JOURNAL OF SCIENCE
SAKARYA UNIVERSITY

Sakarya University Journal of Science
ISSN 1301-4048 | e-ISSN 2147-835X | Period Bimonthly | Founded: 1997 | Publisher Sakarya University |
http://www.saujs.sakarya.edu.tr/

Title: On some Zweier convergent vector valued multiplier spaces

Authors: Ramazan Kama
Recieved: 2018-12-06 00:00:00

Accepted: 2018-12-31 00:00:00

Article Type: Research Article
Volume: 23
Issue: 4
Month: August
Year: 2019
Pages: 541-548

How to cite
Ramazan Kama; (2019), On some Zweier convergent vector valued multiplier spaces.
Sakarya University Journal of Science, 23(4), 541-548, DOI:
10.16984/saufenbilder.492788
Access link
http://www.saujs.sakarya.edu.tr/issue/43328/492788

Sakarya University Journal of Science 23(4), 541-548, 2019

SAKARYA UNIVERSITY

On Zweier convergent vector valued multiplier spaces

Ramazan Kama*[1]

Abstract

In this paper, we introduce the Zweier convergent vector valued multiplier spaces $M_Z^{\infty}(\sum_i T_i x_i)$ and $M_{wZ}^{\infty}(\sum_i T_i x_i)$. We study some topological and algebraic properties on these spaces. Furthermore, we study some inclusion relations concerning these spaces.

Keywords: vector valued multiplier space, Zweier matrix, summing operator, operator valued series.

1. INTRODUCTION

Let $\mathbb{N}$ and $\mathbb{R}$ be the sets of all positive integers and real numbers, respectively. We shall denote the space of all real valued sequences by

$$w = \{x = (x_i) : x_i \in \mathbb{R}\}.$$

Any vector subspace of w is called as a *sequence space*. Let l_∞, c and c_0 denote the spaces of all bounded, convergent and null sequences $x = (x_i)$ with real terms, respectively, normed by $\|x\|_\infty = \sup_{i\in\mathbb{N}}|x_i|$.

A sequence space X with linear topology is called a *K-space* provided each of the maps $p_i: X \to \mathbb{R}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$. If $x \in X$, then $e^i \otimes x$ denote the sequence with x in the i^{th} coordinate and zero in the other coordinates. If $\mathfrak{I} \subset \mathbb{N}$, $\chi_{\mathfrak{I}}$ denote the characteristic function of $\mathfrak{I}$ and $x = (x_i)$ is any sequence, $\chi_{\mathfrak{I}} x$ denote the coordinatewise product of $\chi_{\mathfrak{I}}$ and x. A sequence space X is monoton if $\chi_{\mathfrak{I}} x \in X$ for every $\mathfrak{I} \subset \mathbb{N}$ and $x \in X$.

Let X and Y be sequence spaces and $A = (a_{ni})$ be an infinite matrix of real numbers a_{ni}, where $n, i \in \mathbb{N}$. Then, we say that A defines a matrix mapping from X to Y. If for every sequence $x = (x_i) \in X$ the sequence $Ax = ((Ax)_n)$, the $A-$transform of $x \in X$ in Y, where $(Ax)_n = \sum_k a_{ni} x_i$ for each $n \in \mathbb{N}$. The matrix domain X_A of an infinite matrix A in a sequence space X is defined by

$$X_A = \{x = (x_i) \in w : Ax \in X\}$$

* Corresponding Author email: ramazankama@siirt.edu.tr

[1] Siirt University, Department of Mathematics and Physical Sciences Education, Siirt, Turkey. ORCID: 0000-0003-3520-1227

which is a sequence space [4, 6, 11].

Şengönül [15] defined the sequence $y = (y_k)$ which is frequently used as the $Z^\alpha -$ transformation of the sequence $x = (x_k)$ i.e.

$y_k = \alpha x_k + (1-\alpha)x_{k-1}$,

where $x_{-1} = 0, 1 < k < \infty$ and Z^α denotes the matrix $Z^\alpha = (z_{ij})$ defined by

$$(z_{ij}) = \begin{cases} \alpha, & \text{if } i = j, \\ 1-\alpha, & \text{if } i-1 = j, \\ 0, & \text{otherwise.} \end{cases}$$

Following Başar and Altay [5], Şengönül [15] introduced the Zweier sequence spaces Z and Z_0 as follows:

$$Z = \{x = (x_k) \in w : Z_p x \in c\},$$

$$Z_0 = \{x = (x_k) \in w : Z_p x \in c_0\}.$$

For details on Zweier sequence spaces we also refer to [8–10].

Let X, Y be normed spaces, $L(X,Y)$ be also the space of continuous linear operators from X into Y and $\sum_i T_i$ be a series in $L(X,Y)$. λ be a vector space of $X-$valued sequences which contains $c_{00}(X)$, the space of all sequences which are eventually 0. By $l_\infty(X)$ and $c_0(X)$, we denote the $X-$ valued sequence spaces of bounded and convergence to zero, respectively. The series $\sum_i T_i$ is $\lambda-$ multiplier convergent if the series $\sum_i T_i x_i$ converges in Y for every sequence $x = (x_i) \in \lambda$. The series $\sum_i T_i$ is $\lambda-$ multiplier Cauchy if the series $\sum_i T_i x_i$ is Cauchy in Y for every sequence $x = (x_i) \in \lambda$. For more information about vector valued multiplier spaces and multiplier convergent series, see [2, 7, 8, 13].

Let $\sum_i T_i$ be a series in $L(X,Y)$. Then, we will define the spaces

$$M_Z^\infty(\sum_i T_i x_i) = \{x = (x_i) \in l_\infty(X) : Z - \sum_i T_i x_i \text{ exists}\}$$

and

$$M_{wZ}^\infty(\sum_i T_i x_i) = \{x = (x_i) \in l_\infty(X) : wZ - \sum_i T_i x_i \text{ exists}\}$$

endowed sup norm, where

$$Z - \sum_i T_i x_i = \lim_{n\to\infty} (1-\alpha) \sum_{i=1}^{n-1} T_i x_i^n + \alpha \sum_{i=1}^{n} T_i x_i^n$$

and

$$wZ - \sum_i T_i x_i = \lim_{n\to\infty} (1-\alpha) \sum_{i=1}^{n-1} f(T_i x_i^n) + \alpha \sum_{i=1}^{n} f(T_i x_i^n)$$

$f \in Y^*$ (dual of Y). Notice that $M_Z^\infty(\sum_i T_i x_i) \subset M_{wZ}^\infty(\sum_i T_i x_i) \subset l_\infty(X)$.

In [1, 12], authors introduced some subspaces of l_∞ by means of multiplier convergent series and studied some properties of this spaces. Also, in [3, 14], the above spaces studied in the case of some convergence.

In this paper, we will show that the spaces $M_Z^\infty(\sum_i T_i x_i)$ and $M_{wZ}^\infty(\sum_i T_i x_i)$ are Banach spaces by means of $c_0(X)-$ multiplier convergent series. Also, we will give some characterizations of $l_\infty(X)$ and $c_0(X)-$ multiplier convergent series by using summing operators related to the series $\sum_i T_i$.

2. THE ZWEIER SUMMABILITY SPACE

Before starting this section, we give the following propostion will be used for establishing some results of this study:

Proposition 2.1. $\sum_i T_i$ $c_0(X)-$ multiplier convergent series if and only if the set

$$E = \left\{ \sum_{i}^{n} T_i\, x_i : \|x_i\| \le 1, n \in \mathbb{N} \right\} \quad (1)$$

is bounded [14].

The following theorem gives the completeness of the space $M_Z^\infty(\sum_i T_i x_i)$.

Theorem 2.2. Let X and Y are normed spaces and $\sum_i T_i$ is a series in $L(X,Y)$. If

(i) X and Y are Banach spaces,

(ii) The series $\sum_i T_i$ $c_0(X)-$ multiplier convergent,

then $M_Z^\infty(\sum_i T_i x_i)$ is a Banach space.

Proof. Since the series $\sum_i T_i$ is $c_0(X)$ −multiplier convergent, by Proposition 2.1, there exists M > 0 such that

$$M = \sup\left\{ \left\| \sum_{i}^{n} T_i\, x_i \right\| : \|x_i\| \le 1, n \in \mathbb{N} \right\}.$$

We suppose that (x^m) be a Cauchy sequence in $M_Z^\infty(\sum_i T_i)$. Since $M_Z^\infty(\sum_i T_i) \subset l_\infty(X)$ and $l_\infty(X)$ is a Banach space (since X is a Banach space), there exists $x = (x_i^0) \in l_\infty(X)$ such that $\lim_m x^m = x^0$. We will show that $x^0 \in M_Z^\infty(\sum_i T_i)$.

We take $\varepsilon > 0$. Then, there exists $m_0 \in \mathbb{N}$ such that

$$\|x^m - x^0\| < \frac{\varepsilon}{3M}$$

for $m \ge m_0$. Since $\frac{3M}{\varepsilon}\|x^m - x^0\| < 1$,

$$\frac{3M}{\varepsilon}\left\| (1-\alpha)\sum_{i=1}^{n-1} T_i(x_i^m - x_i^0) + \alpha \sum_{i=1}^{n} T_i(x_i^m - x_i^0) \right\| \le M$$

and so

$$\left\| (1-\alpha)\sum_{i=1}^{n-1} T_i(x_i^m - x_i^0) + \alpha \sum_{i=1}^{n} T_i(x_i^m - x_i^0) \right\| < \frac{\varepsilon}{3} \quad (2)$$

for $m \ge m_0$ and $n \in \mathbb{N}$. On the other hand, since (x^m) is a Cauchy sequence in $M_Z^\infty(\sum_i T_i)$ there exists sequence $(y_m) \subset Y$ such that

$$\left\| (1-\alpha)\sum_{i=1}^{n-1} T_i x_i^m + \alpha \sum_{i=1}^{n} T_i x_i^m - y_m \right\| < \frac{\varepsilon}{3} \quad (3)$$

for $n \ge n_0$. If we take $p > q \ge m_0$, from (2) and (3), then we have $\|y_p - y_q\| < \varepsilon$. Hence, (y_m) is a Cauchy sequence. Let $\lim_m y_m = y_0$ and suppose that $\|y_m - y_0\| < \frac{\varepsilon}{3}$. Consequently,

$$\left\| (1-\alpha)\sum_{i=1}^{n-1} T_i x_i^0 + \alpha \sum_{i=1}^{n} T_i x_i^0 - y_0 \right\|$$

$$\le \left\| (1-\alpha)\sum_{i=1}^{n-1} T_i(x_i^m - x_i^0) + \alpha \sum_{i=1}^{n} T_i(x_i^m - x_i^0) \right\| + \left\| (1-\alpha)\sum_{i=1}^{n-1} T_i x_i^m + \alpha \sum_{i=1}^{n} T_i x_i^m - y_m \right\| + \|y_m - y_0\| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

for $n \ge n_0$. This means that $x^0 \in M_Z^\infty(\sum_i T_i)$.

In the next theorem we show that the converse of above theorem is hold. But, it does not need to be the spaces X and Y are complete.

Theorem 2.3. If $M_Z^\infty(\sum_i T_i)$ is a Banach space, then $\sum_i T_i$ $c_0(X)-$ multiplier convergent series.

Proof. We consider the sequence $x = (x_i) \in c_0(X)$. From the closedness of $M_Z^\infty(\sum_i T_i)$ and

$c_{00}(X) \subset M_Z^\infty(\sum_i T_i)$, the inclusion $c_0(X) \subset M_Z^\infty(\sum_i T_i)$ is hold.

Then, the series $\sum_i T_i x_i$ is subseries Zweier convergent beause of $c_0(X)$ is a monoton space. So, $\sum_i T_i x_i$ is weakly subseries Zweier convergent series. Using Orlicz-Pettis theorem ([1, Theorem 4.1]), we obtain that the series $\sum_i T_i x_i$ is subseries norm convergent, and hence $\sum_i T_i$ is $c_0(X) -$ multiplier convergent.

Remark 2.4. (1) In Theorem 2.2, if Y is not a Banach space, then there exists a sequence $y = (y_i)$ in Y and $F \in Y^{**}\backslash Y$ such that

$$\|y_i\| < \frac{1}{3^i 3^i} \text{ and } \sum_i y_i = F$$

for every $i \in \mathbb{N}$. Also, note that $Z - \sum_i y_i = F$. We take $x_0 \in X$ with $\|x_0\| = 1$. By Hahn-Banach theorem, we choose $x_0^* \in X^*$ such that $x_0^*(x_0) = \|x_0\|$. We denote sequence $T_i \in L(X,Y)$ by $T_i\, x = x_0^*\,(x)3^i\, y_i$ for each $i \in \mathbb{N}$. It is obtain that $\sum_i T_i\ is\ c_0(X) -$ multiplier Cauchy. Consider the sequence $x = (x_0/3^i) \in c_0(X)$. Then $x^n = \sum_{i=1}^n e^i \otimes \mathrm{x}_0\, /3^i \in M_Z^\infty(\sum_i T_i)$ for every $n \in \mathbb{N}$ and $x^n \to x_0/3^i$, but since

$$Z - \sum_i T_i x_i = Z - \sum\nolimits_i \frac{1}{3^i} x_0^*\,(x_0) 3^i y_i$$
$$= Z - \sum_i y_i = F,$$

$M_Z^\infty(\sum_i T_i)$ is not a Banach space.

(2) It is well know that if $\lim_i x_i = x_0$, then $Z - \lim_i x_i = x_0$, and also $\sum_i x_i = x_0$, then $Z - \sum_i x_i = x_0$. Therefore, if

$$M^\infty(\sum_i T_i) = \left\{x = (x_i) \in l_\infty(X) \colon \sum_i T_i x_i \text{ exists}\right\},$$

then we obtain the inclusion $M^\infty(\sum_i T_i) \subset M_Z^\infty(\sum_i T_i)$.

(3) Let X and Y be normed spaces. We denote the summing operator associate with the series $\sum_i T_i$

$$S \colon M_Z^\infty(\sum_i T_i) \to Y,\ \ S(x) = Z - \sum_i T_i x_i .$$

Then, the summing operator S is continuous if and only if the series $\sum_i T_i$ is $c_0(X) -$ multiplier Cauchy. Let us suppose that S is continuous. Since $c_{00}(X) \subset M_Z^\infty(\sum_i T_i)$, and if $x = (x_i) \in c_{00}(X)$ with $\|x\| \leq 1$ such that $x_i = 0$ for all $i > k$, we have that

$$\|S_1 x_1 + \cdots + S_k x_k\| = \|Sx\| \leq \|S\|.$$

Therefore

$$\sup_k \left\{ \left\| \sum_{i=1}^k T_i x_i \right\| \colon \|x_i\| \leq 1, k \in \mathbb{N} \right\} \leq \|S\|$$

and hence, the series $\sum_i T_i$ is $c_0(X) -$ multiplier Cauchy by Proposition 2.1.

Now, suppose that $\sum_i T_i$ is $c_0(X) -$ multiplier Cauchy. Then, by Proposition 2.1, the set $E = \left\{ \left\| \sum_{i=1}^k T_i x_i \right\| \colon \|x_i\| \leq 1, k \in \mathbb{N} \right\}$ is bounded. We take $\|e\| \leq K$ for every $e \in E$. Let $x = (x_i) \in M_Z^\infty\,(\sum_i T_\mathrm{i})$ with $\|x\| \leq 1$. Thus $Z - \sum_{i=1}^k T_i x_i$ exists, and hence

$$\|S_k(x)\| = \left\| Z - \sum_{i=1}^k T_i x_i \right\| \leq K$$

for $k \in \mathbb{N}$. This means that S is continuous.

(4) We suppose that Y is a Banach space. Then, we will show that the summing operator S is compact if and only if the series $\sum_i T_i$ is $l_\infty(X) -$ multiplier convergent. Indeed, let S be compact and $x = (x_i) \in l_\infty(X)$. If we define the following set that is bounded on the space $M_Z^\infty\,(\sum_i T_\mathrm{i})$

$$M = \left\{ \sum_{i \in \Im} e^i \otimes x_i \colon \Im\ is\ finite,\ \ \|x_i\| \leq 1 \right\},$$

then $S(M) = Z - \sum_{i \in \Im} T_i x_i \colon \Im \mathrm{isfinite}, \|x_i\| \leq 1$ is relatively compact. Hence, the series $\sum_i T_i x_i$ is subseries norm Zweier summability ([13, Theorem 2.48]), and so the series $\sum_i T_i x_i$ is subseries norm convergent by Orlicz-Pettis theorem. That is $\sum_i T_i$ is $l_\infty(X)-$ multiplier convergent series.

Conversely, let $\sum_i T_i$ is $l_\infty(X)-$ multiplier convergent series, then $Z-\sum_i T_i x_i$ is uniformly convergent series for $\|x_i\| \le 1$ ([13, Corollary 11.11]). If we define the operators $S_n: M_Z^\infty(\sum_i T_i) \to Y$ by $S_n(x) = Z - \sum_{i=1}^n T_i x_i$ for $n \in \mathbb{N}$, then

$$\|S_n - S\| = \left\| Z - \sum_{i=1}^{n} T_i x_i - Z - \sum_{i=1}^{\infty} T_i x_i \right\| = \left\| Z - \sum_{i=n+1}^{\infty} T_i x_i \right\| \to 0$$

for $\|x_i\| \le 1$, as $n \to \infty$. Therefore, S is compact.

By Theorem 2.2, Theorem 2.3 and Remark 2.4, we can obtain the following corollary:

Corollary 2.5. If X and Y are Banach spaces and $\sum_i T_i$ is a series in $L(X,Y)$, then the following statements are equivalent:

(i) $\sum_i T_i$ $c_0(X)-$ multiplier convergent series.

(ii) $M^\infty \sum_i T_i$ is a Banach space.

(iii) $c_0(X) \subseteq M^\infty \sum_i T_i$.

(iv) $M_Z^\infty(\sum_i T_i)$ is a Banach space.

(v) $c_0(X) \subseteq M_Z^\infty(\sum_i T_i)$.

3. THE WEAK ZWEIER SUMMABILITY SPACE

In this section, we will extend that to the space $M_{wZ}^\infty(\sum_i T_i)$ some of the conclusions obtained in the preceding section for the space $M_Z^\infty(\sum_i T_i)$. We begin this section by the following theorem.

Theorem 3.1. If X and Y are Banach spaces and the series $\sum_i T_i$ $c_0(X)-$multiplier convergent, then $M_{wZ}^\infty(\sum_i T_i x_i)$ is a Banach space.

Proof. Let $(x^m) \subset M_{wZ}^\infty(\sum_i T_i x_i)$ be a Cauchy sequence. Then, $\lim_m x^m = x^0$ in $l_\infty(X)$. We will prove that $x^0 \in M_{wZ}^\infty(\sum_i T_i)$.

If the proof of Theorem 2.2 is followed, then there exists $m_0 \in \mathbb{N}$ such that

$$\left\| (1-\alpha)\sum_{i=1}^{n-1} T_i(x_i^m - x_i^0) + \alpha \sum_{i=1}^{n} T_i(x_i^m - x_i^0) \right\| < \frac{\varepsilon}{3} \qquad (4)$$

for $m \ge m_0$ and $n \in \mathbb{N}$. If $p > q \ge m_0$ are fixed, then a functional $f \in S_{Y^*}$ (unit sphere in Y^*) can be found such that $\|y_p - y_q\| = |f(y_p - y_q)|$. Since (x^m) is a Cauchy sequence in $M_Z^\infty(\sum_i T_i)$, there exists sequence $(y_m) \subset Y$ such that

$$\left\| (1-\alpha)\sum_{i=1}^{n-1} f(T_i x_i^m) + \alpha \sum_{i=1}^{n} f(T_i x_i^m) - f(y_m) \right\| < \frac{\varepsilon}{3} \qquad (5)$$

for $n \ge n_0$. From (4) and (5), we have $\|y_p - y_q\| < \varepsilon$. Thus, (y_m) is a Cauchy sequence. Since Y is a Banach space, there exists $y_0 \in Y$ such that $\|y_m - y_0\| < \frac{\varepsilon}{3}$. Finally, we obtain that the following inequalities,

$$\left| (1-\alpha)\sum_{i=1}^{n-1} f(T_i x_i^0) + \alpha \sum_{i=1}^{n} f(T_i x_i^0) - f(y_0) \right|$$

$$\le \left| (1-\alpha)\sum_{i=1}^{n-1} f(T_i(x_i^m - x_i^0)) + \alpha \sum_{i=1}^{n} f(T_i(x_i^m - x_i^0)) \right|$$

$$+\left|(1-\alpha)\sum_{i=1}^{n-1} f(T_i x_i^m) + \alpha\sum_{i=1}^{n} f(T_i x_i^m) - f(y_m)\right| + |f(y_m) - f(y_0)| < \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3} = \varepsilon$$

for $n \geq n_0$. In the other words, $wZ - \sum_{i=1}^{n} T_i x_i^0 = y_0$, and so $x^0 \in M_{wZ}^{\infty}(\sum_i T_i)$.

Theorem 3.2. If $M_{wZ}^{\infty}(\sum_i T_i)$ is a Banach space, then $\sum_i T_i$ $c_0(X) -$ multiplier convergent series.

Proof. As in the proof of Theorem 2.3, if $M_{wZ}^{\infty}(\sum_i T_i)$ is a Banach space, then $c_0(X) \subset M_{wZ}^{\infty}(\sum_i T_i)$. From the monotonity of $c_0(X)$, the series $\sum_i T_i x_i$ is weakly subseries Zweier convergent and hence $\sum_i T_i$ is $c_0(X) -$ multiplier convergent.

Remark 3.3. (1) In Theorem 3.1, if Y is not a Banach space and consider the sequence $x = (x_0/3^i) \in c_0(X)$, following the Remark 2.4 (1), then we obtain that $wZ - \sum_i T_i x_i = F$ for $F \in Y^{**}$. Thus, $x = (x_0/3^i) \notin M_{wZ}^{\infty}(\sum_i T_i)$. That is, $M_{wZ}^{\infty}(\sum_i T_i)$ is not a Banach space.

(2) Since $w - \sum_i x_i = x_0$ implies $wZ - \sum_i x_i = x_0$, therefore, if

$$M_w^{\infty}\left(\sum_i T_i\right) = \left\{x = (x_i) \in l_{\infty}(X) : w - \sum_i T_i x_i \text{ exists}\right\},$$

then $M_w^{\infty}(\sum_i T_i) \subset M_{wZ}^{\infty}(\sum_i T_i)$.

(3) Let X and Y be normed spaces. We can also define the summing operator associate with the series $\sum_i T_i$

$$S: M_{wZ}^{\infty}\left(\sum_i T_{\mathrm{i}}\right) \to Y, \quad S(x) = wZ - \sum_i T_{\mathrm{i}}\, x_i.$$

As we did Remark 2.4 (3), one can see that the summing operator S is continuous if and only if the series $\sum_i T_i$ is $c_0(X) -$ multiplier Cauchy.

(4) Let Y be a Banach space. If S is compact, from Remark 2.4 (4), then the set $S(M)$ is weakly relatively compact, and hence $\sum_{\mathrm{i}} \mathrm{T_i}$ is $l_{\infty}(X) -$ multiplier convergent series. On the other hand, let us suppose that Y is complete and the series $\sum_{\mathrm{i}} \mathrm{T_i}$ is $l_{\infty}(X) -$ multiplier convergent. Then, $\mathrm{wZ} - \sum_{\mathrm{i}} \mathrm{T_i}\,\mathrm{x_i}$ is uniformly convergent for $\|\mathrm{x_i}\| \leq 1$ ([13, Corollary 11.11]). Therefore, we have that

$$\|S_n - S\| = \left\|wZ - \sum_{i=1}^{n} T_i x_i - wZ - \sum_{i=1}^{\infty} T_i x_i\right\| = \left\|wZ - \sum_{i=n+1}^{\infty} T_i x_i\right\| \to 0$$

for $\|\mathrm{x_i}\| \leq 1$, as $\mathrm{n} \to \infty$, where the operators $S_n: M_{wZ}^{\infty}(\sum_i T_{\mathrm{i}}) \to Y$ is defined by $S_n(x) = wZ - \sum_{i=1}^{n} T_i x_i$ for $n \in \mathbb{N}$. This implies that S is compact.

By the previous theorems and remark above, we can give the following corollaries:

Corollary 3.4. If X and Y are Banach spaces and $\sum_i T_i$ is a series in $L(X,Y)$, then the following conditions are equivalent:

(i) $\sum_i T_i$ $c_0(X) -$ multiplier convergent series.

(ii) $M_w^{\infty} \sum_i T_i$ is a Banach space.

(iii) $c_0(X) \subseteq M_w^{\infty} \sum_i T_i$.

(iv) $M_{wZ}^{\infty}(\sum_i T_i)$ is a Banach space.

(v) $c_0(X) \subseteq M_{wZ}^{\infty} \sum_i T_i$.

Corollary 3.5. If Y is Banach space, then the following are equivalent:

(i) S is compact.

(ii) S is a weakly compact.

(iii) $\sum_i T_i$ is $l_{\infty}(X) -$ multiplier convergent series.

Finally, we will give a sufficient condition for the equivalence of both spaces, which are defined in the introduction.

Proposition 3.6. Let X and Y be normed spaces. If $\sum_i T_i$ is $l_\infty(X)-$ multiplier Cauchy series, $M_Z^\infty\,(\sum_i T_i) = M_{wZ}^\infty\,(\sum_i T_i)$.

Proof. We prove that the inclusion $M_{wZ}^\infty\,(\sum_i T_i) \subset M_Z^\infty\,(\sum_i T_i)$ is hold. If we take $x = (x_i) \in M_{wZ}^\infty\,(\sum_i T_i)$, then there exists $y_0 \in Y$ such that

$$Z - \sum_i f(T_i x_i) = f(y_0)$$

for every $f \in Y^*$. Also, since the series $\sum_i T_i$ is $l_\infty(X)-$ multiplier Cauchy, the series $\sum_i T_i x_i$ is Cauchy in Y. Thus, there exists $F \in Y^{**}$ such that

$$Z - \sum_i T_i x_i = F.$$

If consider the uniqueness of limit, then we have $F = y_0$. Thus, $x = (x_i) \in M_Z^\infty\,(\sum_i T_i)$.

Acknowledgments

This work was supported by the Siirt University Research Fund with Project Number 2018-SİÜEĞT-041.

4. REFERENCES

[1] A. Aizpuru, C. Pérez-Eslava, and J. B. Seoane Sepúlveda, "Matrix summability methods and weakly unconditionally Cauchy series," The Rocky Mountain Journal of Mathematics, vol. 39, no. 2, pp. 367-380, 2009.

[2] F. Albiac and N. J. Kalton, "Topics in Banach Spaces Theory," New York, Springer-Verlag, 2006.

[3] B. Altay and R. Kama, "On Cesàro summability of vector valued multiplier spaces and operator valued series," Positivity, vol. 22, no. 2, pp. 575-586, 2018.

[4] F. Başar, "Summability Theory and Its Applications," Bentham Science Publishers, İstanbul, 2012.

[5] F. Başar and B. Altay, "On the space of sequences of p-bounded variation and related matrix mappings", (English, Ukrainian summary) Ukrain. Mat. Zh., vol. 55, no. 1, pp. 108-118, 2003; reprinted in Ukrainian Math. J., vol. 55, no. 1, pp. 136-14, 2003.

[6] J. Boos and P. Cass, "Classical and Modern Methods in Summability," Oxford University Press, Oxford, 2000.

[7] J. Diestel, "Sequences and Series in Banach Spaces," New York, Springer-Verlag, 1984.

[8] O. Duyar and S. Demiriz, "On vector-valued operator Riesz sequence spaces," Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat., vol. 68, no. 3, pp. 236-247, 2019.

[9] B. Hazarika, T. Karan, and B. K. Singh, "Zweier ideal convergent sequence space defined by Orlicz function," Journal of mathematics and computer science, vol. 8, no. 3, pp. 307-318, 2014.

[10] V. A. Khan, K. Ebadullah, A. Esi, and M. Shafiq, "On some Zweier I-convergent sequence spaces defined by a modulus function," Afrika Matematika, vol. 26, no. 1-2, pp. 115-125, 2015.

[11] V. A. Khan and N. Khan, "Zweier I-convergent double sequence spaces defined by a sequence of moduli," Theory and Applications of Mathematics and Computer Science, vol. 5, no. 2, pp. 194-202, 2015.

[12] M. Mursaleen, "Applied Summability Methods, Springer Briefs," Heidelberg New York Dordrecht London, 2014.

[13] F. J. Pérez-Fernández, F. Benítez-Trujillo, and A. Aizpuru, "Characterizations of completeness of normed spaces through weakly unconditionally Cauchy series,"

Czechoslovak Math. J., vol. 50, no. 4, pp. 889-896, 2000.

[14] C. Swartz, “Multiplier Convergent Series,” Singapore, World Sci. Publ., 2009.

[15] C. Swartz, “Operator valued series and vector valued multiplier spaces,” Casp. J. Math. Sci., vol. 3, no. 2, pp. 277-288, 2014.

[16] M. Şengönül, “On the Zweier sequence space,” Demonstratio Mathematica, vol. 40, no. 1, pp. 181-196, 2007