow Y$  and  $g, h : Y \rightarrow Y$  be two mappings such that

$$(3.2) \quad \int_0^{G(F(a,b),S(p,q),S(c,r))} \varphi(t)dt \leq k \int_0^{(G(ha,gp,gc)+G(hb,gq,gr))} \varphi(t)dt$$

for all  $a, b, c, p, q, r \in Y$  and  $\varphi : [0, +\infty) \rightarrow [0, +\infty)$  is a Lebesgue integrable mapping which is summable such that for each  $\epsilon > 0$ ,  $\int_0^\epsilon \varphi(t)dt > 0$ . Assume that  $F, S$  and  $g, h$  satisfy the following conditions:

(i)  $F(Y \times Y) \subset g(Y)$  and  $S(Y \times Y) \subset h(Y)$

(ii)  $g(Y)$  or  $h(Y)$  is complete and

(iii)  $g$  and  $h$  are G-continuous and pairs  $\{F, h\}$  and  $\{S, g\}$  are commuting mappings.

If  $k \in [0, \frac{1}{8})$ , then there is a unique  $a \in Y$  such that  $F(a, a) = S(a, a) = g(a) = h(a) = a$ .

*Proof.* Let  $a_0, b_0 \in Y$ . Since  $F(Y \times Y) \subset g(Y)$ , choose  $a_1, b_1 \in Y$  such that  $u'_1 = ga_1 = F(a_0, b_0)$  and  $v'_1 = gb_1 = F(b_0, a_0)$ . Again since  $S(Y \times Y) \subset h(Y)$ , choose  $a_2, b_2 \in Y$  such that  $u'_2 = ha_2 = S(a_1, b_1)$  and  $v'_2 = hb_2 = S(b_1, a_1)$ . Continuing this process, we can construct two sequences  $(u'_n)$  and  $(v'_n)$  in  $Y$  such that  $u'_{2n+1} = ga_{2n+1} = F(a_{2n}, b_{2n})$ ,  $v'_{2n} = gb_{2n+1} = F(b_{2n+1}, a_{2n+1})$  and  $u'_{2n+2} = ha_{2n+2} = S(a_{2n+1}, b_{2n+1})$ ,  $v'_{2n+2} = hb_{2n+2} = S(b_{2n+1}, a_{2n+1})$ . For  $n \in \mathbb{N}$ , we have

$$\begin{aligned} \int_0^{G(u'_{2n+1}, u'_{2n+2}, u'_{2n+2})} \varphi(t)dt &= \int_0^{(G(F(a_{2n}, b_{2n}), S(a_{2n+1}, b_{2n+1}), S(a_{2n+1}, b_{2n+1})))} \varphi(t)dt \\ &\leq k \int_0^{(G(ha_{2n}, ga_{2n+1}, ga_{2n+1})+G(hb_{2n}, gb_{2n+1}, gb_{2n+1}))} \varphi(t)dt \\ &= k \int_0^{(G(u'_{2n}, u'_{2n+1}, u'_{2n+1})+G(v'_{2n}, v'_{2n+1}, v'_{2n+1}))} \varphi(t)dt. \end{aligned} \quad (3.3)$$

In the same manner

$$\int_0^{G(v'_{2n+1}, v'_{2n+2}, v'_{2n+2})} \varphi(t)dt \leq k \int_0^{(G(v'_{2n}, v'_{2n+1}, v'_{2n+1})+G(u'_{2n}, u'_{2n+1}, u'_{2n+1}))} \varphi(t)dt. \quad (3.4)$$

We have

$$\begin{aligned}
\int_0^{G(u'_{2n+1}, u'_{2n+2}, u'_{2n+2})} \varphi(t) dt &+ \int_0^{G(v'_{2n+1}, v'_{2n+2}, v'_{2n+2})} \varphi(t) dt \\
&\leq 2k \int_0^{\{G(u'_{2n}, u'_{2n+1}, u'_{2n+1}) + G(v'_{2n}, v'_{2n+1}, v'_{2n+1})\}} \varphi(t) dt \\
&\leq 8k \int_0^{\{G(u'_{2n}, u'_{2n+1}, u'_{2n+1}) + G(v'_{2n}, v'_{2n+1}, v'_{2n+1})\}} \varphi(t) dt, \quad (3.5)
\end{aligned}$$

holds for all  $n \in N$ , again from

$$\begin{aligned}
\int_0^{G(u'_{2n}, u'_{2n+1}, u'_{2n+1})} \varphi(t) dt &\leq 2 \int_0^{G(u'_{2n+1}, u'_{2n}, u'_{2n})} \varphi(t) dt \\
&= 2 \int_0^{(G(F(a_{2n}, b_{2n}), S(a_{2n-1}, b_{2n-1}), S(a_{2n-1}, b_{2n-1})))} \varphi(t) dt \\
&\leq 2k \int_0^{(G(ha_{2n}, ga_{2n-1}, ga_{2n-1}) + G(hb_{2n}, gb_{2n-1}, gb_{2n-1}))} \varphi(t) dt \\
&= 2k \int_0^{(G(u'_{2n}, u'_{2n-1}, u'_{2n-1}) + G(v'_{2n}, v'_{2n-1}, v'_{2n-1}))} \varphi(t) dt \\
&\leq 4k \int_0^{(G(u'_{2n-1}, u'_{2n}, u'_{2n}) + G(v'_{2n-1}, v'_{2n}, v'_{2n}))} \varphi(t) dt \quad (3.6)
\end{aligned}$$

and

$$\begin{aligned}
\int_0^{G(v'_{2n}, v'_{2n+1}, v'_{2n+1})} \varphi(t) dt &\leq 2 \int_0^{G(v'_{2n+1}, v'_{2n}, v'_{2n})} \varphi(t) dt \\
&= 2 \int_0^{(G(F(b_{2n}, a_{2n}), S(b_{2n-1}, a_{2n-1}), S(b_{2n-1}, a_{2n-1})))} \varphi(t) dt \\
&\leq 2k \int_0^{(G(hb_{2n}, gb_{2n-1}, gb_{2n-1}) + G(ha_{2n}, ga_{2n-1}, ga_{2n-1}))} \varphi(t) dt \\
&= 2k \int_0^{(G(v'_{2n}, v'_{2n-1}, v'_{2n-1}) + G(u'_{2n}, u'_{2n-1}, u'_{2n-1}))} \varphi(t) dt \\
&\leq 4k \int_0^{(G(u'_{2n-1}, u'_{2n}, u'_{2n}) + G(v'_{2n-1}, v'_{2n}, v'_{2n}))} \varphi(t) dt. \quad (3.7)
\end{aligned}$$

We have

$$\begin{aligned}
\int_0^{G(u'_{2n}, u'_{2n+1}, u'_{2n+1})} \varphi(t) dt &+ \int_0^{G(v'_{2n}, v'_{2n+1}, v'_{2n+1})} \varphi(t) dt \\
&\leq 8k \int_0^{(G(u'_{2n-1}, u'_{2n}, u'_{2n}) + G(v'_{2n-1}, v'_{2n}, v'_{2n}))} \varphi(t) dt, \quad (3.8)
\end{aligned}$$

holds for all  $n \in N$ . Thus, using (3.5) and (3.8) in (3.3), we get

$$\begin{aligned} \int_0^{G(u'_{2n+1}, u'_{2n+2}, u'_{2n+2})} \varphi(t) dt &\leq k8k \int_0^{(G(u'_{2n-1}, u'_{2n}, u'_{2n}) + G(v'_{2n-1}, v'_{2n}, v'_{2n}))} \varphi(t) dt \\ &\leq k(8k)^2 \int_0^{(G(u'_{2n-2}, u'_{2n-1}, u'_{2n-1}) + G(v'_{2n-2}, v'_{2n-1}, v'_{2n-1}))} \varphi(t) dt \\ &\leq k(8k)^{2n} \int_0^{(G(u'_0, u'_1, u'_1) + G(v'_0, v'_1, v'_1))} \varphi(t) dt \\ &\vdots \\ &\leq (8k)^{2n+1} \int_0^{(G(u'_0, u'_1, u'_1) + G(v'_0, v'_1, v'_1))} \varphi(t) dt, \end{aligned}$$

and also, using (3.5) and (3.8) in (3.6), we get

$$\begin{aligned} \int_0^{G(u'_{2n}, u'_{2n+1}, u'_{2n+1})} \varphi(t) dt &\leq 4k(8k) \int_0^{(G(u'_{2n-2}, u'_{2n-1}, u'_{2n-2}) + G(v'_{2n-1}, v'_{2n-1}, v'_{2n-1}))} \varphi(t) dt \\ &\vdots \\ &\leq (8k)^{2n} \int_0^{(G(u'_0, u'_1, u'_1) + G(v'_0, v'_1, v'_1))} \varphi(t) dt. \end{aligned}$$

Thus for all  $n \in N$ , we have

$$\int_0^{G(u'_n, u'_{n+1}, u'_{n+1})} \varphi(t) dt \leq (8k)^n \int_0^{(G(u'_0, u'_1, u'_1) + G(v'_0, v'_1, v'_1))} \varphi(t) dt.$$

Let  $m, n \in N$  with  $m > n$ , we have

$$\begin{aligned} \int_0^{G(u'_n, u'_m, u'_m)} \varphi(t) dt &\leq \int_0^{G(u'_n, u'_{n+1}, u'_{n+1})} \varphi(t) dt + \int_0^{G(u'_{n+1}, u'_{n+2}, u'_{n+2})} \varphi(t) dt \\ &\quad + \dots + \int_0^{G(u'_{m-1}, u'_m, u'_m)} \varphi(t) dt. \end{aligned}$$

Since  $8k < 1$ , we get

$$\begin{aligned} \int_0^{G(u'_n, u'_m, u'_m)} \varphi(t) dt &\leq \sum_{i=n}^{m-1} (8k)^i \int_0^{(G(u'_0, u'_1, u'_1) + G(v'_0, v'_1, v'_1))} \varphi(t) dt \\ &\leq \frac{(8k)^n}{(1 - 8k)} \int_0^{(G(u'_0, u'_1, u'_1) + G(v'_0, v'_1, v'_1))} \varphi(t) dt. \end{aligned}$$

We have

$$\lim_{n, m \rightarrow +\infty} G(u'_n, u'_m, u'_m) = 0.$$

Thus  $(u'_n)$  is G-Cauchy in  $g(Y)$ . As  $g(Y)$  is G-complete then subsequence  $(u'_{2n+1}) = (ga_{2n+1})$  and  $(v'_{2n+1}) = (gb_{2n+1})$  are convergent to some  $a \in Y$  and  $b \in Y$  respectively. As we know that every sequence and subsequence of a G-Cauchy sequence are convergent to the same point. Hence  $(u'_{2n}) = (ha_{2n})$  and  $(v'_{2n}) = (hb_{2n})$  are also convergent. Since  $g$  and  $h$  are G-continuous, we have

$$(gga_{2n+1}) \rightarrow ga, (hga_{2n+1}) \rightarrow ha, (gha_{2n}) \rightarrow ga, (hha_{2n}) \rightarrow ha$$

and

$$(ggb_{2n+1}) \rightarrow gb, (hgb_{2n+1}) \rightarrow hb, (ghb_{2n}) \rightarrow gb, (hgb_{2n}) \rightarrow hb.$$

Since pairs  $\{F, h\}$  and  $\{S, g\}$  are commutative mappings, we have

$$hga_{2n+1} = hF(a_{2n}, b_{2n}) = F(ha_{2n}, hb_{2n})$$

and

$$gha_{2n} = gS(a_{2n-1}, b_{2n-1}) = S(ga_{2n-1}, ga_{2n-1}).$$

Thus

$$\begin{aligned} \int_0^{G(hga_{2n+1}, gha_{2n}, gha_{2n})} \varphi(t) dt &= \int_0^{G(F(ha_{2n}, hb_{2n}), S(ga_{2n-1}, gb_{2n-1}), S(ga_{2n-1}, gb_{2n-1}))} \varphi(t) dt \\ &\leq k \int_0^{(G(hha_{2n}, gga_{2n-1}, gga_{2n-1}) + G(hhb_{2n}, ggb_{2n-1}, ggb_{2n-1}))} \varphi(t) dt. \end{aligned}$$

Letting  $n \rightarrow +\infty$ , we have

$$\int_0^{G(ha, ga, ga)} \varphi(t) dt = k \int_0^{(G(ha, ga, ga) + G(hb, gb, gb))} \varphi(t) dt.$$

In the same way, we can show that

$$\int_0^{G(hb, gb, gb)} \varphi(t) dt = k \int_0^{(G(hb, gb, gb) + G(ha, ga, ga))} \varphi(t) dt.$$

Thus

$$\int_0^{G(ha, ga, ga)} \varphi(t) dt + \int_0^{G(hb, gb, gb)} \varphi(t) dt = 2k \int_0^{(G(ha, ga, ga) + G(hb, gb, gb))} \varphi(t) dt.$$

Since  $2k < 8k < 1$ , the last equality happens only if

$$\int_0^{G(ha, ga, ga)} \varphi(t) dt = \int_0^{G(hb, gb, gb)} \varphi(t) dt = 0.$$

Hence  $ha = ga$  and  $hb = gb$ . Again

$$\begin{aligned} \int_0^{G(hga_{2n+1}, S(a, b), S(a, b))} \varphi(t) dt &= \int_0^{G(F(ha_{2n}, hb_{2n}), S(a, b), S(a, b))} \varphi(t) dt \\ &\leq k \int_0^{(G(hha_{2n}, ggb_{2n}, ggb_{2n}) + G(hhb_{2n}, ggb_{2n}, ggb_{2n}))} \varphi(t) dt. \end{aligned}$$

Letting  $n \rightarrow +\infty$ , we have

$$\int_0^{G(ha, S(a, b), S(a, b))} \varphi(t) dt \leq k \int_0^{(G(ha, gb, gb) + G(hb, gb, gb))} \varphi(t) dt = 0.$$

Thus, we get

$$\int_0^{G(ha, S(a, b), S(a, b))} \varphi(t) dt = 0.$$

Which implies that  $S(a, b) = ha$ . Similarly we can show that  $S(b, a) = hb$ . By using the same technique, we get

$$\begin{aligned} \int_0^{G(F(a,b),gha_{2n},gha_{2n})} \varphi(t)dt &= \int_0^{G(F(a,b),S(ga_{2n-1},gb_{2n-1}),S(ga_{2n-1},gb_{2n-1}))} \varphi(t)dt \\ &\leq k \int_0^{(G(ha,gg a_{2n-1},gg a_{2n-1})+G(hb,gg b_{2n-1},gg b_{2n-1}))} \varphi(t)dt. \end{aligned}$$

Letting  $n \rightarrow +\infty$ , we have

$$\int_0^{G(F(a,b),ga,ga)} \varphi(t)dt = k \int_0^{(G(ha,ga,ga)+G(hb,gb,gb))} \varphi(t)dt = 0.$$

Thus, we get

$$\int_0^{G(F(a,b),ga,ga)} \varphi(t)dt = 0,$$

which means that  $F(a, b) = ga$ . By using the same method we can show that  $F(b, a) = hb$ . Hence we get  $ga = ha$ ,  $gb = hb$  and  $F(a, b) = gb$ ,  $S(a, b) = ha$ ,  $S(b, a) = hb$ , by using Lemma 3.1 we have

$$F(a, b) = ga = gb = F(b, a) = S(a, b) = ha = hb = S(b, a).$$

Now

$$\begin{aligned} \int_0^{G(ga_{2n+1},ga,ga)} \varphi(t)dt &= \int_0^{G(F(a_{2n},b_{2n}),S(a,b),S(a,b))} \varphi(t)dt \\ &\leq k \int_0^{(G(ha_{2n},ga,ga)+G(hb_{2n},gb,gb))} \varphi(t)dt. \end{aligned}$$

Letting  $n \rightarrow \infty$ , we have,

$$\int_0^{G(a,ga,ga)} \varphi(t)dt = k \int_0^{(G(a,ga,ga)+G(b,gb,gb))} \varphi(t)dt.$$

Similarly, we can show that

$$\int_0^{G(b,gb,gb)} \varphi(t)dt = k \int_0^{(G(b,gb,gb)+G(a,ga,ga))} \varphi(t)dt.$$

Thus

$$\int_0^{G(a,ga,ga)} \varphi(t)dt + \int_0^{G(b,gb,gb)} \varphi(t)dt = 2k \int_0^{(G(a,ga,ga)+G(b,gb,gb))} \varphi(t)dt.$$

Since  $2k < 8k < 1$ , the last equality happens only if

$$\int_0^{G(a,ga,ga)} \varphi(t)dt = \int_0^{G(b,gb,gb)} \varphi(t)dt = 0.$$

Hence  $a = ga$  and  $b = gb$ . Thus, we get

$$F(a, a) = S(a, a) = ga = ha = a.$$

For uniqueness, let  $y \in Y$  with  $y \neq a$  such that

$$F(y, y) = S(y, y) = gy = y.$$

Then

$$\begin{aligned}
 \int_0^{G(a,y,y)} \varphi(t)dt &= \int_0^{G(F(a,a),S(y,y),S(y,y))} \varphi(t)dt \\
 &\leq k \int_0^{G(ha,gy,gy)+G(ha,gy,gy)} \varphi(t)dt \\
 &= k \int_0^{G(a,y,y)+G(a,y,y)} \varphi(t)dt \\
 &= 2k \int_0^{G(a,y,y)} \varphi(t)dt.
 \end{aligned}$$

Since  $2k < 8k < 1$ , we get

$$\int_0^{G(a,y,y)} \varphi(t)dt < \int_0^{G(a,y,y)} \varphi(t)dt.$$

Which is a contradiction. Thus  $F, S, g, h$  have a unique common fixed point.

**3.2. Corollary.** Let  $(Y, G)$  be a G-metric space. Let  $F, S : Y \times Y \rightarrow Y$  and  $g, h : Y \rightarrow Y$  be two mappings such that

$$(3.3) \quad \int_0^{G(F(a,b),S(p,q),S(p,q))} \varphi(t)dt \leq k \int_0^{G(ha,gp,gp)+G(hb,gq,gq)} \varphi(t)dt$$

for all  $a, b, p, q \in Y$  and  $\varphi : [0, +\infty) \rightarrow [0, +\infty)$  is a Lebesgue integrable mapping which is summable such that for each  $\epsilon > 0$ ,  $\int_0^\epsilon \varphi(t)dt > 0$ . Assume that  $F, S$  and  $g, h$  satisfy the following conditions:

(i)  $F(Y \times Y) \subset g(Y)$   $S(Y \times Y) \subset h(Y)$

(ii)  $g(Y)$  or  $h(Y)$  is complete, and

(iii)  $g$  and  $h$  is G-continuous and pairs  $\{F, h\}$  and  $\{S, g\}$  are of commuting mappings.

If  $k \in [0, \frac{1}{8})$ , then there is a unique  $a \in Y$  such that  $F(a, a) = S(a, a) = g(a) = h(a) = a$ .

*Proof.* In Theorem 3.1 by taking  $c = p$  and  $q = r$ .

**3.3. Example.** Let  $Y = [0, 1]$ . Define  $G : Y \times Y \times Y \rightarrow R^+$  by

$$G(a, b, c) = |a - b| + |a - c| + |b - c|$$

for all  $a, b, c \in Y$ . Then  $(Y, G)$  is a complete G-metric space.

Define mappings  $F, S : Y \times Y \rightarrow Y$  and  $g, h : Y \rightarrow Y$  by

$$F(a, b) = \frac{1}{36}ab, \quad S(a, b) = \frac{1}{144}ab \quad \text{and} \quad ga = \frac{1}{4}a, \quad ha = \frac{1}{2}a.$$

Since  $|ab - pq| = |a - p| + |b - q|$  holds for all  $a, b, p, q \in Y$ .

Then the condition of Theorem (3.1) holds, in fact

$$\begin{aligned}
 \int_0^{G(F(a,b),S(p,q),S(c,r))} \varphi(t)dt &= \int_0^{(|\frac{1}{36}ab - \frac{1}{144}pq| + |\frac{1}{144}pq - \frac{1}{144}cr| + |\frac{1}{144}cr - \frac{1}{36}ab|)} \varphi(t)dt \\
 &\leq \frac{1}{9} \int_0^{\{\frac{1}{2}a - \frac{1}{4}p| + |\frac{1}{4}p - \frac{1}{4}c| + |\frac{1}{4}c - \frac{1}{2}a| + |\frac{1}{2}b - \frac{1}{4}q| + |\frac{1}{4}q - \frac{1}{4}r| + |\frac{1}{4}r - \frac{1}{2}b|\}} \varphi(t)dt \\
 &= \frac{1}{9} \int_0^{G(ha,gp,gc)+G(hb,gq,gr)} \varphi(t)dt
 \end{aligned}$$

holds for all  $a, b, c, p, q, r \in Y$ . It is easy to see that  $F, S, g, h$  satisfies all the hypothesis of Theorem 3.1. Thus  $F, S, g, h$  have a unique common fixed point. Here  $F(0, 0) = S(0, 0) = g0 = h0 = 0$ .

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