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Fibonacci operational matrix algorithm for solving differential equations of Lane-Emden type

Musa Çakmak*[1]

Abstract

The aim of this study is to provide an effective and accurate technique for solving differential equations of Lane-Emden type as initial value problems. In this study, a numerical method called Fibonacci polynomial approximation method (FPAM) establish for approximate solution of Lane-Emden type differential equations by using Fibonacci polynomials. A matrix equation can be solved depending on the reduced form of the Lane-Emden type differential equations, which is characterized by an algebraic equation system, with the matrix relations of Fibonacci polynomials and their derivatives and their unknown Fibonacci coefficients. In addition, numerical results are given by comparisons to confirm the reliability of the proposed method for Lane-Emden type differential equations.

Keywords: Fibonacci polynomials, Functional differential equations, Lane-Emden equation, Operational matrix, Polynomial approach

1. INTRODUCTION

The astrophysicist Jonathan H. Lane (1870) and Robert Emden (1907) first studied the Lane-Emden equation. In view of the classical laws of thermodynamics, the thermal behavior of a spherical gas cloud acting under the mutual attraction of its molecules was taken into account [1,2]. The famous Lane-Emden equation was used to model several cases of astrophysics, such as mathematical physics and stellar structure theory, isothermal gas spheres, thermal behavior of a global gas cloud, and the theory of thermionic currents [3,4]. Numerous works has been done on these sorts of problems for many structures. Differential equations such as Lane-Emden type differential equations, delay differential equations, differential-difference equations and pantograph equations have been studied by different authors with different methods. Lane-Emden type differential equations are similar to differential-difference equations and pantograph equations. Therefore, the solution methods are similar. Researchers who want to take a look at these types of differential equations can also look at the following:[4-16]. On the other hand, researchers who want to study the numbers of Fibonacci, which are the basis of fibonacci polynomials used in mathematics and in many fields of art, can look at the following studies: [17-18]. Authors in [6] used Laguerre polynomials to solve the Lane-Emden type differential equation. In this work, we use Fibonacci polynomials for the solution of the same equation. In this study, the numerical algorithm is presented to solve the differential equation of Lane-Emden type by using the matrix relations between Fibonacci polynomials and their derivatives [6]

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$$\psi''(\gamma z+\tau)+q(z)\psi''(\beta z+\eta)$$
$$+p(z)\psi(\alpha z+\mu)=g(z),\quad 0\le z\le 1 \tag{1}$$

under the initial conditions

$$\psi(0)=\omega_0,\quad \psi'(0)=\omega_{1,}\quad 0\le z\le 1 \tag{2}$$

and the solution is defined in the truncated Fibonacci series form

$$\psi(z)=\sum_{n=1}^{N+1}a_nF_n(z),\qquad 0\le z\le 1 \tag{3}$$

In here, N is chosen any positive integer bigger than 0. $a_n, n=1,2,..,N+1$ are unknown Fibonacci coefficients and $F_n(z)$ are the Fibonacci polynomials defined by [17,18]

$$F_n(z)=\sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n-1-i}{i}z^{n-1-2i}\qquad n\ge 2\,. \tag{4}$$

The layout of this article is as follows. Firstly Fibonacci polynomial defined in Section 2 for the next sections. Functional differential equation with variable coefficients, fundamental relations and operational matrices are exhibited in Section 3. The algorithm is based on the FPAM and solution procedure for Lane-Emden type differential equations is expressed in Section 4. To support the operations performed, numerical examples and their comparative results are expressed in Section 5. Finally, this work concludes with a short brief in Section 6.

2. k-FIBONACCI SEQUENCE AND FIBONACCI POLYNOMIALS

Definition 1. [17,18] For any positive number of real k, the $k-$Fibonacci sequence, say $\{F_{k,n}\}_{n\in N}$ is expressed recurrently by $F_{k,n+1}=kF_{k,n}+F_{k,n-1}$ for $n\ge 1$, with initial condition $F_{k,0}=0$; $F_{k,1}=1$.
If k is a real variable z then $F_{k,n}$ = $F_{z,n}$ and they symbolized by the Fibonacci polynomials which is defined by

$$F_{n+1}(z)=\begin{cases}1 & if\ \ n=0\\ z & if\ \ n=1\\ zF_n(z)+F_{n-1}(z) & if\ \ n>1.\end{cases} \tag{5}$$

3. OPERATIONAL MATRICES

To obtain an expansion form of the solution of the Lane-Emden equation, we use the FPAM as follows.
Assume that (1) has a continuous function. Assume that (1) can be defined

$$\psi(z)=\sum_{n=1}^{\infty}a_nF_n(z) \tag{6}$$

Then, a truncated expansion of $N+1-$ Fibonacci polynomials can be written in the vector form

$$\psi_{N+1}(z)=\sum_{n=1}^{N+1}a_nF_n(z)=F(z)A \tag{7}$$

in which respectively, the unknown Fibonacci coefficients column vector A and the Fibonacci row vector $F(z)$ are given, by

$$A=[a_1\ a_2\ \dots\ a_{N+1}]^T_{(N+1)\times 1},$$
$$F=[F_1(z)\ F_2(z)\dots F_{N+1}(z)]_{1\times(N+1)}. \tag{8}$$

The k th order derivatives of (7) can be formulated as in [14,15]

$$\psi_{N+1}{}^{(k)}(z)=\sum_{n=1}^{N+1}a_n{}^{(k)}F_n(z)=F(z)A^{(k)},$$
$$k=0,1,\dots,n, \tag{9}$$

in which $a_n^{(0)}=a_n$, $\psi^{(0)}(z)=\psi(z)$ and

$$A^{(k)}=[a_1{}^{(k)}a_2{}^{(k)}\dots a_{N+1}{}^{(k)}]^T \tag{10}$$

is the coefficient vector of polynomial approximation of k th order derivative. Then,

$$A^{(k+1)}=M^{(k)}A,\quad k=0,1,2,\dots,n \tag{11}$$

where

$$M=\begin{bmatrix} 0 & 1 & 0 & -1 & 0 & 1 & \cdots & \sin\left(\frac{N\pi}{2}\right) \\ 0 & 0 & 2 & 0 & -2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 3 & 0 & -3 & \cdots & 3s\text{in}\left(\frac{N-2}{2}\pi\right) \\ 0 & 0 & 0 & 0 & 4 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & \cdots & 5s\text{in}\left(\frac{N-4}{2}\pi\right) \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & N \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

M is $(N+1)\times(N+1)$ operational matrix for the derivative expressed by [14,15]

$$M=[m_{ij}]=\begin{cases} 0 & j\le i \\ i.\sin\dfrac{(j-i)\pi}{2} & j>i \end{cases}. \qquad (12)$$

Making use of (9) and (11), produces

$$\psi_{N+1}{}^{(k)}(z)=F(z)M^{(k)}A, \quad k=0,1,...,n. \qquad (13)$$

4. TRANSACTIONS FOR THE SOLUTION OF THE EQUATION

Let us take into account the second-order differential equation of Lane-Emden type,

$$\psi''(\gamma z+\tau)+q(z)\psi''(\beta z+\eta)+ p(z)\psi(\alpha z+\mu)=g(z), \quad 0\le z\le 1 \qquad (14)$$

The first step in the solution procedure is to express the collocation points in the field, so that

$$z_i=\frac{i-1}{N}, \quad i=1,2,...,,N+1, \quad 0\le z_i\le 1 \qquad (15)$$

Then, by collocating problem (14) at the point in (15), produces

$$\psi''(\gamma z_i+\tau)+q(z_i)\psi''(\beta z_i+\eta)+ p(z_i)\psi(\alpha z_i+\mu)=g(z_i), \quad i=1,2,...,N+1. \qquad (16)$$

Instead of the equation system in (16), we can write in matrix form as in (17).

$$\psi''(\gamma z+\tau)+Q(z)\psi''(\beta z+\eta)+ P(z)\psi(\alpha z+\mu)=G(z), \qquad (17)$$

where

$$Q=\begin{bmatrix} q(z_1) & 0 & \cdots & 0 \\ 0 & q(z_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & q(z_{N+1}) \end{bmatrix},$$

$$P=\begin{bmatrix} p(z_1) & 0 & \cdots & 0 \\ 0 & p(z_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p(z_{N+1}) \end{bmatrix} \qquad (18)$$

$$F=\begin{bmatrix} F_1(z_1) & F_2(z_1) & \cdots & F_{N+1}(z_1) \\ F_1(z_2) & F_2(z_2) & \cdots & F_{N+1}(z_2) \\ \vdots & \vdots & \ddots & \vdots \\ F_1(z_{N+1}) & F_2(z_{N+1}) & \cdots & F_{N+1}(z_{N+1}) \end{bmatrix}$$

and $G=[g(z_1)\,g(z_2)...g(z_{N+1})]^T$.

Note (An arbitrary point very close to 0 can be selected instead of $z_1=0$, to remove the indefinite in the only $q(z)=\frac{r}{z}$ function at the only $z_1=0$, collocation point. It is seen with the Maple 14 program that arbitrarily selected this point does not affect the result).

Hence, at the collocation points in (15), derivatives of the unknown function ψ can be written in the matrix form as

$$[\psi_{N+1}^{(2)}(z_i)]=F(z_i)M^2A, \quad i=1,2,...,N+1, \qquad (19)$$

or, alternatively,

$$\psi^{(2)}=\begin{bmatrix} \psi_{N+1}^{(2)}(z_1) \\ \psi_{N+1}^{(2)}(z_2) \\ \vdots \\ \psi_{N+1}^{(2)}(z_{N+1}) \end{bmatrix}=FM^2A. \qquad (20)$$

To express the functional terms of (1) as in the form (7), let we put respectively,

$\gamma z_i+\tau,\ \beta z_i+\eta,\ \alpha z_i+\mu$ instead of z in the relation (20) and then obtain

$$\begin{aligned}&[\psi_{N+1}^{(2)}\left(\gamma z_i+\tau\right)]=\dot{F}\left(\gamma z_i+\tau\right)M^2A,\\&[\psi_{N+1}^{(2)}\left(\beta z_i+\eta\right)]=\dot{F}\left(\beta z_i+\eta\right)M^2A,\\&[\psi_{N+1}^{(2)}(\alpha z_i+\mu)]=\dot{F}(\alpha z_i+\mu)M^2A,\\&i=1,2,...,N+1\end{aligned} \tag{21}$$

$\dot{F}$ are Fibonacci operational matrices corresponding (γ,τ), (β,η) and (α,μ) where

$$\dot{F}\left(\sigma_{jk},\rho_{jk}\right)=\begin{bmatrix}F_1\left(\sigma_{jk}z_1+\rho_{jk}\right) & F_2\left(\sigma_{jk}z_1+\rho_{jk}\right) & \cdots & F_{N+1}\left(\sigma_{jk}z_1+\rho_{jk}\right)\\ F_1\left(\sigma_{jk}z_2+\rho_{jk}\right) & F_2\left(\sigma_{jk}z_2+\rho_{jk}\right) & \cdots & F_{N+1}\left(\sigma_{jk}z_2+\rho_{jk}\right)\\ \vdots & \vdots & \ddots & \vdots\\ F_1\left(\sigma_{jk}z_{N+1}+\rho_{jk}\right) & F_2\left(\sigma_{jk}z_{N+1}+\rho_{jk}\right) & \cdots & F_{N+1}\left(\sigma_{jk}z_{N+1}+\rho_{jk}\right)\end{bmatrix}.$$

Therefore, replacing (20) and (21) in (17) gives the fundamental matrix equation for problem (14) as

$$\begin{aligned}&\psi''_{(\gamma,\tau)}+Q\psi''_{(\beta,\eta)}+P\psi_{(\alpha,\mu)}=G\\&\dot{F}_{(\gamma,\tau)}M^2A+Q\dot{F}_{(\beta,\eta)}M^2A+P\dot{F}_{(\alpha,\mu)}A=G\\&\left\{\dot{F}_{(\gamma,\tau)}M^2+Q\dot{F}_{(\beta,\eta)}M^2+P\dot{F}_{(\alpha,\mu)}\right\}A=G\end{aligned} \tag{22}$$

which corresponds to a system of $N+1$ algebraic equation for $a_n, n=1,2,...,N+1$. Thus, we can write

$W=\left[w_{st}\right]=\left\{\dot{F}_{(\gamma,\tau)}M^2+Q\dot{F}_{(\beta,\eta)}M^2+P\dot{F}_{(\alpha,\mu)}\right\}$, for $s=1,2,...,N+1$ and $t=1,2,...,N+1$, so, we get

$$WA=G. \tag{23}$$

Thereby, the augmented matrix of (23) becomes

$$\left[W;G\right]. \tag{24}$$

On the other hand, in view of (13), the conditions (2) can be considered by forming the following matrix equation,

$$U_j=\left[\sum_{k=0}^{1}a_{jk}\psi^{(k)}(0)\right]=[\omega_j],\ j=0,1 \tag{25}$$

where

$$\begin{aligned}&F_{\mu_{nk}}=[F_1(0),F_2(0),...,F_{N+1}(0)]\\&U_j=[u_{\iota,\zeta}]=\sum_{k=0}^{1}F(0)M^kA\end{aligned} \tag{26}$$

Therefore, the augmented matrix of the initial condition is

$$[U_j;\omega_j]=[u_{j1},u_{j2},...,u_{jN+1};\omega_j]. \tag{27}$$

Consequently, if the matrix in (24) is combined with the matrix of (27) and the necessary arrangements are made, then the new augmented matrix is obtained.

$$[W^*:G^*]=\begin{bmatrix}u_{0,1} & u_{0,2} & \cdots & u_{0,N+1} & ; & \omega_0\\ u_{1,1} & u_{1,2} & \cdots & u_{1,N+1} & ; & \omega_1\\ w_{3,1} & w_{3,2} & \cdots & w_{3,N+1} & ; & g(z_3)\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ w_{N-1,1} & w_{N-1,2} & \cdots & w_{N-1,N+1} & ; & g(z_{N-1})\\ w_{N,1} & w_{N,2} & \cdots & w_{N,N+1} & ; & g(z_N)\\ w_{N+1,1} & w_{N+1,2} & \cdots & w_{N+1,N+1} & ; & g(z_{N+1})\end{bmatrix}. \tag{28}$$

Of course, the determination of the W^* matrix in the above form should be non-zero. In this case, after several algebraic operations, unknown Fibonacci coefficients can be easily obtained from the $[W^*:G^*]$ form.

Also the correctness of the method can be easily checked. As truncated Fibonacci series expansion is an approximate solution of (1) with (2), it must satisfy the following equality for $z=z_n\in[0,b],\ n=1,2,...,N+1,$

$$E(z_n)=\left|\begin{aligned}&\psi''\left(\gamma z_n+\tau\right)+q(z_n)\psi''\left(\beta z_n+\eta\right)\\&+p(z_n)\psi(\alpha z_n+\mu)-g\left(z_n\right)\end{aligned}\right| \tag{29}$$

$\cong 0,$

or $E(z_n)\leq 10^{-m_n}$ (m_n is any positive integer). (30)

when $\max(10^{-m_n})=10^{-m}$ (m is any integer) is commanded, the truncation limit N increased until the difference $E(z_n)$ at each of the collocation points becomes smaller than the desired value 10^{-m}.

5. ILLUSTRATIVE EXAMPLES

In this section, some numerical examples are given to illustrate the efficiency and applicability of the proposed method. It is also noticeable that the computations throughout this study are performed in the Maple 14 environment.

Example 1. Take into consideration a second order differential equation of Lane-Emden type [6]

$$\psi''(2z-1)+\frac{2}{z}\psi''(3z)+z\psi(z-1)= z^4-5z^3+7z^2+63z-34,\ 0\le z\le 1 \tag{31}$$

with the initial conditions

$$\psi(0)=\psi'(0)=0. \tag{32}$$

The exact solution of the differential equation given above is known as $\psi(z)=z^3-2z^2$. It is clear from (31) that the coefficients are

$q(z)=\frac{r}{z},\ r=2,\ \gamma=2,\ \tau=-1,$

$\beta=3,\ \eta=0,\ \alpha=1,\ \mu=-1,$

$p(z)=z$

and the function g is

$g(z)=z^4-5z^3+7z^2+63z-34$.

The augmented matrix for treating Eq. (31) at the collocation points $\left\{z_1=0,\ z_2=1/3,\ z_3=2/3,\ z_4=1\right\}$ is

$$[W,G]=\begin{bmatrix} 0 & 20 & 2 & 34 & ; & -34 \\ \frac{1}{3} & \frac{52}{9} & \frac{391}{27} & \frac{2224}{81} & ; & \frac{-1004}{81} \\ \frac{2}{3} & \frac{25}{9} & \frac{398}{27} & \frac{3526}{81} & ; & \frac{796}{81} \\ 1 & 2 & 15 & 64 & ; & 32 \end{bmatrix}. \tag{33}$$

where

$$\dot{F}_{(2,-1)}=\begin{bmatrix} 1 & -1 & 2 & -3 \\ 1 & \frac{-1}{3} & \frac{10}{9} & \frac{-19}{27} \\ 1 & \frac{1}{3} & \frac{10}{9} & \frac{19}{27} \\ 1 & 1 & 2 & 3 \end{bmatrix},$$

$$\dot{F}_{(3,0)}=\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 3 \\ 1 & 2 & 5 & 12 \\ 1 & 3 & 10 & 33 \end{bmatrix},$$

$$\dot{F}_{(1,-1)}=\begin{bmatrix} 1 & -1 & 2 & -3 \\ 1 & \frac{-2}{3} & \frac{13}{9} & \frac{-44}{27} \\ 1 & \frac{-1}{3} & \frac{10}{9} & \frac{-19}{27} \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$M=\begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$Q=\begin{bmatrix} 2/(0/3) & 0 & 0 & 0 \\ 0 & 2/(1/3) & 0 & 0 \\ 0 & 0 & 2/(2/3) & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix},$$

$$P=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 \\ 0 & 0 & 2/3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, G=\begin{bmatrix} -34 \\ -1004/81 \\ 796/81 \\ 32 \end{bmatrix},$$

$$W=\begin{bmatrix} 0 & 20 & 2 & 34 \\ \frac{1}{3} & \frac{52}{9} & \frac{391}{27} & \frac{2224}{81} \\ \frac{2}{3} & \frac{25}{9} & \frac{398}{27} & \frac{3526}{81} \\ 1 & 2 & 15 & 64 \end{bmatrix}. \tag{34}$$

Here we see that the result is unchanged when we place $2/(0.1/3)$ or $2/(0.01/3)$ or $2/(0.001/3)$ instead of only $q(z_1) = 2/(0/3)$ in the only Q matrix. The matrix of the initial conditions (32) for $j = 0,1$ from equation (27) will be

$$[U_j;\varphi_j] = \begin{bmatrix} 1 & 0 & 1 & 0 & ; & 0 \\ 0 & 1 & 0 & 2 & ; & 0 \end{bmatrix}$$

Thus, from (28), the new augmented matrix with the matrix of the conditions above can be obtained as shown below:

$$\left[W^*, G^*\right] = \begin{bmatrix} 1 & 0 & 1 & 0 & ; & 0 \\ 0 & 1 & 0 & 2 & ; & 0 \\ \dfrac{2}{3} & \dfrac{25}{9} & \dfrac{398}{27} & \dfrac{3526}{81} & ; & \dfrac{796}{81} \\ 1 & 2 & 15 & 64 & ; & 32 \end{bmatrix} . \quad (35)$$

Solving this system gives the coefficient vector $A_3 = \begin{bmatrix} 2 & -2 & -2 & 1 \end{bmatrix}^T$ which implies that applying the method for $N = 3$ is enough to have the exact solution $\psi_3(z) = z^3 - 2z$. When the similar work is done for $N = 4$, the coefficient vector $A_4 = \begin{bmatrix} 2 & -2 & -2 & 1 & 0 \end{bmatrix}^T$ is obtained. So that for $N = 4$ the exact result is obtained.

Example 2. Consider a second order differential equation of Lane-Emden type

$$\psi''(z) + \frac{2}{z}\psi'(z) - 2(2z^2 + 3)\psi(z) = 0, \quad (36)$$

$0 \le z \le 1$

with the initial conditions

$$\psi(0) = 1, \ \psi'(0) = 0 \quad (37)$$

The exact solution of (36) is $\psi(z) = e^{(z^2)}$ [6]. The approximate solutions of Example 2 are calculated for different values of N, and the comparison of absolute errors in the range [0,1] is given in Table 1. Moreover, comparisons of absolute errors of present Fibonacci approach in Example 2 are given in Figure 1 and Figure 2, for $N = 8, 10, 12$, respectively. Also, we can obtain approximation solution of Example 2 for $N = 12$ as follows:

$$\begin{aligned}
\psi_{12}(z) = 1 &+ \\
&(13207289613728373974188523761179271569017/13205895650782551762071950699915456216517)z^2 - \\
&(17299311961750053209428090848628274000/13205895650782551762071950699915456216517)z^3 + \\
&(47006252005378988937962071925398271260837/92441269555477862334503654899408193515619)z^4 - \\
&(464573890929036314221895058222086167630/13205895650782551762071950699915456216517)z^5 + \\
&(24628411295828422652055450903989689560948/92441269555477862334503654899408193515619)z^6 - \\
&(2633555102651860204340332882639494899400/13205895650782551762071950699915456216517)z^7 + \\
&(29982045231779054959099181231252464980384/92441269555477862334503654899408193515619)z^8 - \\
&(3698780181391552335628175067585236670720/13205895650782551762071950699915456216517)z^9 + \\
&(17995418880373952227845246369735891763200/92441269555477862334503654899408193515619)z^{10} - \\
&(1003784172045144189200140824492949555200/13205895650782551762071950699915456216517)z^{11} + \\
&(1503097900468685384866636879532934438912/92441269555477862334503654899408193515619)z^{12}.
\end{aligned} \quad (38)$$

Table 1. Comparison of absolute errors of Laguerre polynomial approach [6] and Present Fibonacci approach of Example 2 for $N = 8, 10, 12$.

z	Exact solution	Laguerre polynomial approach N=8	Laguerre polynomial approach N=10	Laguerre polynomial approach N=12
0.0	1.000000	0.105E-12	0.220E-11	0.123E-09
0.1	1.010050	0.194E-03	0.927E-05	0.297E-06
0.2	1.040811	0.415E-03	0.166E-04	0.467E-06
0.3	1.094174	0.545E-03	0.202E-04	0.546E-06
0.4	1.173511	0.634E-03	0.229E-04	0.612E-06
0.5	1.284025	0.722E-03	0.258E-04	0.684E-06
0.6	1.433329	0.823E-03	0.293E-04	0.772E-06
0.7	1.632316	0.948E-03	0.336E-04	0.884E-06
0.8	1.896481	0.111E-02	0.392E-04	0.103E-05
0.9	2.247907	0.132E-02	0.466E-04	0.123E-05
1.0	2.718282	0.160E-02	0.565E-04	0.148E-05
z		**Presented Fibonacci approach N=8**	**Presented Fibonacci approach N=10**	**Presented Fibonacci approach N=12**
0.0		0.0000000	0.0000000	0.0000000
0.1		0.191E-03	0.917E-05	0.322E-06
0.2		0.399E-03	0.159E-04	0.492E-06
0.3		0.498E-03	0.184E-04	0.545E-06
0.4		0.540E-03	0.195E-04	0.570E-06
0.5		0.562E-03	0.201E-04	0.584E-06
0.6		0.574E-03	0.204E-04	0.590E-06
0.7		0.580E-03	0.205E-04	0.595E-06
0.8		0.584E-03	0.206E-04	0.595E-06
0.9		0.586E-03	0.207E-04	0.596E-06
1.0		0.588E-03	0.207E-04	0.598E-06

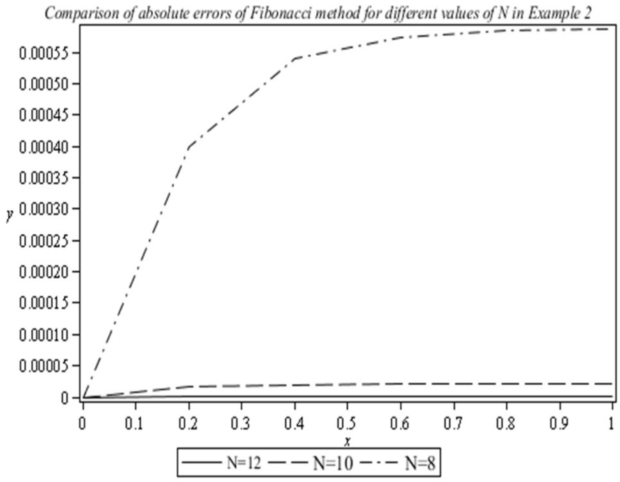

Figure 1. Comparison absolute errors of present Fibonacci approach of Example 2 for $N = 8,10,12$.

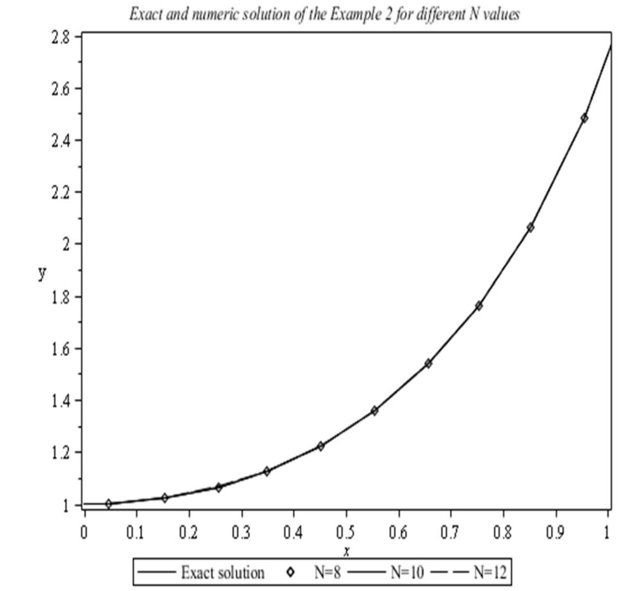

Figure 2. Comparison exact and numeric solution of present Fibonacci approach of Example 2 for $N = 8,10,12$.

Example 3. Take into account a second order differential equation of Lane-Emden type [6]

$$\psi''(3z-1)+\frac{2}{z}\psi'(2z)+z\psi(z+1)=$$

$$z^4+3z^3+3z^2+44z-6, \qquad (39)$$

$$0 \le z \le 1$$

with the initial conditions

$$\psi(0)=1,\ \psi'(0)=0 \qquad (40)$$

The exact solution of the differential equation given above is known as $\psi(z)=z^3+1$. If the solution technique proposed above is used, the unknown Fibonacci coefficients, respectively,

$A_3 = [1\ -2\ \ 0\ \ 1]$, $A_4 = [1\ -2\ \ 0\ \ 1\ \ 0]$

and $A_{11} = [1\ -2\ \ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0]$ are found for $N = 3,4,11$. So that, the exact solution and the approximate solutions obtained are same.

6. CONCLUSION

In this study, the FPAM is presented to differential equations of Lane-Emden type as initial value problems. In short, if we need to explain this technique, Firstly, the truncated Fibonacci polynomial is written in the form of the row vector and the unknown Fibonacci coefficients are written in the form of the column vector. Secondly, the form of matrix of the derivatives of the Fibonacci polynomial is generated using the operational matrix. Then, Lane-Emden type differential equation is converted to matrix form by using collocation points. If the initial values are written in the form of a matrix and necessary arrangements are made, then the expanded matrix is obtained. When the generalized matrix is solved, unknown Fibonacci coefficients are found. So, if these coefficients are applied to the truncated Fibonacci polynomial with unknown coefficients, then the desired solution is found.
Also, explanatory and comparative examples are given to show that FPAM technique is a viable technique. It can be seen from the solved examples that the FPAM can be give better results than some polynomial approach such as Laguerre polynomial approach. The FPAM can be expected an encouraging method for reaching an analytic solution to than Lane-Emden type differential equations.

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