Hacettepe Journal of Mathematics and Statistics
Volume 45 (1) (2016), 245 – 265

# The Weibull-power function distribution with applications

M. H. Tahir[*†], Morad Alizadeh[‡], M. Mansoor[§], Gauss M. Cordeiro[¶] and M. Zubair[‖]

## Abstract

Recently, several attempts have been made to define new models that extend well-known distributions and at the same time provide great flexibility in modelling real data. We propose a new four-parameter model named the *Weibull-power function (WPF)* distribution which exhibits bathtub-shaped hazard rate. Some of its statistical properties are obtained including ordinary and incomplete moments, quantile and generating functions, Rényi and Shannon entropies, reliability and order statistics. The model parameters are estimated by the method of maximum likelihood. A bivariate extension is also proposed. The new distribution can be implemented easily using statistical software packages. We investigate the potential usefulness of the proposed model by means of two real data sets. In fact, the new model provides a better fit to these data than the additive Weibull, modified Weibull, Sarahan-Zaindin modified Weibull and beta-modified Weibull distributions, suggesting that it is a reasonable candidate for modeling survival data.

**Keywords:** Generalized uniform distribution, moments, power function distribution, Weibull-G class.

*2000 AMS Classification:* 60E05, 62E10, 62N05.

[*]Department of Statistics, The Islamia University of Bahawalpur, Bahawalpur 63100, Pakistan.
Email : `mtahir.stat@gmail.com; mht@iub.edu.pk`

[†]Corresponding Author.

[‡]Department of Statistics, Persian Gulf University of Bushehr, Iran.
Email : `moradalizadeh78@gmail.com`

[§]Punjab College, 26-C Shabbir Shaheed Road, Model Town-A, Bahawalpur, Pakistan.
Email : `mansoor.abbasi143@gmail.com`

[¶]Department of Statistics, Federal University of Pernambuco, 50740-540, Recife, PE, Brazil.
Email : `gauss@de.ufpe.br, gausscordeiro@gmail.com`

[‖]Government Degree College Kahrorpacca, Lodhran, Pakistan.
Email : `zubair.stat@yahoo.com`

*Received* : 20.10.2014 *Accepted* : 07.12.2014 *Doi* : 10.15672/HJMS.2014428212

## 1. Introduction

A suitable generalized lifetime model is often of interest in the analysis of survival data, as it can provide insight into characteristics of failure times and hazard functions that may not be available with classical models. Four distributions (exponential, Pareto, power and Weibull) are of interest and very attractive in lifetime literature due to their simplicity, easiness and flexible features to model various types of data in different fields. The *power function distribution* (PFD) is a flexible lifetime model which can be obtained from the Pareto model by using a simple transformation $Y = X^{-1}$ [19] and it is also a special case of the beta distribution. Meniconi and Barry [36] discussed the application of the PFD along with other lifetime models, and concluded that the PFD is better than the Weibull, log-normal and exponential models to measure the reliability of electronic components. The PFD can be used to fit the distribution of certain likelihood ratios in statistical tests. If the likelihood ratio (LR) is based on $n$ iid random variables, it is often found that a useful goodness-of-fit can be obtained by letting $(likelihood\ ratio)^{2/n}$ to have a PFD (see [6]). For introduction and statistical properties of the PFD, the reader is referred to Johnson *et al.* [23, 24], Balakrishnan and Nevzorov [13], Kleiber and Kotz [29] and Forbes *et al.* [21]. The estimation of its parameters is discussed in detail by [55, 56, 9]. The estimation of the sample size for parameter estimation is addressed by Kapadia [26]. Ali *et al.* [8] derived the UMVUE of the mean and the right-tail probability of the PFD. Ali and Woo [6] and Ali *et al.* [7] provided inference on reliability and the ratio of variates in the PFD. Sinha *et al.* [51] proposed a preliminary test estimator for a scale parameter of the PFD.

From a Bayesian point of view, the PFD can be used as a prior when there is limited sample information, and especially in cases where the relationship between the variables is known but the data is scarce (possibly due to high cost of collection). The PFD can also be used as prior distribution for the binomial proportion. Saleem *et al.* [45] performed Bayesian analysis of the mixture of PFDs using complete and censored samples. Rehman *et al.* [41] used Bayes estimation and conjugate prior for the PFD. Kifayat *et al.* [28] analyzed this distribution in the Bayesian context using informative and non-informative priors. Zarrin *et al.* [57] discussed the reliability estimation and Bayesian analysis of the system reliability of the PFD.

Several authors have reported characterization of the PFD based on order statistics and records. Rider [44] first derived the distribution of the product and ratio of the order statistics. Govindarajulu [22] gave the characterization of the exponential and PFD. Exact explicit expression for the single and the product moments of order statistics are obtained by Malik [31]. Ahsanullah [2] defined necessary and sufficient conditions based on PFD order statistics. Kabir and Ahsanullah [25] estimated the location and scale of the PFD using linear function of order statistics. Balakrishnan and Joshi [12] derived some recurrence relations for the single and the product moments of order statistics. Moothathu [38, 39] gave characterizations of the PFD through Lorenz curve. The estimation of the PFD parameters based on record values is studied by Ahsanullah [3]. Saran and Singh [47] developed recurrence relations for the marginal and generating functions of generalized order statistics. Saran and Pandey [46] estimated the parameters of the PFD and proposed a characterization based on $k$th record values. The characterization based on the lower generalized order statistics is given in Ahsanullah [4], and Mbah and Ahsanullah [34]. Chang [16] suggested other characterization by independence of records values. Athar and Faizan [10] derived some recurrence relations for single and product moments of lower generalized order statistics. Tavangar [53] gave a characterization based on dual generalized order statistics. Bhatt [14] proposed a characterization based on any arbitrary non-constant function. Recently, Azedine [11] derived single and double moments of the lower record values, and also established recurrence relations for these single and double moments.

Different versions of the PFD are reported in the literature. Some of them are summarized in Table 1, where $\Pi(x)$ denotes its cumulative distribution function (cdf) and $\pi(x)$ denotes its probability probability function (pdf).

Table 1: Some versions of the PFD.

| S.No./Ref. | $\Pi(x)$ | $\pi(x)$ | Range of variable | Parameters |
|---|---|---|---|---|
| 1./ [11] | $x^\alpha$ | $\alpha\, x^{\alpha-1}$ | $0 < x < 1$ | $\alpha > 0$ |
| 2./ [5] | $(x/\lambda)^\alpha$ | $\alpha\, \lambda^{-\alpha}\, x^{\alpha-1}$ | $0 < x < \lambda$ | $\alpha > 0$ |
| 3./ [18] | $(x\,\beta)^\alpha$ | $\alpha\, \beta^\alpha\, x^{\alpha-1}$ | $0 < x < \beta^{-1}$ | $\alpha, \beta > 0$ |
| 4./ [10] | $(x/\theta)^{\alpha+1}$ | $(\alpha+1)\,\theta^{-(\alpha+1)}\, x^\alpha$ | $0 < x < \theta$ | $\alpha > -1, \theta > 0$ |
| 5./ [47] | $1-(1-x)^\delta$ | $\delta\ (1-x)^{\delta-1}$ | $0 < x < 1$ | $\delta > 0$ |
| 6./ [52] | $\left[\frac{x-\theta}{\sigma}\right]^\nu$ | $\frac{\nu}{\sigma}\left[\frac{x-\theta}{\sigma}\right]^{\nu-1}$ | $\theta < x < \sigma+\theta$ | $\nu, \sigma > 0$ |
| 7./ [46] | $1-\left[\frac{\beta-x}{\beta-\alpha}\right]^\gamma$ | $\frac{\gamma}{\beta-\alpha}\left[\frac{\beta-x}{\beta-\alpha}\right]^{\gamma-1}$ | $\alpha < x < \beta$ | $\gamma > 0$ |
| 8./ [6] | $x^{\left[\frac{\sigma}{1-\sigma}\right]}$ | $\left[\frac{\sigma}{1-\sigma}\right] x^{\left[\frac{\sigma}{1-\sigma}\right]-1}$ | $0 < x < 1$ | $0 < \sigma < 1$ |

A random variable $Z$ has the PFD or the generalized uniform distribution (GUD) [40] with two positive parameters $\alpha$ and $\beta$, if its cdf is given by

$$G(x) = \left[\frac{x}{\alpha}\right]^\beta, \quad 0 < x < \alpha, \tag{1.1}$$

where $\alpha$ is the scale (threshold) parameter and $\beta$ is the shape parameter. The pdf corresponding to (1.1) reduces to

$$g(x) = \left[\frac{\beta}{\alpha}\right]\left[\frac{x}{\alpha}\right]^{\beta-1}, \quad 0 < x < \alpha, \tag{1.2}$$

The distribution (1.1) has the following special cases:

(i) if $\alpha = 1$, the PFD reduces to standard power distribution,

(ii) if $\alpha = 1$ and $\beta = 1$, it reduces to standard uniform distribution,

(iii) if $\beta = 1$, it gives the rectangular distribution [31, 25],

(iv) if $\beta = 2$, it refers to triangular distribution [31, 25],

(v) if $\beta = 3$, it refers to J-shaped distribution [31, 25],

(vi) if $\alpha = 1$ and $Y = X^{-1}$, then $Y$ ~Pareto$(0, \beta)$ [21],

(vii) if $\alpha = 1$ and $Y = -\log X$, then $Y$ ~Exponential$(\beta^{-1})$ [21],

(viii) if $\alpha = 1$ and $Y = -\log(X^\beta - 1)$, then $Y$ ~Logistic$(0, 1)$ [21],

(ix) if $\alpha = 1$ and $Y = [-\log(X^\beta)]^{1/\gamma}$, then $Y$ ~Weibull$(0, \gamma)$ [21],

(x) if $\alpha = 1$ and $Y = -\log[-b \log X]$, then $Y$ ~Gumbel$(0, 1)$ [21],

(xi) if $\alpha = 1$ and $Y = -b\,[X_1/X_2]$, then $Y$ ~Laplace$(0, 1)$ [21].

Henceforth, let $Z$ be a random variable having the PFD with parameters $\alpha$ and $\beta$, say $Z$ ~PFD$(\alpha, \beta)$. Then, the quantile function (qf) is $G^{-1}(u) = \alpha\, u^{1/\beta}$ (for $0 < u < 1$).

The survival function (sf) $\overline{G}(x)$, hazard rate function (hrf) $\tau(x)$, reversed hazard rate function (rhrf) $r(x)$, cumulative hazard rate function (chrf) $V(x)$ and odd ratio (OR) $G(x)/\overline{G}(x)$ of $Z$ are given by $\overline{G}(x) = 1-(x/\alpha)^{\beta} = \frac{\alpha^{\beta}-x^{\beta}}{\alpha^{\beta}}$, $\tau(x) = \frac{\beta\, x^{\beta-1}}{\alpha^{\beta}-x^{\beta}}$, $r(x) = (\beta/x)$, $V(x) = -\log\left[1-(x/\alpha)^{\beta}\right]$ and OR $= \frac{x^{\beta}}{\alpha^{\beta}-x^{\beta}}$, respectively.

The $n$th moment of $Z$ comes from (1.2) as

$$E(Z^n) = \frac{\alpha^n\,\beta}{\beta+n}. \tag{1.3}$$

The mean and variance of $Z$ are

$$E(Z) = [\alpha\beta/(\beta+1)]$$

and

$$Var(Z) = \left\{\beta\alpha^2/[(\beta+2)(\beta+1)^2]\right\},$$

respectively.

The moment generating function (mgf) of $Z$ becomes

$$M_Z(t) = \frac{\beta\Big[\Gamma(\beta)-\Gamma(\beta,-t\,\alpha)\Big]}{(-t)^{\beta}\,\alpha^{\beta}}, \quad t<0, \tag{1.4}$$

where $\Gamma(a;bx) = b^a \int_x^{\infty} w^{a-1}\,\mathrm{e}^{-\mathrm{b\,w}}\mathrm{dw}$ for $a>0$ and $b>0$ and $\Gamma(\cdot;\cdot)$ is the complementary gamma function.

The $n$th incomplete moment of $Z$ can be expressed as

$$m_{(n,Z)}(x) = \frac{\beta}{\alpha^{\beta}}\,\frac{x^{\beta+n}}{\beta+n}. \tag{1.5}$$

In this paper, we propose an extension of the PFD called the *Weibull power function* (for short "WPF") distribution based on the *Weibull-G* class of distributions defined by Bourguignon *et al.* [15]. Zagrafos and Balakrishnan [58] pioneered a versatile and flexible gamma-G class of distributions based on Stacy's generalized gamma model and record value theory. More recently, Bourguignon *et al.* [15] proposed the *Weibull-G* class of distributions influenced by the gamma-G class. Let $G(x;\Theta)$ and $g(x;\Theta)$ denote the cumulative and density functions of a baseline model with parameter vector $\Theta$ and consider the Weibull cdf $\pi_W(x) = 1-\mathrm{e}^{-a\,x^b}$ (for $x>0$) with scale parameter $a>0$ and shape parameter $b>0$. Bourguignon *et al.* [15] replaced the argument $x$ by $G(x;\Theta)/\overline{G}(x;\Theta)$, where $\overline{G}(x;\Theta) = 1-G(x;\Theta)$, and defined the cdf of their class, say Weibull-G$(a,b,\Theta)$, by

$$F(x) = F(x;a,b,\Theta) = a\,b\int_0^{\left[\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right]} x^{b-1}\,\mathrm{e}^{-ax^b}\,dx = 1-\mathrm{e}^{-a\left[\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right]^b},\ x\in\Re. \tag{1.6}$$

The Weibull-G class density function becomes

$$f(x) = f(x;a,b,\Theta) = a\,b\,g(x;\Theta)\,\left[\frac{G(x;\Theta)^{b-1}}{\overline{G}(x;\Theta)^{b+1}}\right]\,\mathrm{e}^{-a\left[\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right]^b}. \tag{1.7}$$

If $b=1$, it corresponds to the *exponential-G* class. An interpretation of equation (1.6) can be given as follows. Let $Y$ be the lifetime variable having a parent G distribution. Then, the odds that an individual will die at time $x$ is $G(x;\Theta)/\overline{G}(x;\Theta)$. We are interested in modeling the randomness of the odds of death using an appropriate parametric distribution, say $F(x)$. So, we can write

$$F(x) = \Pr(X\le x) = F\Big[\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\Big].$$

The paper unfolds as follows. In Section 2, we define a new bathtub shaped model called the *Weibull-power function* (WPF) distribution and discuss the shapes of its density and hrf. In Section 3, some of its statistical properties are investigated. In Section 4, Rényi and Shannon entropies are derived and the reliability is determined in Section 5. The density of the order statistics is obtained in Section 6. The model parameters are estimated by maximum likelihood and a simulation study is performed in Section 7. In Section 8, a bivariate extension of the new family is introduced. Applications to two real data sets illustrate the performance of the new model in Section 9. The paper is concluded in Section 10.

## 2. Model definition

Inserting (1.1) in equation (1.6) gives the WPF cdf as

$$(2.1) \qquad F(x) = F(x; a, b, \alpha, \beta) = 1 - \mathrm{e}^{-a\left[\frac{x^\beta}{\alpha^\beta - x^\beta}\right]^b}, \quad 0 < x < \alpha, \quad a, b, \alpha, \beta > 0.$$

The pdf corresponding to (2.1) is given by

$$(2.2) \qquad f(x) = f(x; a, b, \alpha, \beta) = \frac{a\, b\, \beta\, \alpha^\beta\, x^{\beta\, b-1}}{(\alpha^\beta - x^\beta)^{b+1}}\ \mathrm{e}^{-a\left[\frac{x^\beta}{\alpha^\beta - x^\beta}\right]^b}.$$

Henceforth, let $X \sim \mathrm{WPF}(a, b, \alpha, \beta)$ be a random variable having pdf (2.2). The sf, hrf, rhrf and chrf of $X$ are given by

$$(2.3) \qquad S(x) = S(x; a, b, \alpha, \beta) = \mathrm{e}^{-a\left[\frac{x^\beta}{\alpha^\beta - x^\beta}\right]^b},$$

$$\tau(x) = h(x; a, b, \alpha, \beta) = \frac{a\, b\, \beta\, \alpha^\beta x^{\beta\, b-1}}{(\alpha^\beta - x^\beta)^{b+1}},$$

$$r(x) = r(x; a, b, \alpha, \beta) = \frac{a\, b\, \beta\, \alpha^\beta x^{\beta\, b-1}}{(\alpha^\beta - x^\beta)^{b+1}}\ \frac{\mathrm{e}^{-a\left[\frac{x^\beta}{\alpha^\beta - x^\beta}\right]^b}}{\left[1 - \mathrm{e}^{-a\left[\frac{x^\beta}{\alpha^\beta - x^\beta}\right]^b}\right]}$$

and

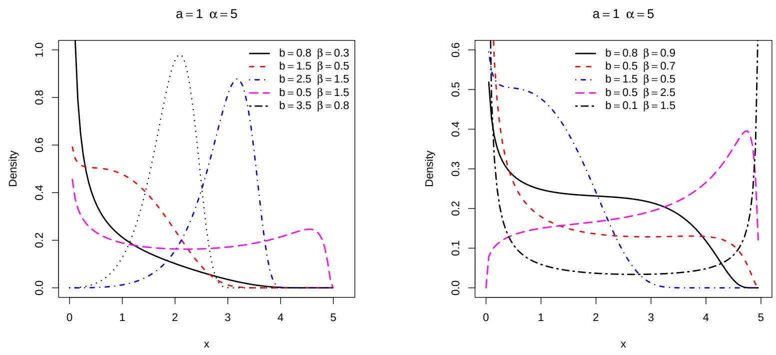
$$V(x) = V(x; a, b, \alpha, \beta) = a\left[\frac{x^\beta}{\alpha^\beta - x^\beta}\right]^b,$$

respectively.

Figures 1 and 2 display some plots of the pdf and hrf of $X$ for some parameter values. Figure 1 indicates that the WPF pdf has various shapes such as symmetric, right-skewed, left-skewed, reversed-J, S, M and bathtub. Also, Figure 2 indicates that the WPF hrf can have bathtub-shaped, J and U shapes.

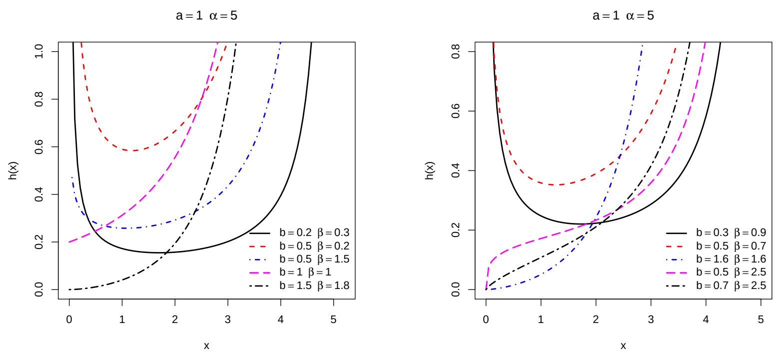
Lemma 2.1 provides some relations of the WPF distribution with the Weibull and exponential distributions.

**2.1. Lemma.** *(Transformation): (a) If a random variable $Y$ follows the Weibull distribution with shape parameter $b$ and scale parameter $a$, then the random variable $X = \alpha\left[\frac{Y}{1+Y}\right]^{1/\beta}$ has the WPF$(a, b, \alpha, \beta)$ distribution.*
*(b) If a random variable $Y$ follows the exponential distribution, then the random variable $X = \alpha\left[\frac{Y^{1/b}}{1+Y^{1/b}}\right]^{1/\beta}$ has the WPF$(a, b, \alpha, \beta)$ distribution.*




(a) (b)

**Figure 1.** Plots of the WPF pdf for some parameters.




(a) (b)

**Figure 2.** Plots of the hazard rate for some parameters.

**2.1. Shape and asymptotics.** The critical points of the density of $X$ are the roots of the equation

$$\frac{b\beta - 1}{x} + \frac{\beta(b+1)x^{\beta-1}}{\alpha^\beta - x^\beta} - \frac{a\, b\, \beta\, \alpha^\beta x^{\beta\, b-1}}{(\alpha^\beta - x^\beta)^{b+1}} = 0. \tag{2.4}$$

The first derivative of the hrf of $X$ is given by

$$\tau'(x) = \frac{x^{b\beta-2}\left\{(\beta+1)x^\beta + (b\,\beta - 1)\alpha^\beta\right\}}{(\alpha^\beta - x^\beta)^{b+2}}. \tag{2.5}$$

The limiting behavior of the pdf and hrf of $X$ are given in the following lemma.

**2.2. Lemma.** *The limits of the pdf and hrf of $X$ when $x \to \alpha^-$ are 0 and $+\infty$. Further, the limits of the pdf and hrf of $X$ when $x \to 0$ are given by*

$$\lim_{x\to 0^+} f(x) = \begin{cases} +\infty & \textit{for } b\,\beta < 1; \\ \frac{a}{\alpha} & \textit{for } b\,\beta = 1; \\ 0 & \textit{for } b\,\beta > 1. \end{cases}$$

$$\lim_{x\to 0^+} \tau(x) = \begin{cases} +\infty & for\ b\,\beta < 1; \\ \frac{a}{\alpha} & for\ b\,\beta = 1; \\ 0 & for\ b\,\beta > 1. \end{cases}$$

The mode of the hrf of $X$ is at $x = 0$ when $\beta\, b \geq 1$ and it occurs at $x = \alpha \left[\frac{1-b\beta}{1+\beta}\right]^{\frac{1}{\beta}}$ when $b\beta < 1$.

**2.3. Theorem.** *The hrf of $X$ is increasing when $b\,\beta \geq 1$ and is bathtub when $b\,\beta < 1$.*

## 3. Mathematical properties

Established algebraic expansions to determine some mathematical properties of the WPF distribution can be more efficient than computing those directly by numerical integration of (2.2), which can be prone to rounding off errors among others. Despite the fact that the cdf and pdf of the WPF distribution require mathematical functions that are widely available in modern statistical packages, frequently analytical and numerical derivations take advantage of certain expansions for its pdf.

**3.1. Quantile function.** The quantile function (qf) of $X$ follows by inverting (2.1) as

$$Q(u) = \alpha \left[ \frac{\left[\frac{-1}{a}\log(1-u)\right]^{\frac{1}{b}}}{1 + \left[\frac{-1}{a}\log(1-u)\right]^{\frac{1}{b}}} \right]^{\frac{1}{\beta}}. \tag{3.1}$$

So, the simulation of the WPF random variable is straightforward. If $U$ is a uniform variate on the unit interval $(0, 1)$, then the random variable $X = Q(U)$ has pdf (2.2).

The analysis of the variability of the the skewness and kurtosis on the shape parameters $\alpha$ and $b$ can be investigated based on quantile measures. The shortcomings of the classical kurtosis measure are well-known. The Bowley skewness [27] based on quartiles is given by

$$B = \frac{Q(3/4) + Q(1/4) - 2\,Q(2/4)}{Q(3/4) - Q(1/4)}.$$

The Moors kurtosis [37] based on octiles is given by

$$M = \frac{Q(3/8) - Q(1/8) + Q(7/8) - Q(5/8)}{Q(6/8) - Q(2/8)}.$$

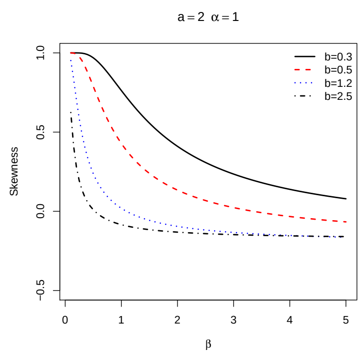
These measures are less sensitive to outliers and they exist even for distributions without moments. In Figure 3, we plot the measures $B$ and $M$ for the WPF distribution. The plots indicate the variability of these measures on the shape parameters $\beta$.

**3.2. Useful expansion.** We use the exponential power series and the expansion

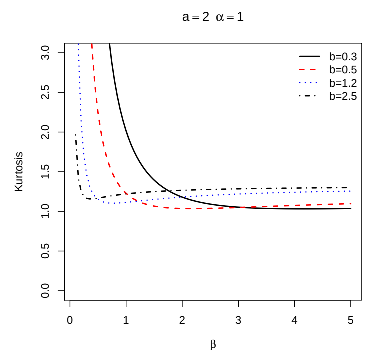
$$\Big[1 - G(x;\Theta)\Big]^{-b} = \sum_{k=0}^{\infty} p_k\, G(x;\Theta)^k,$$

where $p_k = \Gamma(b+k)/[k!\,\Gamma(b)]$. After some algebra, we can easily obtain

$$F(x) = F(x; a, b, \alpha, \beta) = \sum_{\substack{j,k\geq 0 \\ j+k\geq 1}} w_{j,k}\, H(x;\alpha, \beta_{j,k}), \tag{3.2}$$




(a)




(b)

**Figure 3.** Skewness (a) and kurtosis (b) plots for WPF distribution based on quantiles.

where $w_{j,k} = (-a)^j\, p_k/j!$, $\beta_{j,k} = (jb + k)\beta$ and $H(x; \alpha, \beta_{j,k})$ is the cdf of the PFD with scale parameter $\alpha$ and shape parameter $\beta_{j,k}$. Let $Z_{j,k}$ be the random variable with cdf $H(x; \alpha, \beta_{j,k})$. By simple differentiation, we can express the pdf of $X$ as

$$f(x) = f(x; a, b, \alpha, \beta) = \sum_{\substack{j,k\geq 0\\ j+k\geq 1}} w_{j,k}\, h(x; \alpha, \beta_{j,k}), \tag{3.3}$$

where $h(x; \alpha, \beta_{j,k})$ is the pdf of $Z_{j,k}$. Equation (3.3) reveals that the WPF distribution is a mixture of PFDs with the same scale parameter $\alpha$ and different shape parameters. Thus, some WPF mathematical properties can be obtained from those corresponding properties of the PFD.

**3.3. Ordinary and incomplete moments.** The $n$th moment of $X$, say $\mu'_n$ can be expressed from (1.3) and (3.3) as

$$\mu'_n = \alpha^n \sum_{\substack{j,k\geq 0\\ j+k\geq 1}} \frac{\beta_{j,k}\, w_{j,k}}{\beta_{j,k} + n}. \tag{3.4}$$

Setting $n = 1$ in (3.4), we obtain the mean $\mu'_1 = E(X)$. The central moments $(\mu_n)$ and cumulants $(\kappa_n)$ of $X$ are obtained from equation (3.4) as

$$\mu_n = \sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix} (-1)^k\, \mu_1'^k\, \mu'_{n-k} \qquad \text{and} \qquad \kappa_n = \mu'_n - \sum_{k=1}^{n-1} \begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix} \kappa_k\, \mu'_{n-k},$$

respectively, where $\kappa_1 = \mu'_1$ and the notation

$$\begin{Bmatrix} n \\ k \end{Bmatrix}$$

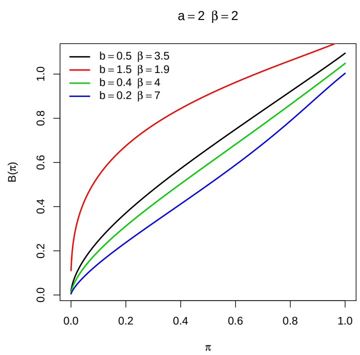
is used to denote the binomial coefficient.

Thus, $\kappa_2 = \mu'_2 - \mu_1'^2$, $\kappa_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2\mu_1'^3$, $\kappa_4 = \mu'_4 - 4\mu'_3\mu'_1 - 3\mu_2'^2 + 12\mu'_2\mu_1'^2 - 6\mu_1'^4$, etc. The skewness and kurtosis can be calculated from the third and fourth standardized cumulants as $\gamma_1 = \kappa_3/\kappa_2^{3/2}$ and $\gamma_2 = \kappa_4/\kappa_2^2$. They are also important to derive Edgeworth expansions for the cdf and pdf of the standardized sum and sample mean of iid random variables having the WPF distribution.

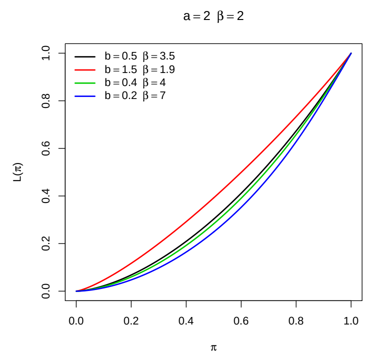
The $n$th incomplete moment of $X$ can be determined from (1.5) and (3.3)

$$m_{(n,X)}(x) = \sum_{\substack{j,k\geq 0 \\ j+k\geq 1}} \frac{\beta_{j,k}}{\alpha^{\beta_{j,k}}} \frac{x^{\beta_{j,k}+n}}{\beta_{j,k}+n}. \tag{3.5}$$

The main application of the first incomplete moment refers to the Bonferroni and Lorenz curves. These curves are very useful in several fields. For a given probability $\pi$, they are defined by $B(\pi) = m_1(q)/(\pi\,\mu_1')$ and $L(\pi) = m_{(1,X)}(q)/\mu_1'$, respectively, where $m_{(1,X)}(q)$ comes from (3.5) with $r = 1$ and $q = Q(\pi)$ is determined from (3.1).




(a)




(b)

**Figure 4.** Plots of the Bonferroni curve (a) and Lorenz curve (b) for the WPF model.

The amount of scatter in a population is measured to some extent by the totality of deviations from the mean and median defined by $\delta_1 = \int_0^\infty |x - \mu_1'|\, f(x)dx$ and $\delta_2(x) = \int_0^\infty |x - M|\, f(x)dx$, respectively, where $\mu_1' = E(X)$ is the mean and $M = Q(0.5)$ is the median. These measures can be expressed as $\delta_1 = 2\mu_1'\, F(\mu_1') - 2m_{(1,X)}(\mu_1')$ and $\delta_2 = \mu_1' - 2m_{(1,X)}(M)$, where $F(\mu_1')$ is given by (2.1) and $m_{(1,X)}(x)$ comes from (3.5) with $n = 1$.

Further applications of the first incomplete moment are related to the mean residual life and mean waiting time given by $s(x; a, b, \alpha, \beta) = [1 - m_{(1,X)}(x)]/S(x) - t$ and $\mu(x; a, b, \alpha, \beta) = t - [m_{(1,X)}(x)/F(x)]$, respectively, where $S(x) = 1 - F(x)$ is obtained from (2.1).

**3.4. Moment generating function.** We obtain the moment generating function (mgf) $M_X(t)$ of $X$ from (3.3) as

$$M(t) \quad = \sum_{\substack{j,k\geq 0 \\ j+k\geq 1}} w_{j,k} \int_0^\alpha \mathrm{e}^{tx}\, h(x; \alpha, \beta_{j,k})\, dx.$$

Based on (1.4), $M(t)$ can be expressed as

$$M(t) \quad = \sum_{\substack{j,k\geq 0 \\ j+k\geq 1}} \frac{w_{j,k}\,\beta_{j,k}}{(-t)^{\beta_{j,k}}\,\alpha^{\beta_{j,k}}} \Big[\Gamma(\beta_{j,k}) - \Gamma(\beta_{j,k}; -t\alpha)\Big],$$

which is the main result of this section.

## 4. Entropies

An entropy is a measure of variation or uncertainty of a random variable $X$. Two popular entropy measures are the Rényi [43] and Shannon [49].

The Rényi entropy of a random variable $X$ with pdf $f(x)$ is defined as

$$I_R(\gamma) = \frac{1}{1-\gamma} \log \left[ \int_0^\infty f^\gamma(x)\, dx \right],$$

for $\gamma > 0$ and $\gamma \neq 1$.

The Shannon entropy of $X$ is defined by $E\{-\log[f(X)]\}$. It is the special case of the Rényi entropy when $\gamma \uparrow 1$. Direct calculation yields

$$\begin{aligned} \mathrm{E}\{-\log[f(X)]\} &= -\log(a\, b\, \beta\, \alpha^\beta) + (1-\beta)\, \mathrm{E}\{\log(X)\} \\ &+ (b+1)\, \mathrm{E}\left[\log(\alpha^\beta - X^\beta)\right] + a\, \mathrm{E}\left[\frac{X^\beta}{\alpha^\beta - X^\beta}\right]^b. \end{aligned}$$

First, we define and compute

$$A(a_1, a_2, a_3; \alpha, \beta, b) = \int_0^\alpha \frac{x^{a_1}}{(\alpha^\beta - x^\beta)^{a_2}}\, \mathrm{e}^{-a_3 \left[\frac{x^\beta}{\alpha^\beta - x^\beta}\right]^b} dx. \tag{4.1}$$

Using the power series and the generalized binomial expansion, and after some algebraic manipulations, we obtain

$$A(a_1, a_2, a_3; \alpha, \beta, b) = \sum_{i,j=0}^\infty \frac{(-1)^{i+j}\, a_3^i\, \alpha^{a_1 - \beta\, a_2}}{[a_1 + \beta\, b\, i + \beta\, j + 1]\; i!} \begin{Bmatrix} -a_2 - b\, i \\ j \end{Bmatrix}.$$

**4.1. Proposition.** *Let $X$ be a random variable with pdf (2.2), then*

$$\mathrm{E}\{\log(X)\} = a\, b\, \beta\, \alpha^\beta \frac{\partial}{\partial t} A(b\beta + t - 1, b+1, a; \alpha, \beta, b)\,|_{t=0},$$

$$\mathrm{E}\left[\log(\alpha^\beta - X^\beta)\right] = a\, b\, \beta\, \alpha^\beta \frac{\partial}{\partial t} A(b\beta - 1, b+1-t, a; \alpha, \beta, b)\,|_{t=0},$$

$$\mathrm{E}\left[\left\{\frac{X^\beta}{\alpha^\beta - X^\beta}\right\}^b\right] = a\, b\, \beta\, \alpha^\beta A(2b\beta - 1, 2b+1, a; \alpha, \beta, b).$$

The simplest formula for the entropy of $X$ is given by

$$\begin{aligned} \mathrm{E}\{-\log[f(X)]\} &= -\log(a\, b\, \beta\, \alpha^\beta) \\ &+ (1-\beta) a\, b\, \beta\, \alpha^\beta \frac{\partial}{\partial t} A(b\beta + t - 1, b+1, a; \alpha, \beta, b)\,|_{t=0} \\ &+ (b+1) a\, b\, \beta\, \alpha^\beta \frac{\partial}{\partial t} A(b\beta - 1, b+1-t, a; \alpha, \beta, b)\,|_{t=0} \\ &+ a^2\, b\, \beta\, \alpha^\beta\, A(2b\beta - 1, 2b+1, a; \alpha, \beta, b). \end{aligned}$$

After some algebraic developments, the Rényi entropy $I_R(\gamma)$ reduces to

$$I_R(\gamma) = \frac{\gamma}{1-\gamma} \log\left[a\, b\, \beta\, \alpha^\beta\right] + \frac{1}{1-\gamma} \log\left\{A\Big[\gamma(\beta\, b - 1), \gamma(b+1), a\gamma; \alpha, \beta, b\Big]\right\}. \tag{4.2}$$

## 5. Reliability

Let $X_1$ and $X_2$ be two continuous and independent WPF random variables with cdfs $F_1(x)$ and $F_2(x)$ and pdfs $f_1(x)$ and $f_2(x)$, respectively. The reliability parameter $R = P(X_1 < X_2)$ is defined by

$$R = P(X_1 < X_2) = \int_0^{\alpha_2} P(X_1 \le X_2 | X_2 = x)\, f_{X_2}(x) dx, \tag{5.1}$$

where $X_1 \sim$WPF$(a_1, b_1, \alpha_1, \beta_1)$ and $X_2 \sim$WPF$(a_2, b_2, \alpha_2, \theta_2)$.

After some algebra, we obtain

$$\begin{aligned} R &= \sum_{\substack{j,k,r,s\geq 0\\ j+k\geq 1, r+s\geq 1}} w_{j,k}^{(1)}\, w_{r,s}^{(2)} \int_0^{\alpha_2} H(x;\alpha_1,\beta_{j,k}^{(1)})\, h(x;\alpha_2,\beta_{r,s}^{(2)})dx \\ &= \sum_{\substack{j,k,r,s\geq 0\\ j+k\geq 1, r+s\geq 1}} w_{j,k}^{(1)}\, w_{r,s}^{(2)}\, \frac{(b_2 r+s)}{\alpha_2\beta_2} \left[\frac{\alpha_2}{\alpha_1}\right]^{\frac{b_1 j+k}{\beta_1}} \left[\frac{b_1 j+k}{\beta_1}+\frac{b_2 r+s}{\beta_2}\right]^{-1}, \end{aligned}$$

where $w_{j,k}^{(1)} = w_{j,k}|_{a=a_1,b=b_1,\beta=\beta_1}$ and $w_{r,s}^{(2)} = w_{r,s}|_{a=a_2,b=b_2,\beta=\beta_2}$.

## 6. Order statistics

Here, we give the density of the $i$th order statistic $X_{i:n}$, $f_{i:n}(x)$ say, in a random sample of size $n$ from the WPF distribution. It is well known that (for $i=1,\ldots,n$)

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i!)}\, f(x)\, F^{i-1}(x)\, \{1-F(x)\}^{n-i}. \tag{6.1}$$

Using the binomial expansion, we can rewrite $f_{i:n}(x)$ as

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i!)}\, f(x) \sum_{j=0}^{n-i} (-1)^j \begin{Bmatrix} n-i \\ j \end{Bmatrix} F(x)^{i+j-1}. \tag{6.2}$$

Using (2.2) in (6.2) to compute $F(x)^{i+j-1}$, we obtain

$$\begin{aligned} f_{i:n}(x) &= \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{n-i}\sum_{k=0}^{j+i-1} \underbrace{(-1)^{j+k} \begin{Bmatrix} n-i \\ j \end{Bmatrix} \begin{Bmatrix} j+i-1 \\ k \end{Bmatrix}}_{t_{j,k}} \\ &\times \quad \frac{a\,b\,\beta\,\alpha^\beta\, x^{\beta\, b-1}}{(\alpha^\beta - x^\beta)^{b+1}}\ \mathrm{e}^{-a(1+k)\left[\frac{x^\beta}{\alpha^\beta - x^\beta}\right]^b}. \end{aligned}$$

The $r$th moment of $X_{i:n}$ can be obtained as

$$\mathrm{E}\,(X_{i:n}^r) = \sum_{j=0}^{n-i}\sum_{k=0}^{j+i-1} t_{j,k}\ A(\beta\, b-1, b+1, a+k;\alpha,\beta), \tag{6.3}$$

where

$$t_{j,k} = \frac{(-1)^{j+k}\, n!}{(i-1)!(n-i)!} \begin{Bmatrix} n-i \\ j \end{Bmatrix} \begin{Bmatrix} j+i-1 \\ k \end{Bmatrix}.$$

After some algebra, the Rényi entropy of $X_{i:n}$ becomes

$$\begin{aligned} I_{R,X_{i:n}}(\gamma) &= \frac{\gamma}{1-\gamma}\log\left[\frac{n!\,a\,b\,\beta\,\alpha^\beta}{(i-1)!(n-i)!}\right] \\ &+ \frac{1}{1-\gamma}\log\Big[\sum_{j,k=0}^{\infty}\sum_{r=0}^{k} t_{j,k,r}^*\ A(\gamma(\beta\, b-1), \gamma(b+1), a(\gamma+r);\alpha,\beta,b)\Big], \end{aligned}$$

where

$$t_{j,k,r}^* = (-1)^{j+k} \begin{Bmatrix} \gamma(n-1) \\ j \end{Bmatrix} \begin{Bmatrix} \gamma(i-1)+j \\ k \end{Bmatrix}.$$

## 7. Estimation

Here, we determine the maximum likelihood estimates (MLEs) of the model parameters of the new family from complete samples only. Let $x_1, \ldots, x_n$ be observed values from the WPF distribution with parameters in $\Theta = (a, b, \beta)$. Then, the total log-likelihood function for $\Theta$ is given by

$$\begin{aligned} \ell_n &= \ell_n(\Theta) = n \log\left[a\,b\,\beta\,\alpha^{\beta}\right] + (\beta\,b - 1)\sum_{i=1}^{n}\log(x_i) \\ &- (b+1)\sum_{i=1}^{n}\log\left(\alpha^{\beta} - x_i^{\beta}\right) - a\sum_{i=1}^{n}\left[\frac{x_i^{\beta}}{\alpha^{\beta} - x_i^{\beta}}\right]^{b}. \end{aligned} \tag{7.1}$$

The log-likelihood function can be maximized either directly by using the SAS (PROC NLMIXED) or the Ox (sub-routine MaxBFGS) program (see [20]), R-language [42] or by solving the nonlinear likelihood equations obtained by differentiating (7.1).

The $\alpha$ is known and we estimate it from the sample maxima. The components of the score function $U_n(\Theta) = (\partial\ell_n/\partial a, \partial\ell_n/\partial b, \partial\ell_n/\partial\beta)^{\top}$ are given by

$$\frac{\partial\ell_n}{\partial a} = \frac{n}{a} - \sum_{i=1}^{n}\left[\frac{x_i^{\beta}}{\alpha^{\beta} - x_i^{\beta}}\right]^{b}$$

$$\frac{\partial\ell_n}{\partial b} = \frac{n}{b} + \beta\sum_{i=1}^{n}\log(x_i) - \sum_{i=1}^{n}\log\left[\alpha^{\beta} - x_i^{\beta}\right] - a\sum_{i=1}^{n}\left[\frac{x_i^{\beta}}{\alpha^{\beta} - x_i^{\beta}}\right]^{b}\log\left[\frac{x_i^{\beta}}{\alpha^{\beta} - x_i^{\beta}}\right]$$

and

$$\begin{aligned} \frac{\partial\ell_n}{\partial\beta} &= \frac{n}{\beta} + n\log(\alpha) + b\sum_{i=1}^{n}\log(x_i) - (b+1)\sum_{i=1}^{n}\left[\frac{\alpha^{\beta}\log(\alpha) - x_i^{\beta}\log(x_i)}{\alpha^{\beta} - x_i^{\beta}}\right] \\ &- a\,b\alpha^{\beta}\sum_{i=1}^{n}\left[\frac{x_i^{b\beta}\log(\frac{x_i}{\alpha})}{(\alpha^{\beta} - x_i^{\beta})^{b+1}}\right]. \end{aligned}$$

Setting these equations to zero and solving them simultaneously yields the MLEs of the three parameters. For interval estimation of the model parameters, we require the $3 \times 3$ observed information matrix $J(\Theta) = \{U_{rs}\}$ (for $r, s = a, b, \beta$), whose elements are listed in Appendix A. Under standard regularity conditions, the multivariate normal $N_3(0, J(\widehat{\Theta})^{-1})$ distribution can be used to construct approximate confidence intervals for the model parameters. Here, $J(\widehat{\Theta})$ is the total observed information matrix evaluated at $\widehat{\Theta}$. Then, the $100(1-\gamma)\%$ confidence intervals for $a$, $b$ and $\beta$ are given by $\hat{a} \pm z_{\alpha^*/2} \times \sqrt{var(\hat{a})}$, $\hat{b} \pm z_{\alpha^*/2} \times \sqrt{var(\hat{b})}$ and $\hat{\beta} \pm z_{\alpha^*/2} \times \sqrt{var(\hat{\beta})}$, respectively, where the $var(\cdot)$'s denote the diagonal elements of $J(\widehat{\Theta})^{-1}$ corresponding to the model parameters, and $z_{\alpha^*/2}$ is the quantile $(1 - \alpha^*/2)$ of the standard normal distribution.

**7.1. Simulation study.** To evaluate the performance of the MLEs of the WPF parameters, a simulation study is conducted for a total of twelve parameter combinations and the process in each case is repeated 200 times. Two different sample sizes $n = 100$ and 300 are considered. The MLEs of the parameters and their standard errors are listed in Table 2. In this simulation study, we take $\alpha = 1$. The figures in Table 2 indicate that the MLEs perform well for estimating the model parameters. Further, as the sample size increases, the biases and standard errors of the estimates decrease.

Table 2: MLEs and standard standard errors for some parameter values

| Sample size | Actual values | | | Estimated values | | | Standard errors | | |
|---|---|---|---|---|---|---|---|---|---|
| $n$ | $a$ | $b$ | $\beta$ | $\tilde{a}$ | $\tilde{b}$ | $\tilde{\beta}$ | $\tilde{a}$ | $\tilde{b}$ | $\tilde{\beta}$ |
| 100 | 0.5 | 0.5 | 1.0 | 0.5124 | 0.5064 | 1.5287 | 0.0138 | 0.0057 | 0.0405 |
| | 0.5 | 1.0 | 1.0 | 0.5928 | 1.0106 | 1.1023 | 0.0361 | 0.0137 | 0.0474 |
| | 0.5 | 1.5 | 2.0 | 0.6161 | 1.4959 | 2.1283 | 0.0399 | 0.0147 | 0.0669 |
| | 1.0 | 1.5 | 2.0 | 1.5224 | 1.4887 | 2.3224 | 0.1386 | 0.0287 | 0.1187 |
| | 1.5 | 1.5 | 2.0 | 1.8829 | 1.5294 | 2.1325 | 0.1543 | 0.0310 | 0.0973 |
| | 2.0 | 1.0 | 1.0 | 2.1982 | 1.0293 | 1.0834 | 0.1271 | 0.0240 | 0.0511 |
| | 2.0 | 0.5 | 1.0 | 1.9921 | 0.5208 | 1.0109 | 0.0525 | 0.0103 | 0.0328 |
| | 2.0 | 0.5 | 2.0 | 1.9807 | 0.5220 | 2.0032 | 0.0509 | 0.0102 | 0.0627 |
| | 2.0 | 0.5 | 1.5 | 1.9977 | 0.5248 | 1.5256 | 0.0539 | 0.0107 | 0.0561 |
| | 2.0 | 0.5 | 0.5 | 1.9794 | 0.5145 | 0.5048 | 0.0475 | 0.0097 | 0.0152 |
| | 2.0 | 1.5 | 0.5 | 2.7821 | 1.5288 | 0.5672 | 0.2529 | 0.0380 | 0.0320 |
| | 2.0 | 2.0 | 0.5 | 2.8568 | 2.0116 | 0.5274 | 0.2984 | 0.0350 | 0.0226 |
| 300 | 0.5 | 0.5 | 1.0 | 0.4999 | 0.5038 | 1.5155 | 0.0046 | 0.0019 | 0.0139 |
| | 0.5 | 1.0 | 1.0 | 0.5301 | 1.0040 | 1.0341 | 0.0105 | 0.0041 | 0.0134 |
| | 0.5 | 1.5 | 2.0 | 0.6161 | 1.4959 | 2.1283 | 0.0230 | 0.0085 | 0.0386 |
| | 1.0 | 1.5 | 2.0 | 1.1401 | 1.5086 | 2.0540 | 0.0454 | 0.0108 | 0.0404 |
| | 1.5 | 1.5 | 2.0 | 1.6533 | 1.5198 | 2.0340 | 0.0565 | 0.0120 | 0.0363 |
| | 1.0 | 1.0 | 2.0 | 1.9977 | 1.0140 | 1.0006 | 0.0384 | 0.0076 | 0.0138 |
| | 2.0 | 0.5 | 1.0 | 1.9912 | 0.5088 | 1.0122 | 0.0178 | 0.0038 | 0.0118 |
| | 2.0 | 0.5 | 2.0 | 2.0012 | 0.5066 | 2.0139 | 0.0160 | 0.0033 | 0.0191 |
| | 2.0 | 0.5 | 1.5 | 2.0412 | 0.4921 | 1.5583 | 0.0153 | 0.0029 | 0.0137 |
| | 2.0 | 0.5 | 0.5 | 2.0110 | 0.5018 | 0.5062 | 0.0170 | 0.0033 | 0.0052 |
| | 2.0 | 1.5 | 0.5 | 2.1140 | 1.5159 | 0.5039 | 0.0722 | 0.0117 | 0.0086 |
| | 2.0 | 2.0 | 0.5 | 2.7353 | 2.0066 | 0.5251 | 0.1319 | 0.0184 | 0.0113 |

## 8. Bivariate extension

Here, we propose an extension of the WPF model using the results of Marshall and Olkin [33].

**8.1. Theorem.** *Let $X_1 \sim WPF(a_1, b, \alpha, \beta)$, $X_2 \sim WPF(a_2, b, \alpha, \beta))$ and $X_3 \sim WPF(a_1, b, \alpha, \beta)$ be independent random variables.*

*Let $X = \min\{X_1, X_3\}$ and $Y = \min\{X_2, X_3\}$. Then, the cdf of the bivariate random variable $(X, Y)$ is given by*

$$F_{X,Y}(x,y) = 1 - \mathrm{e}^{-a_1\left[\frac{x^\beta}{\alpha^\beta - x^\beta}\right]^b - a_2\left[\frac{y^\beta}{\alpha^\beta - y^\beta}\right]^b - a_3\left[\frac{z^\beta}{\alpha^\beta - z^\beta}\right]^b},$$

*where $z = \max\{x, y\}$.*

*The marginal cdf's are given by*

$$F_X(x) \;= 1 - \mathrm{e}^{-(a_1+a_3)\left[\frac{x^\beta}{\alpha^\beta - x^\beta}\right]^b}$$

*and*

$$F_Y(y) = 1 - \mathrm{e}^{-(a_2+a_3)\left[\frac{y^\beta}{\alpha^\beta - y^\beta}\right]^b}.$$

The pdf of $(X, Y)$ is given in the Corollary.

**8.2. Corollary.** *Let $X$ and $Y$ defined as in Theorem 8.1,*

$$f_{X,Y}(x,y) = \begin{cases} f_{\mathrm{WPF}}(x\,; a_1, b, \alpha, \beta)\, f_{\mathrm{WPF}}(y\,; a_2 + a_3, b, \alpha, \beta), & \textit{for } x < y; \\ \\ f_{\mathrm{WPF}}(x\,; a_1 + a_3, b, \alpha, \beta)\, f_{\mathrm{WPF}}(y\,; a_2, b, \alpha, \beta), & \textit{for } x > y; \\ \\ \dfrac{a_3}{a_1 \,+\, a_2 \,+\, a_3}\, f_{\mathrm{WPF}}(x\,; a_1 + a_2 + a_3, b, \alpha, \beta), & \textit{for } x = y. \end{cases}$$

*The marginal pdf's are given by*

$$f_X(x) = \frac{(a_1 + a_3)\, b\, \beta\, \alpha^\beta x^{\beta\, b-1}}{(\alpha^\beta - x^\beta)^{b+1}} \mathrm{e}^{-a\left[\frac{x^\beta}{\alpha^\beta - x^\beta}\right]^b}$$

*and*

$$f_Y(y) = \frac{(a_2 + a_3)\, b\, \beta\, \alpha^\beta y^{\beta\, b-1}}{(\alpha^\beta - y^\beta)^{b+1}} \mathrm{e}^{-a\left[\frac{y^\beta}{\alpha^\beta - y^\beta}\right]^b}.$$

## 9. Applications

In this section, we provide two application to real data in order to illustrate the importance of the WPF distribution. The MLEs of the parameters are determined for the WPF and four other models, and seven goodness-of-fit statistics are computed for checking the adequacy of the all five fitted models.

**9.1. Data set 1: Aarset data.** The first real data set refers to the failure times of 50 items put under a life test. This data set is well-known to exhibit bathtub behavior of the hrf. Aarset [1] first reported these data set which has been analyzed by many authors. The data are: 0.1, 0.2, 1.0, 1.0, 1.0, 1.0, 1.0, 2.0, 3.0, 6.0, 7.0, 11.0, 12.0, 18.0, 18.0, 18.0, 18.0, 18.0, 21.0, 32.0, 36.0, 40.0, 45.0, 45.0, 47.0, 50.0, 55.0, 60.0, 63.0, 63.0, 67.0, 67.0, 67.0, 67.0, 72.0, 75.0, 79.0, 82.0, 82.0, 83.0, 84.0, 84.0, 84.0, 85.0, 85.0, 85.0. 85.0. 85.0. 86.0. 86.0.

**9.2. Data set 2: Device failure times data.** The second real data set refers to 30 devices failure times given in Table 15.1 by Meeker and Escobar [35]. The data are: 275, 13, 147, 23, 181, 30, 65, 10, 300, 173, 106, 300, 300, 212, 300, 300, 300, 2, 261, 293, 88, 247, 28, 143, 300, 23, 300, 80, 245, 266.

We fit the WPF model and other competitive models to both data sets. The other fitted models are: the additive Weibull (AddW) [54], modified-Weibull (MW) [30], Sarhan-Zaindin modified Weibull (SZMW) [48] and beta-modified Weibull (BMW) [50]. Their associated densities are given by:

$$\text{AddW}: \; f_{AddW}(x;\alpha,\beta,\theta,\gamma) = \left(\alpha\,\theta\,x^{\theta-1} + \beta\,\gamma\,x^{\gamma-1}\right) \mathrm{e}^{-\alpha\,x^{\theta}-\beta\,x^{\gamma}}, \quad x>0, \quad \alpha,\beta,\theta,\gamma>0,$$

$$\text{MW}: \; f_{MW}(x;\beta,\gamma,\lambda) = \beta\,(\gamma+\lambda\,x)\;x^{\gamma-1}\,\mathrm{e}^{\lambda\,x}\,\mathrm{e}^{-\beta\;x^{\gamma},\mathrm{e}^{\lambda\,x}}, \; x>0, \; \beta,\gamma,\lambda>0,$$

$$\text{SZMW}: \; f_{SZMW}(x;\alpha,\beta,\gamma) = \left(\alpha+\beta\,x^{\gamma-1}\right)\,\mathrm{e}^{-\alpha\,x-\beta\,x^{\gamma}}, \; x>0, \quad \alpha,\beta,\gamma>0,$$

$$\text{BMW}: \; f_{BMW}(x;a,b,\alpha,\beta,\lambda) = \tfrac{1}{B(a,b)}\,\alpha\,(\beta+\lambda\,x)\;x^{\beta-1}\,\mathrm{e}^{\lambda\,x}\,\mathrm{e}^{-\alpha\,b\,x^{\beta}} \times \left(1-\mathrm{e}^{-\alpha\,x^{\beta}\mathrm{e}^{\lambda x}}\right)^{a-1}, \; x>0, \quad a,b,\alpha,\beta,\lambda>0.$$

The required computations are carried out using a script of the R-language [42], the AdequacyModel, written by Pedro Rafael Diniz Marinho, Cícero Rafael Barros Dias and Marcelo Bourguignon [32] which is freely available. In AdequacyModel package, there exists many maximization algorithms like NR (Newton-Raphson), BFGS (Broyden-Fletcher-Goldfarb-Shanno), BHHH(Berndt-Hall-Hall-Hausman), SANN (Simulated-Annealing), NM (Nelder-Mead) and Limited-Memory quasi-Newton code for Bound-constrained optimization (L-BFGS-B). But here, the MLEs are computed using LBFGS-B method.

The measures of goodness of fit including the log-likelihood function evaluated at the MLEs ($\hat{\ell}$), Akaike information criterion (AIC), consistent Akaike information criterion (CAIC), Hannan-Quinn information criterion (HQIC), Bayesian information criterion (BIC), Anderson-Darling ($A^*$) and Cramér–von Mises ($W^*$) to compare the fitted models. The statistics $W^*$ and $A^*$ are well-defined by Chen and Balakrishnan [17]. In general, the smaller the values of these statistics, the better the fit to the data.

Tables 3 and 5 list the MLEs and their corresponding standard errors (in parentheses) of the model parameters. The numerical values of the statistics $\hat{\ell}$, AIC, CAIC, BIC, HQIC, $W^*$ and $A^*$ are listed in Tables 4 and 6.

Table 3: MLEs and their standard errors (in parentheses) for Aarset data.

| Distribution | $a$ | $b$ | $\alpha$ | $\beta$ | $\theta$ | $\gamma$ | $\lambda$ |
|---|---|---|---|---|---|---|---|
| WPF | 0.7347 | 0.3367 | 86.0 | 1.4898 | - | - | - |
| | (0.2096) | (0.0567) | - | (0.4879) | - | - | - |
| AddW | - | - | 0.0020 | 0.0892 | 1.5164 | 0.3454 | - |
| | - | - | (0.0003) | (0.0424) | (0.0523) | (0.1125) | - |
| MW | - | - | - | 0.0624 | - | 0.3550 | 0.0233 |
| | - | - | - | (0.0266) | - | (0.1126) | (0.0048) |
| SZMW | - | - | 0.0186 | 0.0405 | - | 0.3735 | - |
| | - | - | (0.0038) | (0.0311) | - | (0.1886) | - |
| BMW | 0.2589 | 0.1525 | 0.0034 | 1.0819 | - | - | 0.0401 |
| | (0.0704) | (0.0834) | (0.0015) | (0.2928) | - | - | (0.0122) |

Table 4: The statistics $\hat{\ell}$, AIC, CAIC, BIC, HQIC, $A^*$ and $W^*$ for Aarset data.

| Distribution | $\hat{\ell}$ | AIC | CAIC | BIC | HQIC | $A^*$ | $W^*$ |
|---|---|---|---|---|---|---|---|
| WPF | 205.1732 | 416.3464 | 416.8681 | 422.0824 | 418.5307 | 0.380 | 0.046 |
| AddW | 234.2362 | 476.4725 | 477.3614 | 484.1206 | 479.3849 | 2.174 | 0.343 |
| MW | 227.1552 | 460.3105 | 460.8322 | 466.0465 | 462.4948 | 1.604 | 0.234 |
| SZMW | 239.4842 | 484.9684 | 485.4901 | 490.7045 | 487.1527 | 2.799 | 0.454 |
| BMW | 222.0914 | 454.1827 | 455.5464 | 463.7429 | 457.8233 | 1.276 | 0.169 |

Table 5: MLEs and their standard errors (in parentheses) for Aarset data.

| Distribution | $a$ | $b$ | $\alpha$ | $\beta$ | $\theta$ | $\gamma$ | $\lambda$ |
|---|---|---|---|---|---|---|---|
| WPF | 0.7723 | 0.24487 | 300.0 | 2.8736 | - | - | - |
| | (0.2519) | (0.0553) | - | (1.1351) | - | - | - |
| AddW | - | - | 3.4823E-03 | 1.0000E-10 | 1.0936 | 1.2045E-10 | - |
| | - | - | (1.3515E-03) | (1.1991E-06) | (7.6001E-02) | (9.2675E-11) | - |
| MW | - | - | - | 0.0313 | - | 0.3054 | 0.0081 |
| | - | - | - | (0.0240) | - | (0.1678) | (0.0020) |
| SZMW | - | - | 5.6560E-03 | 1.1789E-05 | - | 7.5972E-03 | - |
| | - | - | (1.0088E-03) | (1.1222E-05) | - | (3.0831E-06) | - |
| BMW | 0.3846 | 0.1832 | 0.0029 | 0.8382 | - | - | 0.0110 |
| | (0.1443) | (0.1305) | (0.0012) | (0.2770) | - | - | (0.0045) |

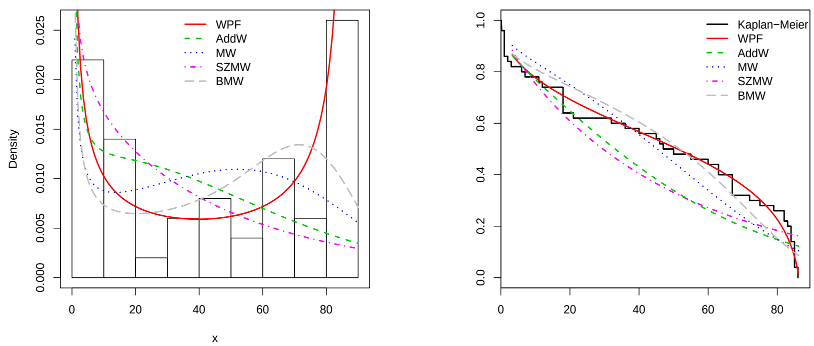
Table 6: The statistics $\hat{\ell}$, AIC, CAIC, BIC, HQIC, $A^*$ and $W^*$ for device failure times data.

| Distribution | $\hat{\ell}$ | AIC | CAIC | BIC | HQIC | $A^*$ | $W^*$ |
|---|---|---|---|---|---|---|---|
| WPF | 152.5768 | 311.1535 | 312.0766 | 315.3571 | 312.4983 | 0.750 | 0.082 |
| AddW | 184.7103 | 377.4206 | 379.0206 | 383.0254 | 379.2136 | 1.872 | 0.314 |
| MW | 178.3303 | 362.6606 | 363.5837 | 366.8642 | 364.0054 | 1.396 | 0.207 |
| SZMW | 185.2905 | 376.5810 | 377.5041 | 380.7846 | 377.9258 | 1.906 | 0.321 |
| BMW | 175.7578 | 361.5157 | 364.0157 | 368.5216 | 363.7569 | 1.262 | 0.182 |

In Tables 4 and 6, we compare the WPF model with the WPF, AddW, MW, SZMW and BMW models. We note that the WPF model gives the lowest values for the $\hat{\ell}$, AIC, CAIC, BIC, HQIC, $A^*$ and $W^*$ statistics for both data sets among the fitted models. So, the WPF model could be chosen as the best model. The histogram of the data sets, and plots the estimated densities and Kaplan-Meier are displayed in Figures 5 and 6. It is clear from Tables 4 and 6 and Figures 5 and 6 that the WPF model provides the best fits to the histogram of these two data sets.

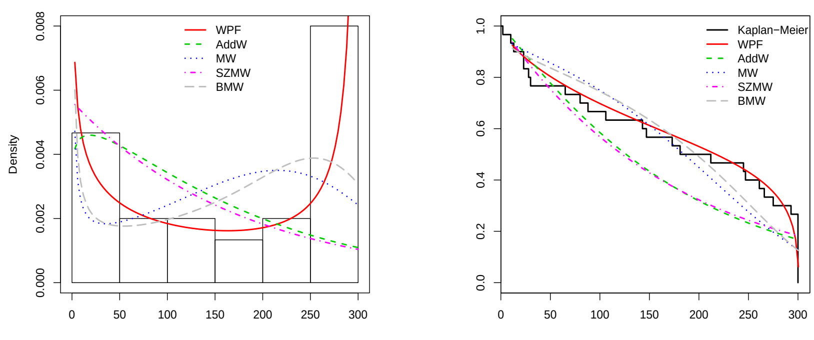
## 10. Concluding remarks

Many new lifetime distributions have been constructed in recent years with a view for better applications in various fields. They usually arise from an adequate transformation of a very-known model. In this paper, we propose a new lifetime model, the Weibull-power function (WPF) distribution, by applying the Weibull-G generator pioneered by Bourguignon *et al.* [15] to the classical power function distribution. We study some of its structural properties including an expansion for the density function and explicit expressions for the ordinary and incomplete moments, generating function, mean deviations, quantile function, entropies, reliability and order statistics. The maximum likelihood method is employed for estimating the model parameters and a simulation study is presented. The WPF model is fitted to two real data sets to illustrate the usefulness of the distribution. It provides consistently a better fit than other competing models. Finally, we hope that the proposed model will attract wider applications in reliability engineering, survival and lifetime data, mortality study and insurance, hydrology, social sciences, economics, among others.




(a) Estimated pdfs (b) Estimated sfs

**Figure 5.** Plots of the estimated pdfs and sfs for the WPF, AddW, MW, SZMW and BMW models for the data set 1.




(a) Estimated pdfs (b) Estimated sfs

**Figure 6.** Plots of the estimated pdfs and sfs for the WPF, AddW, MW, SZMW and BMW models for the data set 2.

## Appendix A

The elements of the $3\times 3$ observed information matrix $J(\Theta)=\{U_{rs}\}$ (for $r,s=a,b,\beta$) are given by

$$
\begin{aligned}
U_{aa} &= -\frac{n}{a^2},\\
U_{ab} &= -\sum_{i=1}^{n}\left[\frac{x_i^{\beta}}{\alpha^{\beta}-x_i^{\beta}}\right]^{b}\log\left[\frac{x_i^{\beta}}{\alpha^{\beta}-x_i^{\beta}}\right],\\
U_{a\beta} &= -b\,\alpha^{\beta}\sum_{i=1}^{n}\left[\frac{x_i^{b\,\beta}\log\,(x_i/\alpha)}{\left(\alpha^{\beta}-x_i^{\beta}\right)^{b+1}}\right],\\
U_{bb} &= -\frac{n}{b^2}-a\sum_{i=1}^{n}\left[\frac{x_i^{\beta}}{\alpha^{\beta}-x_i^{\beta}}\right]^{b}\left\{\log\left[\frac{x_i^{\beta}}{\alpha^{\beta}-x_i^{\beta}}\right]\right\}^{2},\\
U_{b\beta} &= \sum_{i=1}^{n}\log x_i-\sum_{i=1}^{n}\left[\frac{\alpha^{\beta}\,\log\alpha-x_i^{\beta}\log\,x_i}{\alpha^{\beta}-x_i^{\beta}}\right]\\
&\quad -a\,\alpha^{\beta}\sum_{i=1}^{n}\left[\frac{x_i^{b\,\beta}\log\,(x_i/\alpha)}{(\alpha^{\beta}-x_i^{\beta})^{b+1}}\right]\left[1+b\log(x_i^{\beta}/(\alpha^{\beta}-x_i^{\beta}))\right],\\
U_{\beta\beta} &= -\frac{n}{\beta^2}-(b+1)\sum_{i=1}^{n}\left[\frac{\left(\alpha^{\beta}-x_i^{\beta}\right)\left\{\alpha^{\beta}\,(\log\,\alpha)^2-x_i^{\beta}\,(\log\,x_i)^2\right\}}{(\alpha^{\beta}-x_i^{\beta})^2}\right.\\
&\quad \left.-(\alpha^{\beta}\,\log\,\alpha-x_i^{\beta}\,\log\,x_i)^2\right]\\
&\quad -a\,b\alpha^{\beta}\sum_{i=1}^{n}(\alpha^{\beta}-x_i^{\beta})^{b}\left[\frac{\left(\alpha^{\beta}-x_i^{\beta}\right)x_i^{b\beta}\;\{b\,\log\,x_i+\log\,\alpha\}\,\log\,(x_i/\alpha)}{(\alpha^{\beta}-x_i^{\beta})^{2(b+1)}}\right.\\
&\qquad \left.-\frac{(b+1)\,\alpha^{\beta}x_i^{b\beta}\;\log\,(x_i/\alpha)\left(\alpha^{\beta}\,\log\,\alpha-x_i^{\beta}\log\,x_i\right)}{(\alpha^{\beta}-x_i^{\beta})^{2(b+1)}}\right].
\end{aligned}
$$

## Acknowledgments

The authors would like to thank the Editor and the two referees for careful reading and the comments which greatly improved the paper.

## References

[1] Aarset, M.V. *How to identify bathtub hazard rate*, IEEE Transactions on Reliability **36**, 106–108, 1987.

[2] Ahsanullah, M. *A characterization of the power function distribution*, Communications in Statistics **2**, 259–262, 1973.

[3] Ahsanullah, M. *Estimation of the parameters of a power function distribution by record values*, Pakistan Journal of Statistics **5**, 189–194, 1989.

[4] Ahsanullah, M. *On lower generalized order statistics and a characterization of power function distribution*, Statistical Methods **7**, 16–28, 2005.

[5] Ahsanullah, M., Shakil, M. and Golam Kibria, B.M.G. *A characterization of the power function distribution based on lower records*, ProbStat Forum **6**, 68–72, 2013.

[6] Ali, M.M. and Woo, J. *Inference on reliability $P(Y<X)$ in a power function distribution*, Journal of Statistics and Management Science **8**, 681–686, 2005.

[7] Ali, M.M., Woo, J. and Nadarajah, S. *On the ratio of $X/(X+Y)$ for the power function distribution*, Pakistan Journal of Statistics **21**, 131–138, 2005.

[8] Ali, M.M., Woo, J. and Yoon, G.E. *The UMVUE of mean and right-tail probability in a power function distribution*, Estadistica **52**, 1–10, 2000.

[9] Asghar, Z., Shahzad, M.N. and Shehzad, F. *Parameter estimation of power function distribution with moments*, World Applied Science Journal **26**, 205–212, 2013.

[10] Athar, H. and Faizan, M. *Moments of lower generalized order statistics from power function distribution and its characterization*, International Journal of Statistical Sciences **11**, 125–134, 2011.

[11] Azedine, G. *Characterization of the power function distribution based on lower records*, Applied Mathematical Sciences **7**, 5259–5267, 2013.

[12] Balakrishnan, N. and Joshi, P.C. *Moments of order statistics from doubly truncated power function distribution*, Aligarh Journal of Statististics **1**, 98–105, 1981.

[13] Balakrishnan, N. and Nevzorov, V.B. *A Primer on Statistical Distributions* (Wiley, New York, 2003).

[14] Bhatt, M.B. *Characterization of power function distribution through expectation*, Open Journal of Statistics **3**, 441–443, 2013.

[15] Bourguignon, M., Silva, R.B. and Cordeiro, G. M. *The Weibull–G family of probability distributions*, Journal of Data Science **12**, 53–68, 2014.

[16] Chang, S-K. *Characterizations of the power function distribution by the independence of record values*, Journal of Chungcheong Mathematical Society **20**, 139–146, 2007.

[17] Chen, G. and Balakrishnan, N. *A general purpose approximate goodness-of-fit test*, Journal of Quality Technology **27**, 154–161, 1995.

[18] Cordeiro, G.M. and Brito, R.D.S. *The beta power distribution*, Brazalian Journal of Probability and Statistics **26**, 88–112, 2012.

[19] Dallas, A.C. *Characterization of Pareto and power function distribution*, Annals of Mathematical Statistics **28**, 491–497, 1976.

[20] Doornik, J.A. *Ox 5: An Object-Oriented Matrix Programming Language*, fifth edition (Timberlake Consultants, London, 2007).

[21] Forbes, C., Evans, M., Hastings, N. and Peacock, B. *Statistical Distributions*, fourth edition (Wiley, New York, 2011).

[22] Govindarajulu, Z. *Characterization of exponential and power function distribution*, Scandinavian Acturial Journal **3&4**, 132–136, 1966.

[23] Johnson, N.L., Kotz, S. and Balakrishnan, N. *Continuous Univariate Distributions*, Volume 1, Second edition (Wiley, New York, 1994).

[24] Johnson, N.L., Kotz, S. and Balakrishnan, N. *Continuous Univariate Distributions*, Volume 2, Second edition (Wiley, New York, 1995).

[25] Kabir, A.B.M.L. and Ahsanullah, M. *Estimation of location and scale parameters of a power function distribution by linear functions of order statistics*, Communications in Statistics **3**, 463–467, 1974.

[26] Kapadia, C.H. and Kromer, R.E. *Sample size required to estimate a parameter in the power function distribution*, Work of Statistical and Operations Research **29**, 88–96, 1978.

[27] Kenney, J. and Keeping, E. *Mathematics of Statistics*,Volume 1 (Princeton, 1962).

[28] Kifayat, T., Aslam, M. and Ali, S. *Bayesian inference for the parameters of the power function distribution*, Journal of Reliaboility and Statistical Studies **5**, 54–58, 2012.

[29] Kleiber, C. and Kotz, S. *Statistical Size Distributions in Economics and Actuarial Sciences* (Wiley, New York, 2003).

[30] Lai, C.D. and Xie, M. *A modified Weibull distribution*, IEEE Transactions on Reliability **52**, 33–37, 2003.

[31] Malik, H.J. *Exact moments of order statistics for a power function distribution*, Scandinavian Acturial Journal **1-2**, 64–69, 1967.

[32] Marinho, Cícero, P.R.D., Dias, R.B. and Bourguignon, M. AdequacyModel. http://cran.r-project.org/web/packages/AdequacyModel/AdequacyModel.pdf, 2014.

[33] Marshall, A.W. and Olkin, I. *A multivariate exponential distribution*, Journal Of the American Statistical Association **62**, 30–44, 1967.

[34] Mbah, A. and Ahsanullah, M. *Some characterization of power function distribution based on lower generalized order statistics*, Pakistan Journal of Statistics **23**, 139–146, 2007.

[35] Meeker, W.Q. and Escobar, L.A. *Statistical Methods for Reliability Data* (Wiley, New York, 1998).

[36] Meniconi, M. and Barry, D.M. *The power function distribution: A useful and simple distribution to assess electrical component reliability*, Microeolectronics Reliability **36**, 1207–1212, 1996.

[37] Moors, J.J.A. *A quantile alternative for kurtosis*, The Statistician **37**, 25–32, 1998.

[38] Moothathu, T.S.K. *Characterization of power function and Pareto distribution in terms of their Lorenz curve*, Research Review **3**, 118–123, 1984.

[39] Moothathu, T.S.K. *A characterization of power function distribution through a property of the Lorenz curve*, Sankhyã **B48**, 262–265, 1986.

[40] Proctor, J.W. *Estimation of two generalized curves covering the Pearson system*, Proceedings of The American Statistical Association (Computing Section), pp. 287–289, 1987.

[41] Rahman, H., Roy, M.K. and Baizid, A.R. *Bayes estimation under conjugate prior for the case of power function distribution*, American Journal of Mathematics and Statistics **2**, 44–48, 2012.

[42] R Development Core Team, R. A Language and Environment for Statistical Computing, R Foundation for Statistical Computing, Vienna, Austria, 2012.

[43] Rényi, A. *On measures of entropy and information*, in: Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability-I (University of California Press, Berkeley, 1961), 547–561.

[44] Rider, P.R. *Distribution of product and quotient of maximum values in samples from a power function population*, Journal of the American Statistical Association **59**, 877–880, 1964.

[45] Saleem, M., Aslam, M. and Economou, P. *On the Bayesian analysis of the mixture of power function distribution using the complete and the censored sample*, Journal of Applied Statistics **37**, 25–40, 2010.

[46] Saran, J. and Pandey, A. *Estimation of parameters of power function distribution and its characterization by the kth record values*, Statistica **64**, 523–536, 2004.

[47] Saran, J. and Singh, A. *Recurrence relations for marginal and joint moment generating functions of generalized order statistics from power function distribution*, Metron **61**, 27–33, 2003.

[48] Sarhan, A.M. and Zaindain, M. *Modified Weibull distribution*, Applied Sciences **11**, 123–136, 2009.

[49] Shannon, C.E. *A mathematical theory of communication*, Bell System Technical Journal **27**, 379–432, 1948.

[50] Silva, G.O., Ortega, E.M.M. and Cordeiro, G.M. *The beta modified Weibull distribution*, Lifetime Data Analysis **16**, 409–430, 2010.

[51] Sinha, S.K., Singh, P., Singh, D.C. and Singh, R. *Preliminary test estimator for scale parameter of the power function distribution*, Journal of Reliability and Statistical Studies **1**, 18–27, 2008.

[52] Sultan, K.S., Childs„A. and N. Balakrishnan, *Higher order moments of order statistics from the power function distribution and Edgeworth approximate inference*, in: Advances in Stochastic Simulation Methods, N. Balakrishnan, V. B. Melas S. Ermakov (eds.) (Springer, New York, 2000), 245–282.

[53] Tavangar, M. *Power function distribution characterized by dual generalized order statistics*, Journal of the Iranian Statistical Society **10**, 13–27, 2011.

[54] Xie, M., Tang, Y. and Goh, T.N. *A modified Weibull extension with bathtub-shaped failure rate function*, Reliability Engineering and System Safety **76** 279–285, 2002.

[55] Zaka, A. and Akhter, A.S. *Methods for estimating the parameters of power function distribution*, Pakistan Jouranl of Statistics and Operations Research **9**, 213–224, 2013.

[56] Zaka, A., Feroze, N. and Akhter, A.S. *A note on modified estimators for the parameters of power function distribution*, International Journal of Advanced Science and Technology **59**, 71–84, 2013.

[57] Zarrin, S., Saxena, S., Kamal, M. and Arif-ul-Islam. *Reliability computation and Bayesain analysis of system reliability of power function distribution*, International Journal of Advances in Engineering, Science and Technology **2**, 76–86, 2013.

[58] Zografos, K. and Balakrishnan, N. *On families of beta- and generalized gamma-generated distributions and associated inference.* Statistical Methodology **6**, 344–362, 2009.