Advances in the Theory of Nonlinear Analysis and its Applications **3** (2019) No. 1, 18–34.
https://doi.org/10.31197/atnaa.494662
Available online at www.atnaa.org
Research Article

Advances in the Theory of Nonlinear Analysis and its Applications

ISSN: 2587-2648

Peer-Reviewed Scientific Journal

Controllability for Impulsive Fractional Evolution Inclusions with State-Dependent Delay

Khalida Aissani[a], Mouffak Benchohra[b], Juan J. Nieto[c]

[a]*University of Bechar, PO Box 417, 08000, Bechar, Algeria.*

[b]*Laboratory of Mathematics, Djillali Liabes University of Sidi Bel-Abbès PO Box 89, 22000 Sidi Bel-Abbès, Algeria, Department of Mathematics, College of Science, King Saud University, P.O. Box 2455 Riyadh 11451, Saudi Arabia. .*

[c]*Departamento de Estatistica, Análise Matemática e Optimización, Instituto de Matemáticas, Universidade de Santiago de Compostela, 15782, Santiago de Compostela, Spain.*

Abstract

In this paper, sufficient conditions are provided for the controllability of impulsive fractional evolution inclusions with state-dependent delay in Banach spaces. We used a fixed-point theorem for condensing maps due to Bohnenblust–Karlin and the theory of semigroup for the achievement of the results. An Illustrative example is presented.

Keywords: Impulsive fractional evolution, α-resolvent family, solution operator, Caputo fractional derivative, mild solution, state-dependent delay, fixed point, Banach space.
2010 MSC: 26A33, 34A37, 93B05.

1. Introduction

Differential inclusions of fractional order have attracted great interest due to their applications in characterizing many problems in physics, biology, mechanics and so on; see, for instance [2, 3, 4, 46, 47]. The theory of impulsive differential equations is a new and important branch of differential equations, which has an extensive physical background, for instance, we refer to [6, 12, 14, 18, 28, 33, 37, 41].

Email addresses: aissani_k@yahoo.fr (Khalida Aissani), benchohra@yahoo.com (Mouffak Benchohra), juanjose.nieto.roig@usc.es (Juan J. Nieto)

Received December 10, 2018, Accepted: February 11, 2019, Online: February 14, 2019.

One of the basic qualitative behaviors of a dynamical system is controllability, it means that it is possible to steer a dynamical control system from an arbitrary initial state to an arbitrary final state using the set of admissible controls. As a result of its great application, the controllability of such systems all have received more and more attention, we refer the work for more details [7, 9, 11, 13, 15, 19, 31, 32, 40, 44]. Yan [45] established the controllability of fractional-order partial neutral functional integrodifferential inclusions with infinite delay. In [36], the authors provided some sufficient conditions ensuring the existence of mild solution of the problem

$$\begin{aligned} D_t^{\alpha} x(t) &= Ax(t) + f(t, x_{\rho(t,x_t)}, x(t)), && t \in J_k = (t_k, t_{k+1}], k = 0, 1, \ldots, m, \\ \Delta x(t_k) &= I_k(x(t_k^-)), && k = 1, 2, \ldots, m, \\ x(t) &= \phi(t), && t \in (-\infty, 0]. \end{aligned} \tag{1}$$

The controllability of fractional integro-differential equation of the form

$$\begin{aligned} D_t^q x(t) &= Ax(t) + Bu(t) + \int_0^t a(t,s) f(s, x_{\rho(s,x_s)}, x(s)) ds, \quad t \in J = [0, T], \\ x(t) &= \phi(t), \quad t \in (-\infty, 0], \end{aligned} \tag{2}$$

has been considered by Aissani and Benchohra in [8].

Motivated by the papers cited above, in this work, we consider the controllability for a class of impulsive fractional inclusions with state-dependent delay described by

$$\begin{aligned} D_{t_k}^{\alpha} x(t) &\in Ax(t) + F(t, x_{\rho(t,x_t)}, x(t)) + Bu(t), && t \in J_k = (t_k, t_{k+1}],\ k = 0, 1, \ldots, m, \\ \Delta x(t_k) &= I_k(x(t_k^-)), && k = 1, 2, \ldots, m, \\ x(t) &= \phi(t), && t \in (-\infty, 0], \end{aligned} \tag{3}$$

where $D_{t_k}^{\alpha}$ is the Caputo fractional derivative of order $0 < \alpha < 1$, $A : D(A) \subset E \to E$ is the infinitesimal generator of an α-resolvent family $(S_\alpha(t))_{t \geq 0}$, $F : J \times \mathcal{B} \times E \longrightarrow \mathcal{P}(E)$ is a multivalued map ($\mathcal{P}(E)$ is the family of all nonempty subsets of E) and $\rho : J \times \mathcal{B} \to (-\infty, T]$ are appropriated functions, $J = [0, T]$, $T > 0$, B is a bounded linear operator from E into E, the control $u \in L^2(J; E)$, the Banach space of admissible controls. Here, $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = T, I_k : E \to E, k = 1, 2, \ldots, m$, are given functions, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, $x(t_k^+) = \lim_{h \to 0} x(t_k + h)$ and $x(t_k^-) = \lim_{h \to 0} x(t_k - h)$ denote the right and the left limit of $x(t)$ at $t = t_k$, respectively. We denote by x_t the element of $\mathcal{B}$ defined by $x_t(\theta) = x(t + \theta), \theta \in (-\infty, 0]$. Here x_t represents the history up to the present time t of the state $x(\cdot)$. We assume that the histories x_t belongs to some abstract phase space $\mathcal{B}$, to be specified later, and $\phi \in \mathcal{B}$.

2. Preliminaries

In this section, we state some notations, definitions and preliminary facts about fractional calculus and the multivalued analysis.

Let $(E, \|\cdot\|)$ be a Banach space.
$C = C(J, E)$ be the Banach space of continuous functions from J into E with the norm

$$\|y\|_C = \sup \{ \ \|y(t)\| \ : \ t \in J \ \}.$$

By $AC(J, E)$ we denote the space of absolutely continuous function from J into E.
$AC^n(J, E) = \{y \in C^{n-1}(J, E) : y^{(n-1)} \in AC(J, E)\}$.

$L(E)$ be the Banach space of all linear and bounded operators on E.
$L^1(J, E)$ the space of $E-$valued Bochner integrable functions on J with the norm

$$\|y\|_{L^1} = \int_0^T \|y(t)\| dt.$$

Denote by $P_{cl}(X) = \{Y \in P(X) : Y \text{ closed}\}$, $P_b(X) = \{Y \in P(X) : Y \text{ bounded}\}$, $P_{cp}(X) = \{Y \in P(X) : Y \text{ compact}\}$, $P_{cp,c}(X) = \{Y \in P(X) : Y \text{ compact, convex}\}$,
$P_{cl,c}(E) = \{Y \in P(E) : Y \text{ closed, convex}\}$.

A multivalued map $G : X \to P(X)$ is convex (closed) valued if $G(X)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(B) = \cup_{x\in B} G(x)$ is bounded in X for all $B \in P_b(X)$ (i.e. $\sup_{x\in B}\{\sup\{\|y\| : y \in G(x)\}\} < \infty$).

G is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$ the set $G(x_0)$ is a nonempty, closed subset of X, and if for each open set U of X containing $G(x_0)$, there exists an open neighborhood V of x_0 such that $G(V) \subseteq U$.

G is said to be completely continuous if $G(B)$ is relatively compact for every $B \in P_b(X)$. If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e. $x_n \longrightarrow x_*,\ y_n \longrightarrow y_*,\ y_n \in G(x_n)$ imply $y_* \in G(x_*)$). For more details on multivalued maps see the books of Deimling [21], Djebali *et al.* [23], Górniewicz [24] and Hu and Papageorgiou [30] .

Definition 2.1. The multivalued map $F : J \times \mathcal{B} \times E \longrightarrow \mathcal{P}(E)$ is said to be Carathéodory if

(i) $t \longmapsto F(t, x, y)$ is measurable for each $(x, y) \in \mathcal{B} \times E$;

(ii) $(x, y) \longmapsto F(t, x, y)$ is upper semicontinuous for almost all $t \in J$.

Definition 2.2. Let $\alpha > 0$ and $f \in L^1(J, E)$. The Riemann-Liouville integral is defined by

$$I_0^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds.$$

For more details on the Riemann-Liouville fractional derivative, we refer the reader to [20].

Definition 2.3. [38]. The Caputo derivative of order α for a function $f \in AC^n(J, E)$ is defined by

$$D_0^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds = I_0^{n-\alpha} f^{(n)}(t), \quad t > 0,\ n-1 \le \alpha < n.$$

If $0 \le \alpha < 1$, then

$$D_0^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(s)}{(t-s)^{\alpha}} ds.$$

Obviously, the Caputo derivative of a constant is equal to zero.

In order to defined the mild solution of the problems (3) we recall the following definition.

Definition 2.4. A closed and linear operator A is said to be sectorial if there are constants $\omega \in \mathbb{R}, \theta \in [\frac{\pi}{2}, \pi], M > 0$, such that the following two conditions are satisfied:

1. $\sum_{(\theta,\omega)} := \{\lambda \in C : \lambda \neq \omega,\ |arg(\lambda - \omega)| < \theta\} \subset \rho(A)$ ($\rho(A)$ being the resolvent set of A).
2. $\|R(\lambda, A)\|_{L(E)} \le \frac{M}{|\lambda - \omega|}, \quad \lambda \in \sum_{(\theta,\omega)}$.

Sectorial operators are well studied in the literature. For details see [25].

Definition 2.5. [10]. If A is a closed linear operator with domain $D(A)$ defined on a Banach space E and $\alpha > 0$, then we say that A is the generator of an α-resolvent family if there exists $\omega \ge 0$ and a strongly continuous function $S_\alpha : \mathbb{R}_+ \to L(E)$ such that $\{\lambda^\alpha : Re(\lambda) > \omega\} \subset \rho(A)$ and

$$(\lambda^\alpha I - A)^{-1} x = \int_0^\infty e^{-\lambda t} S_\alpha(t) x dt, \quad Re\ \lambda > \omega,\ x \in E.$$

In this case, $S_\alpha(t)$ is called the α-resolvent family generated by A.

Definition 2.6. (see Definition 2.1 in [5]). If A is a closed linear operator with domain $D(A)$ defined on a Banach space E and $\alpha > 0$, then we say that A is the generator of a solution operator if there exist $\omega \geq 0$ and a strongly continuous function $S_\alpha : \mathbb{R}_+ \to L(E)$ such that $\{\lambda^\alpha : Re(\lambda) > \omega\} \subset \rho(A)$ and

$$\lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1}x = \int_0^\infty e^{-\lambda t} S_\alpha(t)x dt, \quad Re\ \lambda > \omega,\ x \in E,$$

in this case, $S_\alpha(t)$ is called the solution operator generated by A.

In this paper, we will employ an axiomatic definition for the phase space $\mathcal{B}$ which is similar to those introduced by Hale and Kato [26]. Specifically, $\mathcal{B}$ will be a linear space of functions mapping $(-\infty, 0]$ into E endowed with a seminorm $\|\cdot\|_{\mathcal{B}}$, and satisfies the following axioms:

(A1) If $x :\ (-\infty, T\,] \longrightarrow E$ is such that $x_0 \in \mathcal{B}$, then for every $t \in J$, $x_t \in \mathcal{B}$ and

$$\|x(t)\| \leq C\|x_t\|_{\mathcal{B}},$$

where $C > 0$ is a constant.

(A2) There exist a continuous function $C_1(t) > 0$ and a locally bounded function $C_2(t) \geq 0$ in $t \geq 0$ such that

$$\|x_t\|_{\mathcal{B}} \leq C_1(t) \sup_{s\in[0,t]} \|x(s)\| + C_2(t)\|x_0\|_{\mathcal{B}},$$

for $t \in [0, T]$ and x as in $(A1)$.

(A3) The space $\mathcal{B}$ is complete.

Example 2.7. The phase space $C_r \times L^p(g, X)$.

Let $r \geq 0, 1 \leq p < \infty$, and let $g : (-\infty, -r) \to \mathbb{R}$ be a nonnegative measurable function which satisfies the conditions $(g-5), (g-6)$ in the terminology of [29]. Briefly, this means that g is locally integrable and there exists a nonnegative, locally bounded function Λ on $(-\infty, 0]$, such that $g(\xi + \theta) \leq \Lambda(\xi)g(\theta)$, for all $\xi \leq 0$ and $\theta \in (-\infty, -r)\backslash N_\xi$, where $N_\xi \subseteq (-\infty, -r)$ is a set with Lebesgue measure zero.

The space $C_r \times L^p(g, X)$ consists of all classes of functions $\varphi : (-\infty, 0] \to X$, such that φ is continuous on $[-r, 0]$, Lebesgue-measurable, and $g\|\varphi\|^p$ on $(-\infty, -r)$. The seminorm in $\|.\|_{\mathcal{B}}$ is defined by

$$\|\varphi\|_{\mathcal{B}} = \sup_{\theta\in[-r,0]} \|\varphi(\theta)\| + \left(\int_{-\infty}^{-r} g(\theta)\|\varphi(\theta)\|^p d\theta\right)^{\frac{1}{p}}.$$

The space $\mathcal{B} = C_r \times L^p(g, X)$ satisfies axioms **(A1)**, **(A2)**, **(A3)**. Moreover, for $r = 0$ and $p = 2$, this space coincides with $C_0 \times L^2(g, X), H = 1, M(t) = \Lambda(-t)^{\frac{1}{2}}, K(t) = 1 + \left(\int_{-r}^0 g(\tau)d\tau\right)^{\frac{1}{2}}$, for $t \geq 0$ (see [29], Theorem 1.3.8 for details).

Let $S_{F,x}$ be a set defined by

$$S_{F,x} = \{v \in L^1(J, E) : v(t) \in F(t, x_{\rho(t,x_t)}, x(t)) \text{ a.e. } t \in J\}.$$

Lemma 2.8. *[34]. Let $F : J \times \mathcal{B} \times E \longrightarrow P_{cp,c}(E)$ be an L^1-Carathéodory multivalued map and let Ψ be a linear continuous mapping from $L^1(J, E)$ to $C(J, E)$, then the operator*

$$\Psi \circ S_F : C(J, E) \longrightarrow P_{cp,c}(C(J, E)),$$
$$x \quad \longmapsto (\Psi \circ S_F)(x) := \Psi(S_{F,x})$$

is a closed graph operator in $C(J, E) \times C(J, E)$.

The next result is known as the Bohnenblust–Karlin's fixed point theorem.

Lemma 2.9. *(Bohnenblust–Karlin [17]). Let X be a Banach space and $D \in P_{cl,c}(X)$. Suppose that the operator $G : D \to P_{cl,c}(D)$ is upper semicontinuous and the set $G(D)$ is relatively compact in X. Then G has a fixed point in D.*

3. Main Result

In this section, we prove our main result. We need the following lemma ([42]).

Lemma 3.1. *Consider the Cauchy problem*

$$\begin{aligned} D_t^{\alpha}x(t) &= Ax(t) + F(t) + Bu(t), \qquad 0 < \alpha < 1, \\ x(0) &= x_0, \end{aligned} \tag{4}$$

where F is a function satisfying the uniform Hölder condition with exponent $\beta \in (0,1]$ and A is a sectorial operator, then the Cauchy problem (4) has a unique mild solution which is given by

$$x(t) = T_{\alpha}(t)x_0 + \int_0^t S_{\alpha}(t-s)F(s)ds + \int_0^t S_{\alpha}(t-s)Bu(s),$$

where

$$T_{\alpha}(t) = \frac{1}{2\pi i}\int_{\hat{B}_r} e^{\lambda t}\frac{\lambda^{\alpha-1}}{\lambda^{\alpha} - A}d\lambda,$$

$$S_{\alpha}(t) = \frac{1}{2\pi i}\int_{\hat{B}_r} e^{\lambda t}\frac{1}{\lambda^{\alpha} - A}d\lambda,$$

$\hat{B}_r$ denotes the Bromwich path, $S_{\alpha}(t)$ is called the α-resolvent family and $T_{\alpha}(t)$ is the solution operator, generated by A.

Theorem 3.2. *[42]. If $\alpha \in (0,1)$ and $A \in \mathbb{A}^{\alpha}(\theta_0, \omega_0)$, then for any $x \in E$ and $t > 0$, we have*

$$\|T_{\alpha}(t)\|_{L(E)} \leq Me^{\omega t} \text{ and } \|S_{\alpha}(t)\|_{L(E)} \leq Ce^{\omega t}(1 + t^{\alpha-1}),\ t > 0,\ \omega > \omega_0.$$

Let

$$\widetilde{M}_T = \sup_{0 \leq t \leq T}\|T_{\alpha}(t)\|_{L(E)}, \qquad \widetilde{M}_s = \sup_{0 \leq t \leq T} Ce^{\omega t}(1 + t^{\alpha-1}),$$

so we have

$$\|T_{\alpha}(t)\|_{L(E)} \leq \widetilde{M}_T,\ \|S_{\alpha}(t)\|_{L(E)} \leq t^{\alpha-1}\widetilde{M}_s.$$

Let us consider the set

$$\begin{aligned} \mathcal{B}_1 = \Big\{ & x : (-\infty, T] \to E \text{ such that } x|_{J_k} \in C(J_k, E) \text{ and there exist} \\ & x(t_k^+) \text{ and } x(t_k^-) \text{ with } x(t_k) = x(t_k^-),\ x_0 = \phi, k = 1, 2, \ldots, m\Big\}, \end{aligned}$$

endowed with the seminorm

$$\|x\|_{\mathcal{B}_1} = \sup\{|x(s)| : s \in [0,T]\} + \|\phi\|_{\mathcal{B}},\ x \in \mathcal{B}_1,$$

where $x|_{J_k}$ is the restriction of x to $J_k = (t_k, t_{k+1}], k = 1, 2, \ldots, m$.
From Lemma 3.1, we define the mild solution of system (3) as follows:

Definition 3.3. A function $x : (-\infty, T] \to E$ is called a mild solution of (3) if the restriction of $x(\cdot)$ to the interval $J_k, (k = 0, 1, \ldots, m)$ is continuous and there exists $v(\cdot) \in L^1(J_k, E)$, such that $v(t) \in$

$F(t, x_{\rho(t,x_t)}, x(t))$ *a.e* $t \in [0, T]$, and x satisfies the following integral equation:

$$x(t) = \begin{cases} \phi(t), & t \in (-\infty, 0]; \\ \int_0^t S_\alpha(t-s)v(s)ds + \int_0^t S_\alpha(t-s)Bu(s)ds, & t \in [0, t_1]; \\ T_\alpha(t-t_1)(x(t_1^-) + I_1(x(t_1^-))) + \int_{t_1}^t S_\alpha(t-s)v(s)ds & \\ + \int_{t_1}^t S_\alpha(t-s)Bu(s)ds, & t \in (t_1, t_2]; \\ \vdots & \\ T_\alpha(t-t_m)(x(t_m^-) + I_m(x(t_m^-))) + \int_{t_m}^t S_\alpha(t-s)v(s)ds & \\ + \int_{t_m}^t S_\alpha(t-s)Bu(s)ds, & t \in (t_m, T]. \end{cases} \tag{5}$$

Definition 3.4. The problem (3) is said to be controllable on the interval J if for every initial function $\phi \in \mathcal{B}$ and $x_1 \in E$ there exists a control $u \in L^2(J, E)$ such that the mild solution $x(\cdot)$ of (3) satisfies $x(T) = x_1$.

Set

$$R(\rho^-) = \{\rho(s, \varphi) : (s, \varphi) \in J \times \mathcal{B}, \rho(s, \varphi) \leq 0\}.$$

We always assume that $\rho : J \times \mathcal{B} \to (-\infty, T]$ is continuous. Additionally, we introduce following hypothesis:

(H_φ) The function $t \to \varphi_t$ is continuous from $\mathcal{R}(\rho^-)$ into $\mathcal{B}$ and there exists a continuous and bounded function $L^\phi : \mathcal{R}(\rho^-) \to (0, \infty)$ such that

$$\|\phi_t\|_{\mathcal{B}} \leq L^\phi(t)\|\phi\|_{\mathcal{B}} \quad \text{for every } t \in \mathcal{R}(\rho^-).$$

Remark 3.5. The condition (H_φ), is frequently verified by continuous and bounded functions. For more details see, e.g., [29].

Remark 3.6. In the rest of this section, C_1^* and C_2^* are the constants

$$C_1^* = \sup_{s \in J} C_1(s) \text{ and } C_2^* = \sup_{s \in J} C_2(s).$$

Lemma 3.7. *[27] If $x : (-\infty, T] \to X$ is a function such that $x_0 = \phi$, then*

$$\|x_s\|_{\mathcal{B}} \leq (C_2^* + L^\phi)\|\phi\|_{\mathcal{B}} + C_1^* \sup\{|y(\theta)|; \theta \in [0, max\{0, s\}]\}, \; s \in \mathcal{R}(\rho^-) \cup J,$$

where $L^\phi = \sup_{t \in \mathcal{R}(\rho^-)} L^\phi(t)$.

Let us list the following assumptions.

(H1) The resolvent family $S_\alpha(t)$ is compact for $t > 0$.

(H2) The multivalued map $F : J \times \mathcal{B} \times E \longrightarrow P_{cp,cv}(E)$ is Carathéodory.

(H3) There exist a function $\mu \in L^1(J, \mathbb{R}^+)$ and a continuous nondecreasing function $\psi : \mathbb{R}^+ \to (0, +\infty)$ such that

$$\|f(t, v, w)\| \leq \mu(t)\psi(\|v\|_{\mathcal{B}} + \|w\|), \quad (t, v, w) \in J \times \mathcal{B} \times E.$$

(H4) $I_k : E \to E$ is continuous, and there exists $\Omega > 0$ such that

$$\Omega = \max_{1 \leq k \leq m} \{\|I_k(x)\|_E, \; x \in D_r\}.$$

(H5) The linear operator $W : L^2(J,E) \to E$ defined by

$$Wu = \int_0^T S_\alpha(T-s)Bu(s)ds,$$

has a pseudo inverse operator $\tilde{W}^{-1}$, which takes values in $L^2(J,E)/\ker W$ and there exist two positive constants M_1 and M_2 such that

$$\|B\|_{L(E)} \le M_1, \ \|\tilde{W}^{-1}\|_{L(E)} \le M_2. \tag{6}$$

Remark 3.8. The question of the existence of the operator $\tilde{W}^{-1}$ and of its inverse is discussed in the paper by Quinn and Carmichael (see [39]).

Theorem 3.9. *Assume that* $(H_\varphi), (H1)-(H5)$ *hold. Then the IVP (3) is controllable on* $(-\infty, T]$.

Proof. We transform the problem (3) into a fixed-point problem. Consider the multivalued operator $N : \mathcal{B}_1 \longrightarrow \mathcal{P}(\mathcal{B}_1)$ defined by $N(h) = \{h \in \mathcal{B}_1\}$ with

$$h(t) = \begin{cases} \phi(t), & t \in (-\infty, 0]; \\ \displaystyle\int_0^t S_\alpha(t-s)v(s)ds + \int_0^t S_\alpha(t-s)Bu(s)ds, & t \in [0, t_1]; \\ T_\alpha(t-t_1)(x(t_1^-) + I_1(x(t_1^-))) + \displaystyle\int_{t_1}^t S_\alpha(t-s)v(s)ds & \\ + \displaystyle\int_{t_1}^t S_\alpha(t-s)Bu(s)ds, & t \in (t_1, t_2]; \\ \vdots, & \\ T_\alpha(t-t_m)(x(t_m^-) + I_m(x(t_m^-))) + \displaystyle\int_{t_m}^t S_\alpha(t-s)v(s)ds & \\ + \displaystyle\int_{t_m}^t S_\alpha(t-s)Bu(s)ds, & t \in (t_m, T]. \end{cases}$$

Using hypothesis $(H5)$ for an arbitrary function $x(\cdot)$ define the control

$$u(t) = \begin{cases} \tilde{W}^{-1}\Big[x_1 - \displaystyle\int_0^T S_\alpha(T-s)v(s)ds\Big](t), & t \in [0, t_1]; \\ \tilde{W}^{-1}\Big[x_1 - T_\alpha(T-t_1)(x(t_1^-) + I_1(x(t_1^-))) & \\ \quad - \displaystyle\int_{t_1}^T S_\alpha(T-s)v(s)ds\Big](t), & t \in (t_1, t_2]; \\ \vdots, & \\ \tilde{W}^{-1}\Big[x_1 - T_\alpha(T-t_m)(x(t_m^-) + I_m(x(t_m^-))) & \\ \quad - \displaystyle\int_{t_m}^T S_\alpha(T-s)v(s)ds\Big](t), & t \in (t_m, T]. \end{cases}$$

It is clear that the fixed points of the operator N are mild solutions of the problem (3).
Let us define $y(\cdot) : (-\infty, T] \longrightarrow E$ as

$$y(t) = \begin{cases} \phi(t), & t \in (-\infty, 0]; \\ 0, & t \in J. \end{cases}$$

Then $y_0 = \phi$. For each $z \in C(J,E)$ with $z(0) = 0$, we denote by $\overline{z}$ the function defined by

$$\overline{z}(t) = \begin{cases} 0, & t \in (-\infty, 0]; \\ z(t), & t \in J. \end{cases}$$

Let $x_t = y_t + \overline{z}_t, t \in (-\infty, T]$. It is easy to see that $x(\cdot)$ satisfies (5) if and only if $z_0 = 0$ and for $t \in J$, we have

$$z(t) = \begin{cases} \int_0^t S_\alpha(t-s)v(s)ds + \int_0^t S_\alpha(t-s)Bu(s)ds, & t \in [0,t_1]; \\ T_\alpha(t-t_1)\left[y(t_1^-) + \overline{z}(t_1^-) + I_1(y(t_1^-) + \overline{z}(t_1^-))\right] \\ + \int_{t_1}^t S_\alpha(t-s)v(s)ds + \int_{t_1}^t S_\alpha(t-s)Bu(s)ds, & t \in (t_1,t_2]; \\ \vdots, \\ T_\alpha(t-t_m)\left[y(t_m^-) + \overline{z}(t_m^-) + I_m(y(t_m^-) + \overline{z}(t_m^-))\right] \\ + \int_{t_m}^t S_\alpha(t-s)v(s)ds + \int_{t_m}^t S_\alpha(t-s)Bu(s)ds, & t \in (t_m, T], \end{cases}$$

where $v(s) \in S_{F, y_{\rho(s,y_s+\overline{z}_s)} + \overline{z}_{\rho(s,y_s+\overline{z}_s)}}$.
Let

$$\mathcal{B}_2 = \{z \in \mathcal{B}_1 \quad \text{such that} \quad z_0 = 0\}.$$

For any $z \in \mathcal{B}_2$, we have

$$\begin{aligned} \|z\|_{\mathcal{B}_2} &= \sup_{t\in J} \|z(t)\| + \|z_0\|_{\mathcal{B}} \\ &= \sup_{t\in J} \|z(t)\|. \end{aligned}$$

Thus $(\mathcal{B}_2, \|\cdot\|_{\mathcal{B}_2})$ is a Banach space. We define the operator $P : \mathcal{B}_2 \longrightarrow \mathcal{P}(\mathcal{B}_2)$ by : $P(z) = \{h \in \mathcal{B}_2\}$ with

$$h(t) = \begin{cases} \int_0^t S_\alpha(t-s)v(s)ds + \int_0^t S_\alpha(t-s)Bu(s)ds, & t \in [0,t_1]; \\ T_\alpha(t-t_1)\left[y(t_1^-) + \overline{z}(t_1^-) + I_1(y(t_1^-) + \overline{z}(t_1^-))\right] \\ + \int_{t_1}^t S_\alpha(t-s)v(s)ds + \int_{t_1}^t S_\alpha(t-s)Bu(s)ds, & t \in (t_1,t_2]; \\ \vdots, \\ T_\alpha(t-t_m)\left[y(t_m^-) + \overline{z}(t_m^-) + I_m(y(t_m^-) + \overline{z}(t_m^-))\right] \\ + \int_{t_m}^t S_\alpha(t-s)v(s)ds + \int_{t_m}^t S_\alpha(t-s)Bu(s)ds, & t \in (t_m, T], \end{cases}$$

where $v(s) \in S_{F, y_{\rho(s,y_s+\overline{z}_s)} + \overline{z}_{\rho(s,y_s+\overline{z}_s)}}$.
It is clear that the operator N has a fixed point if and only if P has a fixed point. So let us prove that P has a fixed point. We shall show that the operators P satisfy all conditions of Lemma 2.9. For better readability, we break the proof into a sequence of steps.
Choose

$$\begin{aligned} r &> \widetilde{M}_T(r+\Omega)\left(1 + \widetilde{M}_S M_1 M_2 \frac{T^\alpha}{\alpha}\right) + \widetilde{M}_S M_1 M_2 \frac{T^\alpha}{\alpha}\|x_1\| \\ &+ \left(1 + \widetilde{M}_S M_1 M_2 \frac{T^\alpha}{\alpha}\right) \widetilde{M}_S \frac{T^\alpha}{\alpha} \psi((C_2^* + L^\phi)\|\phi\|_{\mathcal{B}} + (C_1^*+1)r)\|\mu\|_{L^1}, \end{aligned}$$

and consider the set

$$D_r = \{z \in \mathcal{B}_2 : z(0) = 0, \|z\|_{\mathcal{B}_2} \le r\}.$$

It is clear that D_r is a closed, convex, bounded set in $\mathcal{B}_2$.
Step 1: P is convex for each $z \in \mathcal{B}_2$.

Indeed, if h_1 and h_2 belong to P, then there exist $v_1, v_2 \in S_{F,y_{\rho(s,y_s+\overline{z}_s)}+\overline{z}_{\rho(s,y_s+\overline{z}_s)}}$ such that, for $t \in J$ and $i = 1, 2$, we have

$$h_i(t) = \begin{cases} \int_0^t S_\alpha(t-s)v_i(s)ds \\ + \int_0^t S_\alpha(t-s)B\tilde{W}^{-1}\Big[x_1 - \int_0^T S_\alpha(T-\tau)v_i(\tau)d\tau\Big]ds, & t \in [0, t_1]; \\ T_\alpha(t-t_1)\left[y(t_1^-) + \overline{z}(t_1^-) + I_1(y(t_1^-) + \overline{z}(t_1^-))\right] + \int_{t_1}^t S_\alpha(t-s)v_i(s)ds \\ + \int_{t_1}^t S_\alpha(t-s)B\tilde{W}^{-1}\Big[x_1 - T_\alpha(T-t_1)[y(t_1^-) + \overline{z}(t_1^-) \\ + I_1(y(t_1^-) + \overline{z}(t_1^-))] - \int_{t_1}^T S_\alpha(T-\tau)v_i(\tau)d\tau\Big]ds, & t \in (t_1, t_2]; \\ \vdots, \\ T_\alpha(t-t_m)\left[y(t_m^-) + \overline{z}(t_m^-) + I_m(y(t_m^-) + \overline{z}(t_m^-))\right] + \int_{t_m}^t S_\alpha(t-s)v_i(s)ds \\ + \int_{t_m}^t S_\alpha(t-s)B\tilde{W}^{-1}\Big[x_1 - T_\alpha(T-t_m)[y(t_m^-) + \overline{z}(t_m^-) \\ + I_m(y(t_m^-) + \overline{z}(t_m^-))] - \int_{t_m}^T S_\alpha(T-\tau)v_i(\tau)d\tau\Big]ds, & t \in (t_m, T]. \end{cases}$$

Let $d \in [0, 1]$. Then for each $t \in [0, t_1]$, we get

$$\begin{aligned} dh_1(t) + (1-d)h_2(t) &= \int_0^t S_\alpha(t-s)\left[dv_1(s) + (1-d)v_2(s)\right]ds + \int_0^t S_\alpha(t-s)B\tilde{W}^{-1} \\ &\times \Big[x_1 - \int_0^T S_\alpha(T-\tau)\left(dv_1(\tau) + (1-d)v_2(\tau)\right)d\tau\Big]ds. \end{aligned}$$

Similarly, for any $t \in (t_i, t_{i+1}], i = 1, \dots, m$, we have

$$\begin{aligned} dh_1(t) + (1-d)h_2(t) &= \int_{t_i}^t S_\alpha(t-s)\left[dv_1(s) + (1-d)v_2(s)\right]ds \\ &+ T_\alpha(t-t_i)\left[y(t_i^-) + \overline{z}(t_i^-) + I_i(y(t_i^-) + \overline{z}(t_i^-))\right] \\ &+ \int_{t_i}^t S_\alpha(t-s)B\tilde{W}^{-1}\Big[x_1 - T_\alpha(T-t_i)[y(t_i^-) + \overline{z}(t_i^-) \\ &+ I_i(y(t_i^-) + \overline{z}(t_i^-))] - \int_{t_i}^T S_\alpha(T-\tau)\left(dv_1(\tau) + (1-d)v_2(\tau)\right)d\tau\Big]ds. \end{aligned}$$

Since $S_{F,y_{\rho(s,y_s+\overline{z}_s)}+\overline{z}_{\rho(s,y_s+\overline{z}_s)}}$ is convex (because F has convex values), we get

$$dh_1 + (1-d)h_2 \in P(z).$$

Step 2: $P(D_r) \subset D_r$. Let $h \in P(z)$ and $z \in D_r$, for $t \in [0, t_1]$, we have

$$\begin{aligned}
\|h(t)\| &\leq \int_0^t \|S_\alpha(t-s)\|_{L(E)}\|v(s)\|ds + \int_0^t \|S_\alpha(t-s)\|_{L(E)}\|Bu(s)\|ds \\
&\leq \widetilde{M}_S \int_0^t (t-s)^{\alpha-1}\mu(s)\psi(\|y_{\rho(s,y_s+\overline{z}_s)} + \overline{z}_{\rho(s,y_s+\overline{z}_s)}\| + \|y(s)+\overline{z}(s)\|)ds \\
&+ \widetilde{M}_S M_1 M_2 \int_0^t (t-s)^{\alpha-1}\left[\|x_1\| + \widetilde{M}_S \int_0^T (T-\tau)^{\alpha-1}\|v(\tau)\|d\tau\right] ds \\
&\leq \widetilde{M}_S \int_0^t (t-s)^{\alpha-1}\mu(s)\psi(\|y_{\rho(s,y_s+\overline{z}_s)} + \overline{z}_{\rho(s,y_s+\overline{z}_s)}\| + \|y(s)+\overline{z}(s)\|)ds \\
&+ \widetilde{M}_S M_1 M_2 \int_0^t (t-s)^{\alpha-1}\Big[\|x_1\| \\
&+ \widetilde{M}_S \int_0^T (T-\tau)^{\alpha-1}\mu(\tau)\psi(\|y_{\rho(\tau,y_\tau+\overline{z}_\tau)} + \overline{z}_{\rho(\tau,y_\tau+\overline{z}_\tau)}\| + \|y(\tau)+\overline{z}(\tau)\|)|d\tau\Big] ds \\
&\leq \widetilde{M}_S \frac{T^\alpha}{\alpha}\psi((C_2^* + L^\phi)\|\phi\|_{\mathcal{B}} + (C_1^*+1)r)\int_0^t \mu(s)ds + \widetilde{M}_S M_1 M_2 \frac{T^\alpha}{\alpha}\|x_1\| \\
&+ \widetilde{M}_S^2 M_1 M_2 \frac{T^{2\alpha}}{\alpha^2}\psi((C_2^* + L^\phi)\|\phi\|_{\mathcal{B}} + (C_1^*+1)r)\int_0^t \mu(s)ds \\
&\leq \widetilde{M}_S M_1 M_2 \frac{T^\alpha}{\alpha}\|x_1\| + \left(1 + \widetilde{M}_S M_1 M_2 \frac{T^\alpha}{\alpha}\right)\widetilde{M}_S \frac{T^\alpha}{\alpha} \\
&\times \psi((C_2^* + L^\phi)\|\phi\|_{\mathcal{B}} + (C_1^*+1)r)\|\mu\|_{L^1}.
\end{aligned}$$

Moreover, when $t \in (t_i, t_{i+1}], i = 1, \ldots, m$, we have the estimate

$$\begin{aligned}
\|h(t)\| &\leq \|T_\alpha(t-t_i)\left[z(t_i^-) + I_i(z(t_i^-))\right]\|_E + \int_{t_i}^t \|S_\alpha(t-s)\|_{L(E)}\|v(s)\|ds \\
&+ \int_{t_i}^t \|S_\alpha(t-s)\|_{L(E)}\|B\tilde{W}^{-1}\Big[x_1 - T_\alpha(T-t_i)[z(t_i^-) + I_i(z(t_i^-))] \\
&- \int_{t_i}^T S_\alpha(T-\tau)v(\tau)d\tau\Big]\|ds \\
&\leq \widetilde{M}_T(r+\Omega) + \widetilde{M}_S \int_0^t (t-s)^{\alpha-1}\mu(s)\psi(\|y_{\rho(s,y_s+\overline{z}_s)} + \overline{z}_{\rho(s,y_s+\overline{z}_s)}\| + \|y(s)+\overline{z}(s)\|)ds \\
&+ \widetilde{M}_S M_1 M_2 \int_0^t (t-s)^{\alpha-1}\Big[\|x_1\| + \widetilde{M}_T(r+\Omega) + \widetilde{M}_S \int_0^T (T-\tau)^{\alpha-1}\|v(\tau)\|d\tau\Big] ds \\
&\leq \widetilde{M}_T(r+\Omega)\left(1 + \widetilde{M}_S M_1 M_2 \frac{T^\alpha}{\alpha}\right) + \widetilde{M}_S M_1 M_2 \frac{T^\alpha}{\alpha}\|x_1\| \\
&+ \left(1 + \widetilde{M}_S M_1 M_2 \frac{T^\alpha}{\alpha}\right)\widetilde{M}_S \frac{T^\alpha}{\alpha}\psi((C_2^* + L^\phi)\|\phi\|_{\mathcal{B}} + (C_1^*+1)r)\|\mu\|_{L^1} < r.
\end{aligned}$$

Step 3: P maps bounded sets of D_r into equicontinuous sets of D_r.
Let $\tau_1, \tau_2 \in [0, t_1]$, with $\tau_1 < \tau_2$, we have

$$\|h(\tau_2) - h(\tau_1)\| \leq Q_1 + Q_2,$$

where

$$\begin{aligned}
Q_1 &= \int_{\tau_1}^{\tau_2} \|S_\alpha(\tau_2 - s)\left(v(s) + Bu(s)\right)\|ds \\
Q_2 &= \int_0^{\tau_1} \|\left(S_\alpha(\tau_2 - s) - S_\alpha(\tau_1 - s)\right)\left(v(s) + Bu(s)\right)\|ds.
\end{aligned}$$

Actually, Q_1 and Q_2 tend to 0 as $\tau_1 \to \tau_2$ independently of $z \in D_r$. Indeed, in view of (H3) and (6), we have

$$\begin{aligned}
Q_1 &= \int_{\tau_1}^{\tau_2} \|S_\alpha(\tau_2 - s)\,(v(s) + Bu(s))\,\| ds \\
&\le \int_{\tau_1}^{\tau_2} \|S_\alpha(\tau_2 - s)\|_{L(E)} \|v(s)\| ds + \int_{\tau_1}^{\tau_2} \|S_\alpha(\tau_2 - s)\|_{L(E)} \|Bu(s)\| ds \\
&\le \frac{\widetilde{M}_s(\tau_2 - \tau_1)^\alpha}{\alpha} \psi((C_2^* + L^\phi)\|\phi\|_{\mathcal{B}} + (C_1^* + 1)r)\|\mu\|_{L^1} \\
&+ \frac{M_1 M_2 \widetilde{M}_s(\tau_2 - \tau_1)^\alpha}{\alpha} \left[\|x_1\| + \widetilde{M}_s \frac{T^\alpha}{\alpha} \psi((C_2^* + L^\phi)\|\phi\|_{\mathcal{B}} + (C_1^* + 1)r)\right] \|\mu\|_{L^1}.
\end{aligned}$$

Hence, we deduce that

$$\lim_{\tau_1 \to \tau_2} Q_1 = 0.$$

Also,

$$\begin{aligned}
Q_2 &= \int_0^{\tau_1} \|\,(S_\alpha(\tau_2 - s) - S_\alpha(\tau_1 - s))\,(v(s) + Bu(s))\,\| ds \\
&\le \int_0^{\tau_1} \|\,(S_\alpha(\tau_2 - s) - S_\alpha(\tau_1 - s))\,\|_{L(E)} \left(\|v(s)\| + \|Bu(s)\|\right) ds \\
&\le \int_0^{\tau_1} \|\,(S_\alpha(\tau_2 - s) - S_\alpha(\tau_1 - s))\,\|_{L(E)} \|v(s)\| ds \\
&+ M_1 \int_0^{\tau_1} \|\,(S_\alpha(\tau_2 - s) - S_\alpha(\tau_1 - s))\,\|_{L(E)} \|u(s)\| ds \\
&\le \psi((C_2^* + L^\phi)\|\phi\|_{\mathcal{B}} + (C_1^* + 1)r)\|\mu\|_{L^1} \int_0^{\tau_1} \|\,(S_\alpha(\tau_2 - s) - S_\alpha(\tau_1 - s))\,\|_{L(E)} ds \\
&+ M_1 M_2 \left[\|x_1\| + \widetilde{M}_s \frac{T^\alpha}{\alpha} \psi((C_2^* + L^\phi)\|\phi\|_{\mathcal{B}} + (C_1^* + 1)r)\|\mu\|_{L^1}\right] \\
&\times \int_0^{\tau_1} \|S_\alpha(\tau_2 - s) - S_\alpha(\tau_1 - s)\|_{L(E)} ds.
\end{aligned}$$

Since $\|S_\alpha(\tau_2 - s) - S_\alpha(\tau_1 - s)\|_{L(E)} \le 2\widetilde{M}_s(t_1 - s)^{\alpha - 1} \in L^1(J, \mathbb{R}_+)$ for $s \in [0, t_1]$ and $S_\alpha(\tau_2 - s) - S_\alpha(\tau_1 - s) \to 0$ as $\tau_1 \to \tau_2, S_\alpha$ is strongly continuous. This implies that

$$\lim_{\tau_1 \to \tau_2} Q_2 = 0.$$

Similarly, for $\tau_1, \tau_2 \in (t_i, t_{i+1}], i = 1, \ldots, m$, we have

$$\begin{aligned}
\|h(\tau_2) - h(\tau_1)\| &\le \|T_\alpha(\tau_2 - t_i) - T_\alpha(\tau_1 - t_i)\|_{L(E)} \left[\|z(t_i^-)\| + \|I_i(z(t_i^-))\|\right] + Q_1' + Q_2' \\
&\le \|T_\alpha(\tau_2 - t_i) - T_\alpha(\tau_1 - t_i)\|_{L(E)} (r + \Omega) + Q_1' + Q_2',
\end{aligned}$$

where

$$\begin{aligned}
Q_1' &= \int_{\tau_1}^{\tau_2} \|S_\alpha(\tau_2 - s)\,(v(s) + Bu(s))\,\| ds \\
&\le \frac{\widetilde{M}_s(\tau_2 - \tau_1)^\alpha}{\alpha} \psi((C_2^* + L^\phi)\|\phi\|_{\mathcal{B}} + (C_1^* + 1)r)\|\mu\|_{L^1} + \frac{M_1 M_2 \widetilde{M}_s(\tau_2 - \tau_1)^\alpha}{\alpha} \\
&\times \left[\|x_1\| + \widetilde{M}_T(r + \Omega) + \widetilde{M}_s \frac{T^\alpha}{\alpha} \psi((C_2^* + L^\phi)\|\phi\|_{\mathcal{B}} + (C_1^* + 1)r)\right] \|\mu\|_{L^1}.
\end{aligned}$$

Hence, we deduce that $\lim_{\tau_1\to\tau_2} Q_1' = 0$,

$$\begin{aligned} Q_2' &= \int_0^{\tau_1} \|S_\alpha(\tau_2 - s) - S_\alpha(\tau_1 - s)\,(v(s) + Bu(s))\,\| ds \\ &\leq \psi((C_2^* + L^\phi)\|\phi\|_{\mathcal{B}} + (C_1^* + 1)r)\|\mu\|_{L^1} \int_0^{\tau_1} \|\,(S_\alpha(\tau_2 - s) - S_\alpha(\tau_1 - s))\,\| ds \\ &+ M_1M_2\left[\|x_1\| + \widetilde{M}_T(r+\Omega) + \widetilde{M}_s\frac{T^\alpha}{\alpha}\psi((C_2^* + L^\phi)\|\phi\|_{\mathcal{B}} + (C_1^* + 1)r)\|\mu\|_{L^1}\right] \\ &\times \int_0^{\tau_1} \|S_\alpha(\tau_2 - s) - S_\alpha(\tau_1 - s)\|_{L(E)} ds. \end{aligned}$$

As $\|S_\alpha(\tau_2 - s) - S_\alpha(\tau_1 - s)\|_{L(E)} \leq 2\widetilde{M}_s(t_1 - s)^{\alpha-1} \in L^1(J, \mathbb{R}_+)$ for $s \in [0, t_1]$ and $S_\alpha(\tau_2 - s) - S_\alpha(\tau_1 - s) \to 0$ as $\tau_1 \to \tau_2$, since S_α is strongly continuous. This implies that $\lim_{\tau_1\to\tau_2} Q_2' = 0$. Since T_α is also strongly continuous, so $T_\alpha(\tau_2 - t_i) - T_\alpha(\tau_1 - t_i) \to 0$ as $\tau_1 \to \tau_2$. Thus, from the above inequalities, we have

$$\lim_{\tau_1\to\tau_2} \|h(\tau_2) - h(\tau_1)\| = 0.$$

So, $P(D_r)$ is equicontinuous.
Step 4: The set $(PD_r)(t)$ is relatively compact for each $t \in J$, where

$$(PD_r)(t) = \{h(t) : h \in P(D_r)\}.$$

Let $0 < t \leq s \leq t_1$ be fixed and let ε be a real number satisfying $0 < \varepsilon < t$. For $z \in D_r$ we define

$$h_\varepsilon(t) = \int_0^{t-\varepsilon} S_\alpha(t-s)v(s)ds + \int_0^{t-\varepsilon} S_\alpha(t-s)Bu(s)ds,$$

where $v \in S_{F,y_{\rho(s,y_s+\overline{z}_s)}+\overline{z}_{\rho(s,y_s+\overline{z}_s)}}$. Using the compactness of $S_\alpha(t)$ for $t > 0$, we deduce that the set

$$H_\varepsilon = \{h_\varepsilon(t) : h_\varepsilon \in P(D_r)\}$$

is relatively compact in E. Moreover,

$$\|h(t) - h_\varepsilon(t)\| \leq \left\|\int_{t-\varepsilon}^t S_\alpha(t-s)v(s)ds\right\| + \left\|\int_{t-\varepsilon}^t S_\alpha(t-s)Bu(s)ds\right\|.$$

Similarly, for any $t \in (t_i, t_{i+1}]$ with $i = 1, \dots, m$. Let $t_i < t \leq s \leq t_{i+1}$ be fixed and let ε be a real number satisfying $0 < \varepsilon < t$. For $z \in D_r$ we define

$$\begin{aligned} h_\varepsilon(t) &= T_\alpha(t - t_i)\left[y(t_i^-) + \overline{z}(t_i^-) + I_i(y(t_i^-) + \overline{z}(t_i^-))\right] \\ &+ \int_{t_i}^{t-\varepsilon} S_\alpha(t-s)v(s)ds + \int_{t_i}^{t-\varepsilon} S_\alpha(t-s)Bu(s)ds, \end{aligned}$$

where $v \in S_{F,y_{\rho(s,y_s+\overline{z}_s)}+\overline{z}_{\rho(s,y_s+\overline{z}_s)}}$. Since $S_\alpha(t)$ is a compact operator, the set

$$H_\varepsilon = \{h_\varepsilon(t) : h \in P(D_r)\}$$

is relatively compact. Moreover,

$$\|h(t) - h_\varepsilon(t)\| \leq \left\|\int_{t-\varepsilon}^t S_\alpha(t-s)v(s)ds\right\| + \left\|\int_{t-\varepsilon}^t S_\alpha(t-s)Bu(s)ds\right\|.$$

On the other hand, using the continuity of the operator $T_\alpha(t)$, it follows that $(PD_r)(t)$ is relatively compact in E, for every $t \in [0, T]$.
As a consequence of Step 2 to 4 together with Arzelá–Ascoli theorem we can conclude that P is completely continuous.
Step 5: P has a closed graph.

Let $z_n \to z_*, h_n \in P(z_n)$ with $h_n \to h_*$. We shall prove that $h_* \in P(z_*)$.
In fact $h_n \in P(z_n)$ means that there is exists $v_n \in S_{F,y_n\rho(s,y_{ns}+\overline{z}_{ns})+\overline{z}_n\rho(s,y_{ns}+\overline{z}_{ns})}$ such that, for each $t \in [0, t_1]$,

$$h_n(t) = \int_0^t S_\alpha(t-s)v_n(s)ds + \int_0^t S_\alpha(t-s)Bu_n(s)ds,$$

where

$$u_n(t) = \tilde{W}^{-1}\left[x_1 - \int_0^T S_\alpha(T-s)v_n(s)ds\right](t).$$

We must show that there exists $v_* \in S_{F,y_*\rho(s,y_{*s}+\overline{z}_{*s})+\overline{z}_*\rho(s,y_{*s}+\overline{z}_{*s})}$ such that, for each $t \in [0, t_1]$,

$$h_*(t) = \int_0^t S_\alpha(t-s)v_*(s)ds + \int_0^t S_\alpha(t-s)Bu_*(s)ds,$$

where

$$u_*(t) = \tilde{W}^{-1}\left[x_1 - \int_0^T S_\alpha(T-s)v_*(s)ds\right](t).$$

Consider the following linear continuous operator$\Upsilon : L^1([0, t_1], E) \longrightarrow C([0, t_1], E)$ defined by

$$(\Upsilon v)(t) = \int_0^t S_\alpha(t-s)\left[v(s) + B\tilde{W}^{-1}\left(x_1 - \int_0^T S_\alpha(T-\tau)v(\tau)d\tau\right)(s)\right]ds.$$

By Lemma 2.8, we know that $\Upsilon o S_F$ is a closed graph operator. Moreover, for every $t \in [0, t_1]$, we obtain

$$h_n(t) \in \Upsilon(S_{F,y_n\rho(s,y_{ns}+\overline{z}_{ns})+\overline{z}_n\rho(s,y_{ns}+\overline{z}_{ns})}).$$

Since $z_n \to z_*$ and $h_n \to h_*$, it follows, that for every $t \in [0, t_1]$,

$$h_*(t) = \int_0^t S_\alpha(t-s)v_*(s)ds + \int_0^t S_\alpha(t-s)Bu_*(s)ds,$$

for some $v_* \in S_{F,y_*\rho(s,y_{*s}+\overline{z}_{*s})+\overline{z}_*\rho(s,y_{*s}+\overline{z}_{*s})}$.
Similarly, for any $t \in (t_i, t_{i+1}], i = 1, \ldots, m$, we have

$$\begin{aligned} h_n(t) &= T_\alpha(t-t_i)\left[y_n(t_i^-) + \overline{z}_n(t_i^-) + I_i(y_n(t_i^-) + \overline{z}_n(t_i^-))\right] \\ &+ \int_{t_i}^t S_\alpha(t-s)v_n(s) + \int_{t_i}^t S_\alpha(t-s)Bu_n(s)ds, \end{aligned}$$

where

$$\begin{aligned} u_n(t) &= \tilde{W}^{-1}\Big[x_1 - T_\alpha(T-t_i)\left(y_n(t_i^-) + \overline{z}_n(t_i^-) + I_i(y_n(t_i^-) + \overline{z}_n(t_i^-))\right) \\ &\quad - \int_{t_i}^T S_\alpha(T-s)v_n(s)ds\Big](t). \end{aligned}$$

We shall prove that there exists $v_* \in S_{F,y_*\rho(s,y_{*s}+\overline{z}_{*s})+\overline{z}_*\rho(s,y_{*s}+\overline{z}_{*s})}$ such that, for each $t \in (t_i, t_{i+1}]$,

$$\begin{aligned} h_*(t) &= T_\alpha(t-t_i)\left[y_*(t_i^-) + \overline{z}_*(t_i^-) + I_i(y_*(t_i^-) + \overline{z}_*(t_i^-))\right] \\ &+ \int_{t_i}^t S_\alpha(t-s)v_*(s)ds + \int_{t_i}^t S_\alpha(t-s)Bu_*(s)ds, \end{aligned}$$

where

$$\begin{aligned} u_*(t) &= \tilde{W}^{-1}\Big[x_1 - T_\alpha(T-t_i)\left(y_*(t_i^-) + \overline{z}_*(t_i^-) + I_i(y_*(t_i^-) + \overline{z}_*(t_i^-))\right) \\ &\quad - \int_{t_i}^T S_\alpha(T-s)v_*(s)ds\Big](t). \end{aligned}$$

Denote

$$\widehat{u}(t) = \tilde{W}^{-1}\left[x_1 - T_\alpha(T-t_i)\left(y(t_i^-) + \overline{z}(t_i^-) + I_i(y(t_i^-) + \overline{z}(t_i^-))\right)\right](t).$$

Since I_i and $\tilde{W}^{-1}$ are continuous, we have

$$\widehat{u}_n(t) \longrightarrow \widehat{u}_*(t), \quad for \quad t \in (t_i, t_{i+1}], i = 1, \dots, m.$$

Clearly, we have

$$\begin{aligned}&\Bigg\|\left(h_n(t) - T_\alpha(t-t_i)\left[y_n(t_i^-) + \overline{z}_n(t_i^-) + I_i(y_n(t_i^-) + \overline{z}_n(t_i^-))\right] - \int_{t_i}^t S_\alpha(t-s)B\widehat{u}_n(s)ds\right)\\ &-\left(h_*(t) - T_\alpha(t-t_i)\left[y_*(t_i^-) + \overline{z}_*(t_i^-) + I_i(y_*(t_i^-) + \overline{z}_*(t_i^-))\right] - \int_{t_i}^t S_\alpha(t-s)B\widehat{u}_*(s)ds\right)\Bigg\|\\ &\to 0 \quad as\ n \to \infty.\end{aligned}$$

Consider the linear continuous operator $\Upsilon : L^1((t_i, t_{i+1}], E) \longrightarrow C((t_i, t_{i+1}], E)$,

$$\begin{aligned} v \longmapsto (\Upsilon v)(t) &= \int_{t_i}^t S_\alpha(t-s)\Big[v(s) + B\tilde{W}^{-1}\left(x_1 - T_\alpha(T-t_i)\left(y_n(t_i^-) + \overline{z}_n(t_i^-)\right.\right.\\ &+ \left.\left. I_i(y_n(t_i^-) + \overline{z}_n(t_i^-))\right) - \int_{t_i}^T S_\alpha(T-\tau)v(\tau)\,d\tau\right)(s)\Big]\,ds.\end{aligned}$$

In view of Lemma 2.8, we deduce that $\Upsilon o S_F$ is a closed graph operator. Also, from the definition of Υ, we have that, for every $t \in (t_i, t_{i+1}], i = 1, \dots, m$,

$$\left(h_n(t) - T_\alpha(t-t_i)\left[y_n(t_i^-) + \overline{z}_n(t_i^-) + I_i(y_n(t_i^-) + \overline{z}_n(t_i^-))\right]\right) \in \Upsilon(S_{F, y_n\rho(s, y_{ns} + \overline{z}_{ns}) + \overline{z}_n\rho(s, y_{ns} + \overline{z}_{ns})}).$$

Since $z_n \to z_*$, for some $v_* \in S_{F, y_*\rho(s, y_{*s} + \overline{z}_{*s}) + \overline{z}_*\rho(s, y_{*s} + \overline{z}_{*s})}$ it follows from Lemma 2.8 that, for every $t \in (t_i, t_{i+1}]$, we have

$$\begin{aligned} h_*(t) &= T_\alpha(t-t_i)\left[y_*(t_i^-) + \overline{z}_*(t_i^-) + I_i(y_*(t_i^-) + \overline{z}_*(t_i^-))\right]\\ &+ \int_{t_i}^t S_\alpha(t-s)v_*(s)ds + \int_0^t S_\alpha(t-s)Bu_*(s)ds.\end{aligned}$$

Therefore P has a closed graph.

Hence by Lemma 2.9, P has a fixed point z on D_r, which is the mild solution of the system (3), then problem (3) is controllable on $(-\infty, T]$. This completes the proof of the theorem.

4. An Example

Consider the impulsive fractional integro-differential inclusion:

$$\begin{aligned} &\frac{\partial_t^q}{\partial t^q}v(t,\zeta) \in \frac{\partial^2}{\partial \zeta^2}v(t,\zeta) + \int_{-\infty}^t a_1(s-t)v(s - \rho_1(t)\rho_2(|v(t-s,\zeta)|), \xi)ds + t^2 \sin|v(t,\zeta)|\\ &\qquad + \mu(t,\zeta), \qquad\qquad t \in (t_k, t_{k+1}],\ \zeta \in [0,\pi],\\ &v(t,0) = v(t,\pi) = 0, \qquad\qquad t \in [0,T],\\ &v(t,\zeta) = v_0(\theta,\zeta), \qquad\qquad \theta \in (-\infty, 0],\ \zeta \in [0,\pi],\\ &\Delta v(t_k)(\zeta) = \int_{-\infty}^{t_k} p_k(t_k - y)dy\cos(v(t_k)(\zeta)), \qquad k = 1, 2, \dots, m.\end{aligned} \tag{7}$$

where $0 < q < 1, \mu : [0,T] \times [0,\pi] \to [0,\pi], p_k : \mathbb{R} \to \mathbb{R}, k = 1, 2, \dots, m$, and $a_1 : \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a convex-valued multivalued map, and $\rho_i : [0, +\infty) \to [0, +\infty),\ i = 1, 2$ are continuous functions.

Set $E = L^2([0,\pi])$ and $D(A) \subset E \to E$ be the operator $A\omega = \omega''$ with domain

$$D(A) = \{\omega \in E : \omega, \omega' \text{ are absolutely continuous, } \omega'' \in E, \omega(0) = \omega(\pi) = 0\}.$$

Then

$$A\omega = \sum_{n=1}^{\infty} n^2(\omega, \omega_n)\omega_n, \quad \omega \in D(A),$$

where $\omega_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx), n \in \mathbb{N}$ is the orthogonal set of eigenvectors of A. It is well known that A is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\geq 0}$ in E and is given by

$$T(t)\omega = \sum_{n=1}^{\infty} e^{-n^2 t}(\omega, \omega_n)\omega_n, \quad \forall \omega \in E, \ \textit{and every } t > 0.$$

From these expressions, it follows that $\{T(t)\}_{t\geq 0}$ is a uniformly bounded compact semigroup, so that $R(\lambda, A) = (\lambda - A)^{-1}$ is a compact operator for all $\lambda \in \rho(A)$, that is, $A \in \mathbb{A}^{\alpha}(\theta_0, \omega_0)$.
For the phase space, we choose $\mathcal{B} = C_0 \times L^2(g, X)$, see Example 2.7 for details.

Set

$$x(t)(\zeta) = v(t, \zeta), \quad t \in [0, T], \ \zeta \in [0, \pi].$$

$$\phi(\theta)(\zeta) = v_0(\theta, \zeta), \quad \theta \in (-\infty, 0], \ \zeta \in [0, \pi].$$

$$F(t, \varphi, x(t))(\zeta) = \int_{-\infty}^{0} a_1(s)\varphi(s, \xi)ds + t^2 \sin|x(t)(\zeta)|, \quad t \in [0, T], \ \zeta \in [0, \pi].$$

$$\rho(t, \varphi) = s - \rho_1(s)\rho_2(|\varphi(0)|).$$

$$I_k(x(t_k^-))(\zeta) = \int_{-\infty}^{0} p_k(t_k - y)dy \cos(x(t_k)(\zeta)), \quad k = 1, 2, \ldots, m.$$

$$Bu(t)(\zeta) = \mu(t, \zeta).$$

Under the above conditions, we can represent the system (7) in the abstract form (3). Assume that the operator $W : L^2(J, E) \to E$ defined by

$$Wu(\cdot) = \int_0^T S_\alpha(T - s)\mu(s, \cdot)ds,$$

has a bounded invertible operator $\tilde{W}^{-1}$ in $L^2(J, E)/\ker W$.

The following result is a direct consequence of Theorem 3.9.

Proposition 4.1. *Let $\varphi \in \mathcal{B}$ be such that (H_φ) holds, and assume that the above conditions are fulfilled, then system (7) is controllable on $(-\infty, T]$.*

Acknowledgement: The work of J.J. Nieto has been partially supported by Agencia Estatal de Investigacion (AEI) of Sapin under grant MTM2016-75140-P, co-financed by the European Community fund FEDER, and XUNTA de Galicia under grant GRC2015-004.

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