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On Lacunary ideal convergence of some sequences

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Abstract: In this paper, new classes of lacunary ideal convergent and lacunary ideal bounded sequences combining an infinite matrix and an Orlicz function are defined. Some properties of these spaces are investigated and also some inclusion relations are obtained.

Keywords: Invariant means, ideal convergence, Lacunary sequence, Orlicz functions.

1 Introduction

The idea of statistical convergence of real sequences was presented by Fast [2] as a generalization of ordinary convergence. It is a practical tool to study the problems related to convergence of numerical sequences by means of the concept of density. Subsequently, Kostyrko [11] extended the set of statistical convergent sequences to ideal convergent sequences by the aid of ideal $\mathscr{I}$ which is a family of subsets of natural numbers $\mathbb{N}$. Since then, ideal convergence has been studied by many researchers including Kostyrko et al [12], Mursaleen and Alotaibi [22], Mursaleen and Mohiuddine [23,24], Mursaleen et al [25], Das [1], Komisarski [10], Lahiri and Das [15], Şahiner et al [28], Gürdal et al [5,7,6], Tripathy and Hazarika [29,30] some of which are on topological spaces and normed spaces.

A family $\mathscr{I}$ of subsets of a non-empty set X is called an *ideal* on X if for each $A,B \in \mathscr{I}$, we have $A \cup B \in \mathscr{I}$ and for each $A \in \mathscr{I}$ and $B \subseteq A$, we have $B \in \mathscr{I}$. If $X \notin \mathscr{I}$, it is called a non-trivial ideal. A non-trivial ideal is said to be admissible if it contains all finite subsets of X. Throughout the study, by $\mathscr{I}$, we mean an admissible ideal on $\mathbb{N}$.

Recall that a sequence $x = (x_k)$ of reel numbers is said to be ideal convergent to a real number l if for every $\varepsilon > 0$ the set $\{k \in \mathbb{N} : |x_k - l| \geq \varepsilon\}$ belongs to the ideal ([11]). A sequence $x = (x_k)$ of real numbers is called ideal bounded if there is a $K > 0$ such that $\{k \in \mathbb{N} : |x_k| > K\} \in \mathscr{I}$ ([12]).

By ω, we denote the space of all real valued sequences. Let $\Theta = (\theta_r)$ be an increasing sequences of positive integers such that $\theta_0 = 0$ and $h_r = \theta_r - \theta_{r-1} \to \infty$ as $r \to \infty$. Then Θ is called a lacunary sequence and the intervals $(\theta_{r-1} - \theta_r]$ specified by Θ is detoned by Λ_r for all $r \in \mathbb{N}$. Using lacunary sequences, Tripathy et al [31] introduced the space of lacunary ideal convergent sequences

$$\left\{ x = (x_j) \in \omega : \{r \in \mathbb{N} : \frac{1}{h_r} \sum_{j \in \Lambda_r} |x_j - L| \geq \varepsilon\} \in \mathscr{I} \text{ for every } \varepsilon > 0 \text{ and some } L \right\}$$

which is more extensive than lacunary strongly convergent sequence space defined by Freedman et al [3].

A function $M : [0,\infty) \to [0,\infty)$ is called an *Orlicz function* if M is continuous, nondecreasing and convex with $M(0) = 0$,

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$M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. By convexity of M and $M(0) = 0$, it is obtained that $M(\lambda x) \leq \lambda M(x)$ for all $\lambda \in (0,1)$.

In [14], it is said that M satisfies Δ_2-condition for all $x \in [0,\infty)$ if there exists a constant $K > 0$ such that $M(2x) \leq KM(x)$ and it can be easily seen that $K > 2$. Also this is equivalent to the satisfaction of condition $M(Lx) \leq KLM(x)$, where $L > 1$.

By using the idea of Orlicz function, Lindenstrauss and Tzafriri [16] defined *Orlicz sequence space*

$$\ell_M = \left\{ x \in \omega : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}$$

which is a Banach space with the norm

$$\|x\| = \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}.$$

In the literature many authors defined various types of ideal convergent sequence spaces by using Orlicz functions, infinite matrices, lacunary sequences. Some of them can be found in [4,8,9,19,20,21,27].

Let σ be an injective mapping from the set of the positive integers to itself such that $\sigma^p(n) \neq n$ for all positive integers n and p, where $\sigma^p(n) = \sigma(\sigma^{p-1}(n))$. An invariant mean or a σ-mean is a continuous linear functional defined on the space of all bounded sequences ℓ_∞ satisfying following conditions for all $x = (x_n) \in \ell_\infty$,

1.if $x_n \geq 0$ for all n, then $\varphi(x) \geq 0$,
2.$\varphi(e) = 1$, where $e = (1,1,1,\ldots)$,
3.$\varphi(Sx) = \varphi(x)$, where $Sx = (x_{\sigma(n)})$.

V_σ denotes the set of bounded sequences all of whose invariant means are equal which is also called as the space of σ-convergent sequences. In [26], it is defined by

$$V_\sigma = \{x \in \ell_\infty : \lim_k t_{kn}(x) = l, \text{ uniformly in } n, l = \sigma - \lim x\},$$

where

$$t_{kn}(x) = \frac{x_n + x_{\sigma^1(n)} + \ldots + x_{\sigma^k(n)}}{k+1}.$$

σ-mean is called a Banach limit if σ is the translation mapping $n \to n+1$. In this case, V_σ becomes the set of almost convergent sequences which is denoted by $\hat{c}$ and defined in [17] as

$$\hat{c} = \{x \in \ell_\infty : \lim_k d_{kn}(x) \text{ exists uniformly in } n\},$$

where

$$d_{kn}(x) = \frac{x_n + x_{n+1} + \ldots + x_{n+k}}{k+1}.$$

The space of strongly almost convergent sequences was defined by Maddox [18] as follow:

$$[\hat{c}] = \{x \in \ell_\infty : \lim_k d_{kn}(|x - le|) \text{ exists uniformly in } n \text{ for some } l\}.$$

Let $p=(p_k)$ be a sequence of positive real numbers such that $0<h=\inf p_k\leq p_k\leq H=\sup p_k<\infty$. For each $k\in\mathbb{N}$ the inequalities

$$|\alpha_k+\beta_k|^{p_k}\leq D\{|\alpha_k|^{p_k}+|\beta_k|^{p_k}\} \tag{1}$$

and

$$|\alpha|^{p_k}\leq \max\{1,|\alpha|^H\} \tag{2}$$

hold, where $\alpha,\alpha_k,\beta_k\in\mathbb{C}$ and $D=\max\left\{1,2^{H-1}\right\}$.

Let $A=(a_{ij})$ be an infinite matrix of complex numbers a_{ij}, where $i,j\in\mathbb{N}$. We write $Ax=(A_i(x))$ if $A_i(x)=\sum_{j=1}^{\infty}a_{ij}x_j$ converges for each $i\in\mathbb{N}$. Throughout the text, by $t_{kn}(Ax)$, we mean

$$t_{kn}(Ax)=\frac{A_n(x)+A_{\sigma^1(n)}(x)+...+A_{\sigma^k(n)}(x)}{k+1}$$

for all $k,n\in\mathbb{N}$.

A sequence space X is called as *solid (or normal)* if $(\gamma_k x_k)\in X$ whenever $(x_k)\in X$ and (γ_k) is a sequence of scalars such that $|\gamma_k|\leq 1$ for all $k\in\mathbb{N}$.

Let X be a sequence space and $K=\{k_1<k_2<...\}\subseteq\mathbb{N}$. The sequence space $Z_K^X=\{(x_{k_n})\in\omega:(x_n)\in X\}$ is called *K-step space* of X.

A *canonical preimage* of a sequence $(x_{k_n})\in Z_K^X$ is a sequence $(y_n)\in\omega$ defined by

$$y_n=\begin{cases}x_n, & \text{if } n\in\mathbb{N},\\ 0, & \text{otherwise.}\end{cases}$$

A sequence space X is *monotone* if it contains the canonical preimages of all its step spaces.

Lemma 1. *([13],p.53) If a sequence space X is solid, then X is monotone.*

2 Main results

Throughout the study, $p=(p_k)$ and $q=(q_k)$ be bounded sequences of positive real numbers. Let A be an infinite matrix and M be an Orlicz function. We define the following spaces:

$$\mathscr{I}-N^0_{\theta,\sigma}(M,A,p)=\left\{x\in\omega:\mathscr{I}-\lim_r\frac{1}{h_r}\sum_{k\in I_r}\left[M\left(\frac{|t_{kn}(Ax)|}{\rho}\right)\right]^{p_k}=0\text{ for all }n\in\mathbb{N}\text{ and some }\rho>0\right\},$$

$$\mathscr{I}-N_{\theta,\sigma}(M,A,p)=\left\{x\in\omega:\mathscr{I}-\lim_r\frac{1}{h_r}\sum_{k\in I_r}\left[M\left(\frac{|t_{kn}(Ax-le)|}{\rho}\right)\right]^{p_k}=0\text{ for all }n\in\mathbb{N},\text{ some }\rho>0\text{ and }l\in\mathbb{C}\right\},$$

$$\mathscr{I}-N^\infty_{\theta,\sigma}(M,A,p)=\left\{x\in\omega:\frac{1}{h_r}\sum_{k\in I_r}\left[M\left(\frac{|t_{kn}(Ax)|}{\rho}\right)\right]^{p_k}\text{ is }\mathscr{I}\text{-bounded for all }n\in\mathbb{N}\text{ and some }\rho>0\right\}.$$

If we take $p_k = 1$ for all $k \in \mathbb{N}$, then the above spaces are denoted by $\mathscr{I} - N^0_{\theta,\sigma}(M,A)$, $\mathscr{I} - N_{\theta,\sigma}(M,A)$, $\mathscr{I} - N^\infty_{\theta,\sigma}(M,A)$, respectively.

Theorem 1. *The spaces $\mathscr{I} - N^0_{\theta,\sigma}(M,A,p)$, $\mathscr{I} - N_{\theta,\sigma}(M,A,p)$, $\mathscr{I} - N^\infty_{\theta,\sigma}(M,A,p)$ are linear spaces.*

Proof. We prove the result for the space $\mathscr{I} - N^0_{\theta,\sigma}(M,A,p)$. The proofs for the other spaces follow similarly. Let x and y be any two elements of the space $\mathscr{I} - N^0_{\theta,\sigma}(M,A,p)$. Then there exist $\rho_1 > 0$ and $\rho_2 > 0$ such that

$$\mathscr{I} - \lim_r \frac{1}{h_r} \sum_{k\in\Lambda_r} \left[M\left(\frac{|t_{kn}(Ax)|}{\rho_1} \right) \right]^{p_k} = 0$$

and

$$\mathscr{I} - \lim_r \frac{1}{h_r} \sum_{k\in\Lambda_r} \left[M\left(\frac{|t_{kn}(Ay)|}{\rho_2} \right) \right]^{p_k} = 0.$$

That is, for every $\varepsilon > 0$, we have that

$$T_1 = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k\in\Lambda_r} \left[M\left(\frac{|t_{kn}(Ax)|}{\rho_1} \right) \right]^{p_k} \geq \frac{\varepsilon}{2D} \right\} \in \mathscr{I}$$

and

$$T_2 = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k\in\Lambda_r} \left[M\left(\frac{|t_{kn}(Ay)|}{\rho_2} \right) \right]^{p_k} \geq \frac{\varepsilon}{2D} \right\} \in \mathscr{I}.$$

Since M is non-decreasing and convex, it follows from inequality 1 that

$$\begin{aligned} \frac{1}{h_r} \sum_{k\in\Lambda_r} \left[M\left(\frac{|t_{kn}(A(ax+by))|}{\rho} \right) \right]^{p_k} &\leq \frac{1}{h_r} \sum_{k\in\Lambda_r} \left[M\left(\frac{|t_{kn}(Ax)|}{\rho_1} \right) + M\left(\frac{|t_{kn}(Ay)|}{\rho_2} \right) \right]^{p_k} \\ &\leq D \left\{ \frac{1}{h_r} \sum_{k\in\Lambda_r} \left[M\left(\frac{|t_{kn}(Ax)|}{\rho_1} \right) \right]^{p_k} + \frac{1}{h_r} \sum_{k\in\Lambda_r} \left[M\left(\frac{|t_{kn}(Ay)|}{\rho_2} \right) \right]^{p_k} \right\}, \end{aligned}$$

where $\rho = \max\{2|a|\rho_1, 2|b|\rho_2\}$ and $a, b \in \mathbb{C}$.

If $r \notin T_1 \cup T_2$, then we obtain that

$$\frac{1}{h_r} \sum_{k\in\Lambda_r} \left[M\left(\frac{|t_{kn}(A(ax+by))|}{\rho} \right) \right]^{p_k} < \varepsilon.$$

Therefore the inclusion

$$\begin{aligned} \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k\in\Lambda_r} \left[M\left(\frac{|t_{kn}(A(ax+by))|}{\rho} \right) \right]^{p_k} \geq \varepsilon \right\} &\subseteq \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k\in\Lambda_r} \left[M\left(\frac{|t_{kn}(Ax)|}{\rho_1} \right) \right]^{p_k} \geq \frac{\varepsilon}{2D} \right\} \\ &\cup \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k\in\Lambda_r} \left[M\left(\frac{|t_{kn}(Ay)|}{\rho_2} \right) \right]^{p_k} \geq \frac{\varepsilon}{2D} \right\} \end{aligned}$$

holds. Since the sets in the right side of the inclusion belong to the ideal, this implies that

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k\in\Lambda_r} \left[M\left(\frac{|t_{kn}(A(ax+by))|}{\rho} \right) \right]^{p_k} \geq \varepsilon \right\} \in \mathscr{I}$$

which means $ax + by \in \mathscr{I} - N^0_{\theta,\sigma}(M,A,p)$. Hence we conclude that $\mathscr{I} - N^0_{\theta,\sigma}(M,A,p)$ is a linear space.

Theorem 2. *Let M_1 and M_2 Orlicz functions , then we have*

$$\mathscr{I}-N^0_{\theta,\sigma}(M_1,A,p)\cap\mathscr{I}-N^0_{\theta,\sigma}(M_2,A,p)\subseteq\mathscr{I}-N^0_{\theta,\sigma}(M_1+M_2,A,p),$$

$$\mathscr{I}-N_{\theta,\sigma}(M_1,A,p)\cap\mathscr{I}-N_{\theta,\sigma}(M_2,A,p)\subseteq\mathscr{I}-N_{\theta,\sigma}(M_1+M_2,A,p),$$

$$\mathscr{I}-N^\infty_{\theta,\sigma}(M_1,A,p)\cap\mathscr{I}-N^\infty_{\theta,\sigma}(M_2,A,p)\subseteq\mathscr{I}-N^\infty_{\theta,\sigma}(M_1+M_2,A,p).$$

Proof. Let $x\in\mathscr{I}-N^0_{\theta,\sigma}(M_1,A,p)\cap\mathscr{I}-N^0_{\theta,\sigma}(M_2,A,p)$. Then, given any $\varepsilon>0$ we have

$$T_1=\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{k\in\Lambda_r}\left[M_1\left(\frac{|t_{kn}(Ax)|}{\rho}\right)\right]^{p_k}\geq\frac{\varepsilon}{2D}\right\}\in\mathscr{I}$$

and

$$T_2=\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{k\in\Lambda_r}\left[M_2\left(\frac{|t_{kn}(Ax)|}{\rho}\right)\right]^{p_k}\geq\frac{\varepsilon}{2D}\right\}\in\mathscr{I}$$

for some $\rho>0$. Let $r\notin T_1\cup T_2$. By the following inequality

$$\frac{1}{h_r}\sum_{k\in\Lambda_r}\left[(M_1+M_2)\left(\frac{|t_{kn}(A(x)|}{\rho}\right)\right]^{p_k}=\frac{1}{h_r}\sum_{k\in\Lambda_r}\left[M_1\left(\frac{|t_{kn}(A(x)|}{\rho}\right)+M_2\left(\frac{|t_{kn}(A(x)|}{\rho}\right)\right]^{p_k}$$
$$\leq D\left\{\frac{1}{h_r}\sum_{k\in\Lambda_r}\left[M_1\left(\frac{|t_{kn}(A(x)|}{\rho}\right)\right]^{p_k}+\frac{1}{h_r}\sum_{k\in\Lambda_r}\left[M_2\left(\frac{|t_{kn}(A(x)|}{\rho}\right)\right]^{p_k}\right\}$$

and definition of ideal, we obtain that $\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{k\in\Lambda_r}\left[(M_1+M_2)\left(\frac{|t_{kn}(A(x)|}{\rho}\right)\right]^{p_k}\geq\varepsilon\right\}\in\mathscr{I}$. This means $x\in\mathscr{I}-N^0_{\theta,\sigma}(M_1+M_2,A,p)$ and completes the prof. The proof for the other cases follows similarly.

Theorem 3. *Let M_1 and M_2 be Orlicz functions and M_2 satisfy Δ_2 condition. Then the inclusions*

$$\mathscr{I}-N^0_{\theta,\sigma}(M_1,A,p)\subseteq\mathscr{I}-N^0_{\theta,\sigma}(M_2\circ M_1,A,p),$$

$$\mathscr{I}-N_{\theta,\sigma}(M_1,A,p)\subseteq\mathscr{I}-N_{\theta,\sigma}(M_2\circ M_1,A,p),$$

$$\mathscr{I}-N^\infty_{\theta,\sigma}(M_1,A,p)\subseteq\mathscr{I}-N^\infty_{\theta,\sigma}(M_2\circ M_1,A,p)$$

hold.

Proof. Let $x\in\mathscr{I}-N^0_{\theta,\sigma}(M_1,A,p)$. Then, for some $\rho>0$ we have that $\mathscr{I}-\lim_r\frac{1}{h_r}\sum_{k\in\Lambda_r}\left[M_1\left(\frac{|t_{kn}(Ax)|}{\rho}\right)\right]^{p_k}=0$. Given any $\varepsilon>0$ choose δ with $0<\delta<1$ such that $M_2(u)<\varepsilon$ whenever $0\leq u\leq\delta$.

Firstly, assume that $M_1\left(\frac{|t_{kn}(Ax)|}{\rho}\right)>\delta$. Since M_2 satisfies Δ_2 condition, there exists $K>1$ such that

$$M_2\left(M_1\left(\frac{|t_{kn}(Ax)|}{\rho}\right)\delta^{-1}2\right)\leq KM_1\left(\frac{|t_{kn}(Ax)|}{\rho}\right)\delta^{-1}M_2(2). \tag{3}$$

By using inequalities (2) and (3), we obtain that

$$\begin{aligned}\frac{1}{h_r}\sum_{k\in\Lambda_r}\left[M_2\left(M_1\left(\frac{|t_{kn}(Ax)|}{\rho}\right)\right)\right]^{p_k} &\leq \frac{1}{h_r}\sum_{k\in\Lambda_r}\left[M_2\left(1+M_1\left(\frac{|t_{kn}(Ax)|}{\rho}\right)\delta^{-1}\right)\right]^{p_k}\\ &\leq \frac{1}{h_r}\sum_{k\in\Lambda_r}\left[\frac{1}{2}M_2(2)+\frac{1}{2}M_2\left(M_1\left(\frac{|t_{kn}(Ax)|}{\rho}\right)\delta^{-1}2\right)\right]^{p_k}\\ &\leq \frac{1}{h_r}\sum_{k\in\Lambda_r}\left[\frac{1}{2}M_2(2)+\frac{1}{2}KM_1\left(\frac{|t_{kn}(Ax)|}{\rho}\right)\delta^{-1}M_2(2)\right]^{p_k}\\ &\leq \frac{1}{h_r}\sum_{k\in\Lambda_r}\left[KM_1\left(\frac{|t_{kn}(Ax)|}{\rho}\right)\delta^{-1}M_2(2)\right]^{p_k}\\ &\leq \max\left\{1,(K\delta^{-1}M_2(2))^H\right\}\frac{1}{h_r}\sum_{k\in\Lambda_r}\left[M_1\left(\frac{|t_{kn}(Ax)|}{\rho}\right)\right]^{p_k}.\end{aligned}$$

The following inclusion follows from the last inequality

$$\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{k\in\Lambda_r}\left[M_2\left(M_1\left(\frac{|t_{kn}(Ax)|}{\rho}\right)\right)\right]^{p_k}\geq\varepsilon\right\}\subseteq\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{k\in\Lambda_r}\left[M_1\left(\frac{|t_{kn}(Ax)|}{\rho}\right)\right]^{p_k}\geq\frac{\varepsilon}{\max\left\{1,(K\delta^{-1}M_2(2))^H\right\}}\right\}.$$

Hence the set $\left\{k\in\mathbb{N}:\frac{1}{h_r}\sum_{k\in\Lambda_r}\left[M_2\left(M_1\left(\frac{|t_{kn}(Ax)|}{\rho}\right)\right)\right]^{p_k}\geq\varepsilon\right\}$ belongs to ideal.

Now, suppose that $M_1\left(\frac{|t_{kn}(A(x)|}{\rho}\right)\leq\delta$. Since M_2 is continuous, for all $\varepsilon>0$ we have $M_2\left(M_1\left(\frac{|t_{kn}(Ax)|}{\rho}\right)\right)<\varepsilon$. This yields $\frac{1}{h_r}\sum_{k\in\Lambda_r}\left[M_2\left(M_1\left(\frac{|t_{kn}(Ax)|}{\rho}\right)\right)\right]^{p_k}\leq\max\{\varepsilon,\varepsilon^h\}$. Thus, we conclude that $\mathscr{I}-\lim_r\frac{1}{h_r}\sum_{k\in\Lambda_r}\left[M_2\left(M_1\left(\frac{|t_{kn}(Ax)|}{\rho}\right)\right)\right]^{p_k}=0$ as $\varepsilon\to0$ and $r\to\infty$.

This completes the proof and the other cases can be proved similarly.

Theorem 4. *If* $\sup_k[M(u)]^{p_k}<\infty$ *for all* $u\geq0$*, then the inclusion*

$$\mathscr{I}-N_{\theta,\sigma}(M,A,p)\subseteq\mathscr{I}-N^{\infty}_{\theta,\sigma}(M,A,p)$$

holds.

Proof. Let $x\in\mathscr{I}-N_{\theta,\sigma}(M,A,p)$. The inequality

$$\frac{1}{h_r}\sum_{k\in\Lambda_r}\left[M\left(\frac{|t_{kn}(A(x)|}{\rho}\right)\right]^{p_k}\leq D\left\{\frac{1}{h_r}\sum_{k\in\Lambda_r}\left[M\left(\frac{|t_{kn}(Ax-le)|}{\rho_1}\right)\right]^{p_k}+\frac{1}{h_r}\sum_{k\in\Lambda_r}\left[M\left(\frac{|t_{kn}(le)|}{\rho_1}\right)\right]^{p_k}\right\}$$

holds by (1), where $\rho=2\rho_1$. By hypothesis, we have $\frac{1}{h_r}\sum_{k\in\Lambda_r}\left[M\left(\frac{|t_{kn}(le)|}{\rho_1}\right)\right]^{p_k}<\infty$. Hence, we obtain

$$\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{k\in\Lambda_r}\left[M\left(\frac{|t_{kn}(Ax)|}{\rho}\right)\right]^{p_k}\geq K\right\}\subseteq\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{k\in\Lambda_r}\left[M\left(\frac{|t_{kn}(Ax-le)|}{\rho_1}\right)\right]^{p_k}\geq\varepsilon\right\}$$

for some $K>0$. This completes the proof.

Theorem 5. *Let* $0<p_k\leq q_k<\infty$ *for every* $k\in\mathbb{N}$ *and* $(\frac{q_k}{p_k})$ *be bounded. Then we have*

$$\mathscr{I}-N^0_{\theta,\sigma}(M,A,q)\subseteq\mathscr{I}-N^0_{\theta,\sigma}(M,A,p),$$

$$\mathscr{I}-N_{\theta,\sigma}(M,A,q)\subseteq\mathscr{I}-N_{\theta,\sigma}(M,A,p).$$

*Proof.*It can be easily proved with the same technique as in Theorem 4 in [9].

Theorem 6. *The spaces $\mathscr{I}-N^0_{\theta,\sigma}(M,A,p)$ and $\mathscr{I}-N^\infty_{\theta,\sigma}(M,A,p)$ are solid.*

Proof. We prove the result for the space $\mathscr{I}-N^0_{\theta,\sigma}(M,A,p)$. For $\mathscr{I}-N^\infty_{\theta,\sigma}(M,A,p)$, the result can be proved similarly.

Let $x\in\mathscr{I}-N^0_{\theta,\sigma}(M,A,p)$. Then for every $\varepsilon>0$, we have

$$\left\{k\in\mathbb{N}:\frac{1}{h_r}\sum_{k\in\Lambda_r}\left[M\left(\frac{|t_{kn}(Ax)|}{\rho}\right)\right]^{p_k}\geq\varepsilon\right\}\in\mathscr{I}.$$

If $\lambda=(\lambda_k)$ is a sequence of scalars such that $|\lambda_k|\leq 1$ for all $k\in\mathbb{N}$, then the following inequality holds:

$$\frac{1}{h_r}\sum_{k\in\Lambda_r}\left[M\left(\frac{|t_{kn}(A\lambda x)|}{\rho}\right)\right]^{p_k}\leq\frac{1}{h_r}\sum_{k\in\Lambda_r}\left[M\left(\frac{|t_{kn}(Ax)|}{\rho}\right)\right]^{p_k}.$$

Hence we obtain

$$\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{k\in\Lambda_r}\left[M\left(\frac{|t_{kn}(A\lambda x)|}{\rho}\right)\right]^{p_k}\geq\varepsilon\right\}\subseteq\left\{k\in\mathbb{N}:\frac{1}{h_r}\sum_{k\in\Lambda_r}\left[M\left(\frac{|t_{kn}(Ax)|}{\rho}\right)\right]^{p_k}\geq\varepsilon\right\}$$

and so

$$\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{k\in\Lambda_r}\left[M\left(\frac{|t_{kn}(A\lambda x)|}{\rho}\right)\right]^{p_k}\geq\varepsilon\right\}\in\mathscr{I}$$

which means $\lambda x\in\mathscr{I}-N^0_{\theta,\sigma}(M,A,p)$.

We conclude that the space $\mathscr{I}-N^0_{\theta,\sigma}(M,A,p)$ is solid.

Corollary 1. *The spaces $\mathscr{I}-N^0_{\theta,\sigma}(M,A,p)$ and $\mathscr{I}-N^\infty_{\theta,\sigma}(M_1,A,p)$ are monotone.*

Proof. The proof follows from Lemma 1.

Theorem 7. *If $\lim_k p_k>0$ and $x\to l(\mathscr{I}-N_{\theta,\sigma}(M,A,p))$, then l is unique.*

Proof. Let $\lim_k p_k=p_0>0$. Assume that $x\to l(\mathscr{I}-N_{\theta,\sigma}(M,A,p))$ and $x\to l'(\mathscr{I}-N_{\theta,\sigma}(M,A,p))$. Then there exist $\rho_1,\rho_2>0$ such that

$$\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{k\in\Lambda_r}\left[M\left(\frac{|t_{kn}(Ax-le)|}{\rho_1}\right)\right]^{p_k}\geq\frac{\varepsilon}{2D}\right\}\in\mathscr{I}$$

and

$$\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{k\in\Lambda_r}\left[M\left(\frac{|t_{kn}(Ax-l'e)|}{\rho_2}\right)\right]^{p_k}\geq\frac{\varepsilon}{2D}\right\}\in\mathscr{I}$$

for every $\varepsilon>0$. Put $\rho=\max\{2\rho_1,2\rho_2\}$. Hence we have

$$\frac{1}{h_r}\sum_{k\in\Lambda_r}\left[M\left(\frac{|t_{kn}(le-l'e)|}{\rho}\right)\right]^{p_k}\leq D\left\{\frac{1}{h_r}\sum_{k\in\Lambda_r}\left[M\left(\frac{|t_{kn}(Ax-le)|}{\rho_1}\right)\right]^{p_k}+\frac{1}{h_r}\sum_{k\in\Lambda_r}\left[M\left(\frac{|t_{kn}(Ax-l'e)|}{\rho_2}\right)\right]^{p_k}\right\}.$$

This inequality implies that $\left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in \Lambda_r} \left[M\left(\frac{|t_{kn}(le-l'e)|}{\rho}\right)\right]^{p_k} \geq \varepsilon\right\} \in \mathscr{I}$; that is, $\mathscr{I} - \lim_r \frac{1}{h_r} \sum_{k \in \Lambda_r} \left[M\left(\frac{|t_{kn}(le-l'e)|}{\rho}\right)\right]^{p_k} = 0$. Also we have

$$\left[M\left(\frac{|t_{kn}(le-l'e)|}{\rho}\right)\right]^{p_k} \to \left[M\left(\frac{|t_{kn}(le-l'e)|}{\rho}\right)\right]^{p_0}$$

as $k \to \infty$ and so $\left[M\left(\frac{|t_{kn}(le-l'e)|}{\rho}\right)\right]^{p_0} = 0$. Hence, we conclude that $l = l'$.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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