Hacettepe Journal of Mathematics and Statistics
Volume 44 (2) (2015), 295 – 315

# Atanassov's intuitionistic fuzzy grade of complete hypergroups of order less than or equal to 6

Bijan Davvaz * , E. Hassani Sadrabadi† and Irina Cristea‡

**Abstract**

The length of the sequence of join spaces and Atanassov's intuitionistic fuzzy sets associated with a hypergroupoid $H$ is called the intuitionistic fuzzy grade of $H$. In this paper, we consider the class of the complete hypergroups of order less than or equal to 6, determining their intuitionistic fuzzy grade.

*2000 AMS Classification:* 20N20; 03E72.

**Keywords:** Fuzzy set, Atanassov's intuitionistic fuzzy set, Complete hypergroup, Fuzzy grade, Join space.

Received 07 /05 /2011 : Accepted 04 /02 /2013 Doi : 10.15672/HJMS.2015449427

## 1. Introduction

The study of the connections between hyperstructures and fuzzy sets [31] (or Atanassov's intuitionistic fuzzy sets [1, 2]) opens a new field of research in fuzzy algebraic structures theory, theory initiated by Rosenfeld [28]: he showed that many results concerning groups may be extended in a natural way to fuzzy groups. The notion of fuzzy group has been generalized by Davvaz [19], introducing the concept of fuzzy subhypergroup of a hypergroup. Later on, this subject has been studied in depth also in connection with other structures, like rings [22], modules [20], $n$-ary hypergroups [21], complete hypergroups, etc. For example, Cristea and Darafsheh [16, 17], investigating a particular fuzzy subhypergroup of a complete hypergroup, have found a new decomposition of the group $\mathbb{Z}_n$, when $n \in \{p, p^2, pq\}$, for $p$ and $q$ prime numbers. The books [3, 10, 23, 30] are surveys of the theory of algebraic hyperstructures and their applications.

*Department of Mathematics, Yazd University, Yazd, Iran.
Email: davvaz@yazd.ac.ir
†Department of Mathematics, Yazd University, Yazd, Iran.
Email:hassanipma@yahoo.com
‡Centre for Systems and Information Technologies, University of Nova Gorica, SI-5000, Vipavska 13, Nova Gorica, Slovenia.
Email:irinacri@yahoo.co.uk Corresponding Author.

Two fundamental relations between hyperstructures and fuzzy sets were considered by P.Corsini; he associated a join space with a fuzzy set [4], and then a fuzzy set with a hypergroupoid $H$ [5]. These connections lead to a sequence of fuzzy sets and join spaces, which ends if two consecutive join spaces are isomorphic. The length of this sequence is called the fuzzy grade of the hypergroupoid $H$. Till now, one determined the fuzzy grade of the i.p.s. hypergroups of order less than or equal to 7 [6, 7], of the complete hypergroups or 1-hypergroups which are not complete [8, 14]. Moreover, several properties of the above sequence has been determined in the general case [29], and also for the direct product of two hypergroupoids [15]. Corsini et al. studied the same sequence associated with a hypergraph [11, 12], and with multivalued functions [13]. Cristea and Davvaz [18] extended the notion of fuzzy grade of a hypergroupoid to that of intuitionistic fuzzy grade.

The study of the fuzzy grade and intuitionistic fuzzy grade of remarkable classes of finite hypergroups helps us to identify some important properties which could be generalized for any finite hypergroup. For example, calculating intuitionistic fuzzy grade of the i.p.s. hypergroups of order 7, we noticed that, some times, the sequence of join spaces associated with an i.p.s. hypergroup is cyclic (see [24, 25]).

The study conducted in this note shows that the intuitionistic fuzzy grade of a complete hypergroup $H$ of cardinality $n$ depends, not only on the decomposition of $n$, as in the case of the fuzzy grade of a complete hypergroup, but also on the group used in the construction of $H$. We believe that the aspects treated in this particular case serve as a foundation, starting point for further research on the intuitionistic fuzzy grade of an arbitrary finite complete hypergroup.

Inspired and motivated by the above achievements, in this paper, we will construct the sequences of join spaces and Atanassov's intuitionistic fuzzy sets associated with the complete hypergroups of order less than or equal to 6. Our aim is to determine their intuitionistic fuzzy grades in order to make a comparison with their fuzzy grades determined by Cristea [14].

To do so, the paper is organized in the following way. In Section 2 we present some basic notions concerning hypergroups and a short description of the complete hypergroups. In Section 3 we present a brief introduction about the sequence of join spaces and Atanassov's intuitionistic fuzzy sets associated with a hypergroupoid. Section 4 includes the sequences of the intuitionistic fuzzy grades of the complete hypergroups of order less than or equal to 6. Finally, Section 5 concludes the paper, giving also some future lines of our research.

## 2. Preliminaries

In this paper, we adopt the terminology and notation used in [4, 5, 14, 18, 24, 25]. We consider $\langle H, \circ \rangle$ to be a hypergroupoid, where $H$ denotes a non-empty set, $\mathcal{P}^*(H)$ stands for the set of all non-empty subsets of $H$ and $\circ : H^2 \to \mathcal{P}^*(H)$ is a hyperoperation. The image of the pair $(x, y) \in H \times H$ is denoted by $x \circ y$. If $A$ and $B$ are nonempty subsets of $H$, then $A \circ B = \bigcup\limits_{\substack{a \in A \\ b \in B}} a \circ b$.

For the sake of convenience and completeness of our presentation, we recall some basic definitions and properties concerning hypergroups. More details on this argument can be found in the books [3, 10].

**2.1. Definition.** A *hypergroup* is a hypergroupoid $\langle H, \circ \rangle$ which satisfies the following conditions:

($i$) For any $(a, b, c) \in H^3$, $(a \circ b) \circ c = a \circ (b \circ c)$ (the *associativity*),

(*ii*) For any $a \in H, H \circ a = a \circ H = H$ (the *reproducibility*).
If, for any $(x, y) \in H^2, x \circ y = H$, then the hypergroup $H$ is called *total hypergroup*.

For each pair $(a, b) \in H^2$, we denote: $a/b = \{x \in H \mid a \in x \circ b\}$ and $b\backslash a = \{y \in H \mid a \in b \circ y\}$.

**2.2. Definition.** A commutative hypergroup $\langle H, \circ \rangle$ is called a *join space* if, for any four elements $a, b, c, d \in H$, such that $a/b \cap c/d \neq \emptyset$, it follows that $a \circ d \cap b \circ c \neq \emptyset$.

The notion of join space, introduced by Prenowitz, was used by Prenowitz and Jantosciak [27] for the reconstruction, from an algebraic point of view, of several branches of geometry: the projective, the descriptive and the spherical geometry.

**2.3. Definition.** Let $\langle H, \circ \rangle$ and $\langle H', \circ' \rangle$ be two hypergroups and $f : H \to H'$ an application from $H$ in $H'$. We say that

(*i*) $f$ is a *homomorphism* if, for all $(x, y) \in H^2$, $f(x \circ y) \subseteq f(x) \circ' f(y)$.
(*ii*) $f$ is a *good homomorphism* if, for all $(x, y) \in H^2$, $f(x \circ y) = f(x) \circ' f(y)$.

We say that the two hypergroups are *isomorphic*, and we write $H \simeq H'$, if there is a good homomorphism between them which is also a bijection.

The relation $\beta$ on a hypergroupoid $\langle H, \circ \rangle$ is defined as follows:

$$a\beta b \Longleftrightarrow \exists n \in \mathbb{N}^*, \exists (x_1, x_2, \ldots, x_n) \in H^n : a \in \prod_{i=1}^{n} x_i \ni b.$$

Notice that $\beta$ is a reflexive and a symmetric relation on $H$, but generally, not a transitive one. Let us denote by $\beta^*$ the transitive closure of $\beta$. It is well known that, if $H$ is a hypergroup, then $\beta^* = \beta$ and $H/\beta$ is a group[3].

One of the most important notions in hypergroup theory is that of the heart of a hypergroup $H$. Studying its properties one determines completely the structure of the hypergroup $H$.

**2.4. Definition.** The *heart* of a hypergroup $H$ is $\omega_H = \{x \in H \mid \varphi_H(x) = 1\}$, where $\varphi_H : H \longrightarrow H/\beta$ is the canonical projection and 1 is the identity of the group $H/\beta$.

**2.5. Definition.** A hypergroup $H$ is called 1-*hypergroup* if the cardinality of its heart equals 1.

**2.6. Definition.** Let $\langle H, \circ \rangle$ be a hypergroup and $A$ be a non-empty subset of $H$. We say that $A$ is a *complete part* of $H$ if the following implication holds:

$$\forall n \in \mathbb{N}^*, \forall (x_1, x_2, \ldots, x_n) \in H^n, \prod_{i=1}^{n} x_i \cap A \neq \emptyset \Longrightarrow \prod_{i=1}^{n} x_i \subset A.$$

The *complete closure* of $A$ in $H$ is the intersection of all the complete parts of $H$, containing $A$; it is denoted by $\mathcal{C}(A)$.

**2.7. Definition.** A hypergroup $\langle H, \circ \rangle$ is called *complete* if, for any $(x, y) \in H^2$, $\mathcal{C}(x \circ y) = x \circ y$.

The following result concerning the complete hypergroups will be used in the sequel.

**2.8. Theorem.** *Any complete hypergroup may be constructed as the union* $H = \bigcup_{g \in G} A_g$, *where:*

(*i*) *$G$ is a group.*
(*ii*) *The family $\{A_g \mid g \in G\}$ is a partition of $G$.*
(*iii*) *If $(a, b) \in A_{g_1} \times A_{g_2}$, then $a \circ b = A_{g_1 g_2}$.*

For a complete hypergroup $H$, it is known that $\omega_H = A_e$, where $e$ is the identity of the group $G$, and it coincides with the set of identities of $H$. Therefore, by the above representation, we say that any complete hypergroup of order $n$ is characterized by an $m$-tuple denoted $[k_1, k_2, \ldots, k_m]$, where $m = |G|$, $2 \leq m \leq n-1$, $G = \{g_1, g_2, \ldots, g_m\}$ and, for any $i \in \{1, 2, \ldots, m\}, k_i = |A_{g_i}|$. With other words, for determining all the non-isomorphic complete hypergroups of order $n$, it is enough to know the structure of the non-isomorphic groups of order $m$, $2 \leq m \leq n-1$, and all the $m$-decompositions of $n$, i.e. all the ordered systems of natural numbers $[k_1, k_2, \ldots, k_m]$ such that $k_i \geq 1$, $k_1 + k_2 + \ldots + k_m = n$ and $k_2 \leq k_3 \leq \ldots \leq k_m$, for $1 \leq i \leq m$.( see [14])

## 3. Intuitionistic fuzzy grade of hypergroups

In this section, first we recall the construction of the sequence of join spaces and Atanassov's intuitionistic fuzzy sets associated with a hypergroupoid $H$, and then the formulas for the membership functions associated with a complete hypergroup. In this paper $H$ denotes a finite hypergroupoid.

For simplicity, we denote an Atanassov's intuitionistic fuzzy set (by short intuitionistic fuzzy set) $A = \{(x, \mu_A(x), \lambda_A(x)) \mid x \in X\}$, where, for any $x \in X$, the *degree of membership* of $x$ (namely $\mu_A(x)$) and the *degree of non-membership* of $x$ (namely $\lambda_A(x)$) verify the relation $0 \leq \mu_A(x) + \lambda_A(x) \leq 1$, by $A = (\mu, \lambda)$.

For any hypergroupoid $\langle H, \circ \rangle$, Cristea and Davvaz [18] defined an intuitionistic fuzzy set $A = (\bar{\mu}, \bar{\lambda})$ in the following way: for any $u \in H$, one considers:

$$(3.1) \qquad \bar{\mu}(u) = \frac{\displaystyle\sum_{(x,y)\in Q(u)} \frac{1}{|x \circ y|}}{n^2}, \quad \bar{\lambda}(u) = \frac{\displaystyle\sum_{(x,y)\in \bar{Q}(u)} \frac{1}{|x \circ y|}}{n^2},$$

where $Q(u) = \{(a,b) \in H^2 \mid u \in a \circ b\}, \bar{Q}(u) = \{(a,b) \in H^2 \mid u \notin a \circ b\}$. If $Q(u) = \emptyset$, we set $\bar{\mu}(u) = 0$ and similarly, if $\bar{Q}(u) = \emptyset$ we set $\bar{\lambda}(u) = 0$. It is clear that, for any $u \in H$, $0 \leq \bar{\mu}(u) + \bar{\lambda}(u) \leq 1$.

Now, let $A = (\bar{\mu}, \bar{\lambda})$ be an intuitionistic fuzzy set on $H$. One may associate with $H$ two join spaces $\langle {}_0H, \circ_{\bar{\mu}\wedge\bar{\lambda}} \rangle$ and $\langle {}^0H, \circ_{\bar{\mu}\vee\bar{\lambda}} \rangle$, where, for any fuzzy set $\alpha$ on $H$, the hyperproduct "$\circ_\alpha$", introduced by Corsini [4], is defined as

$$(3.2) \qquad x \circ_\alpha y = \{u \in H \mid \alpha(x) \wedge \alpha(y) \leq \alpha(u) \leq \alpha(x) \vee \alpha(y)\}.$$

Using repeatedly the formulas (3.1) and (3.2), one obtains two sequences of join spaces and intuitionistic fuzzy sets associated with $H$, denoted by $({}_iH = \langle {}_iH, \circ_{\bar{\mu}_i\wedge\bar{\lambda}_i} \rangle; \bar{A}_i = (\bar{\mu}_i, \bar{\lambda}_i))_{i\geq 0}$ and $({}^iH = \langle {}^iH, \circ_{\bar{\mu}_i\vee\bar{\lambda}_i} \rangle; \bar{A}_i = (\bar{\mu}_i, \bar{\lambda}_i))_{i\geq 0}$.

The lengths of these sequences are called the lower, and respectively, the upper intuitionistic fuzzy grade of $H$, more exactly:

**3.1. Definition.** (see [18]) A set $H$ endowed with an intuitionistic fuzzy set $A = (\mu, \lambda)$ has the *lower (upper) intuitionistic fuzzy grade* $m$, $m \in \mathbb{N}^*$, and we write $l.i.f.g.(H) = m$ (resp. $u.i.f.g.(H) = m$) if, for any $i$, $0 \leq i < m-1$, the join spaces $\langle {}_iH, \circ_{\bar{\mu}_i\wedge\bar{\lambda}_i} \rangle$ and $\langle {}_{i+1}H, \circ_{\bar{\mu}_{i+1}\wedge\bar{\lambda}_{i+1}} \rangle$ (resp. $\langle {}^iH, \circ_{\bar{\mu}_i\vee\bar{\lambda}_i} \rangle$ and $\langle {}^{i+1}H, \circ_{\bar{\mu}_{i+1}\vee\bar{\lambda}_{i+1}} \rangle$) associated with $H$ are not isomorphic (where ${}_0H = \langle {}_0H, \circ_{\bar{\mu}\wedge\bar{\lambda}} \rangle$ and ${}^0H = \langle {}^0H, \circ_{\bar{\mu}\vee\bar{\lambda}} \rangle$) and for any $s$, $s \geq m$, ${}_sH$ is isomorphic with ${}_{m-1}H$ (resp. ${}^sH$ is isomorphic with ${}^{m-1}H$).

It is important to know that, if we start the construction of the above sequences with a hypergroupoid $\langle H, \circ \rangle$, and not with a set $H$ endowed with an intuitionistic fuzzy set, then we obtain only one sequence of join spaces because, in this case, the join spaces $\langle {}_0H, \circ_{\bar{\mu}\wedge\bar{\lambda}} \rangle$ and $\langle {}^0H, \circ_{\bar{\mu}\vee\bar{\lambda}} \rangle$ are isomorphic (see [18]). In order to explain this situation, one introduces a new concept.

**3.2. Definition.** (see [18]) We say that a hypergroupoid $H$ has the *intuitionistic fuzzy grade* $m, m \in \mathbb{N}^*$, and we write $i.f.g.(H) = m$, if $l.i.f.g.(H) = m$.

A natural question appears: When are these join spaces non-isomorphic? It is clear that it has to be answered for two consecutive join spaces in the built sequence, since in the case of isomorphism, the sequence ends. In order to solve this problem one introduces some notations. Let $({}_iH = \langle {}_iH, \circ_{\bar{\mu}_i \wedge \bar{\lambda}_i} \rangle; \bar{A}_i = (\bar{\mu}_i, \bar{\lambda}_i))_{i \geq 0}$ be the sequence of join spaces and intuitionistic fuzzy sets associated with a hypergroupoid $H$. Then, for any $i$, there are $r$, namely $r = r_i$, and a partition $\Pi = \{{}^iC_j\}_{j=1}^r$ of ${}_iH$ such that, for any $j \geq 1, x, y \in {}^iC_j \Longleftrightarrow \bar{\mu}_i(x) \wedge \bar{\lambda}_i(x) = \bar{\mu}_i(y) \wedge \bar{\lambda}_i(y)$. For $x \in H$, we denote $\lambda(x) = i_j$, when $x \in {}^iC_j$. On the set of the classes $\{{}^iC_j\}_{j=1}^r$ we define the following ordering relation:

$i_j < i_k$ if, for elements $x \in {}^iC_j$ and $y \in {}^iC_k$,

$$\bar{\mu}_i(x) \wedge \bar{\lambda}_i(x) < \bar{\mu}_i(y) \wedge \bar{\lambda}_i(y) \text{ (therefore } \lambda(x) < \lambda(y)).$$

With any ordered chain $({}^iC_{j_1}, {}^iC_{j_2}, \dots, {}^iC_{j_r})$ one associates an ordered $r$-tuple of the type $(k_{j_1}, k_{j_2}, \dots, k_{j_r})$, where $k_{j_l} = |{}^iC_{j_l}|$, for all $l, 1 \leq l \leq r$.

**3.3. Theorem.** *(see [9]) Let ${}_iH$ and ${}_{i+1}H$ be the join spaces associated with $H$ determined by the membership functions $\bar{\mu}_i \wedge \bar{\lambda}_i$ and $\bar{\mu}_{i+1} \wedge \bar{\lambda}_{i+1}$, where ${}^iH = \bigcup_{l=1}^{r_1} C_l$, ${}^{i+1}H = \bigcup_{l=1}^{r_2} C'_l$ and $(k_1, k_2, \dots, k_{r_1})$ is the $r_1$-tuple associated with ${}_iH$, $(k'_1, k'_2, \dots, k'_{r_2})$ is the $r_2$-tuple associated with ${}_{i+1}H$. The join spaces ${}_iH$ and ${}_{i+1}H$ are isomorphic if and only if $r_1 = r_2$ and $(k_1, k_2, \dots, k_{r_1}) = (k'_1, k'_2, \dots, k'_{r_1})$ or $(k_1, k_2, \dots, k_{r_1}) = (k'_{r_1}, k'_{r_1-1}, \dots, k'_1)$.*

Now we recall the formulas for the membership functions $\bar{\mu}$ and $\bar{\lambda}$ associated with a complete hypergroup.

Let $H = \bigcup_{g \in G} A_g$ be a complete hypergroup of cardinality $n$. By Theorem 2.8, it is obvious that, for any $u \in H$, there exists a unique $g_u \in G$ such that $u \in A_{g_u}$. Moreover, we define on $H$ the following equivalence $u \sim v \Longleftrightarrow \exists g \in G : u, v \in A_g$. Thereby one obtains that

$$\bar{\mu}(u) = \frac{|Q(u)|}{|A_{g_u}|} \cdot \frac{1}{n^2}, \quad \bar{\lambda}(u) = \left( \sum_{v \notin \hat{u}} \frac{|Q(v)|}{|A_{g_v}|} \right) \cdot \frac{1}{n^2}. \tag{3.3}$$

We end this section with a useful result concerning the complete hypergroups generated by a group $G$ isomorphic with the additive group $\mathbb{Z}_2$.

**3.4. Proposition.** *(see [18]) $H = \bigcup_{g \in G} A_g$ be a complete hypergroup of cardinality $n$. If the group $G$ is isomorphic with the additive group $\mathbb{Z}_2$, then $i.f.g.(H) = 1$.*

## 4. Intuitionistic fuzzy grade of the complete hypergroups of order less than or equal to 6

Cristea [14] listed all the forty complete hypergroups of order less than or equal to 6, calculating their fuzzy grade. In this section we determine the intuitionistic fuzzy grade of them. When the group which generates the complete hypergroup is isomorphic with the additive group $\mathbb{Z}_2$, by Proposition 3.4 it follows that $i.f.g.(H) = 1$ and in this case we do not list the table of the complete hypergroups (the reader may see it in [14]).

**4.1. Theorem.** *Let $H$ be a complete hypergroup of order $n \leq 6$.*

($i$) *There are two non-isomorphic complete hypergroups of order 3 having* $i.f.g.(H) = 1$.

($ii$) *There are five non-isomorphic complete hypergroups of order* 4*: for three of them, one finds that* $i.f.g.(H) = 1$*, and for other two that* $i.f.g.(H) = 2$.

($iii$) *There are twelve non-isomorphic complete hypergroups of order* 5*: nine of them have* $i.f.g.(H) = 1$*, and three of them have* $i.f.g.(H) = 3$.

($iv$) *There are twenty one non-isomorphic complete hypergroups of order* 6*: sixteen of them with* $i.f.g.(H) = 1$*, three of them with* $i.f.g.(H) = 2$ *and for two of them one finds that* $i.f.g.(H) = 3$.

*Proof.* We will denote, in the following tables, for any $s \in \{1, 2, \ldots, 5\}$, $B_s = H \setminus \{a_s\}$ and $B_0 = H \setminus \{e\}$. Let $H$ be a complete hypergroup of order $n \leq 6$, denoted by $H = \{e, a_1, \ldots, a_n\}$, with $3 \leq n \leq 5$, that is $H = \bigcup_{g \in G} A_g$.

($i$) If the hypergroup $H$ is of order 3, then it is obvious that $G \simeq (\mathbb{Z}_2, +)$, so there are only two complete hypergroups with the associated 2-tuple of the form $[1, 2]$ or $[2, 1]$. Thus, by Proposition 3.4, it follows that $i.f.g.(H_i) = 1$, for $i \in \{1, 2\}$.

($ii$) Let us suppose $H$ of order 4.

($a$) Setting $G \simeq (\mathbb{Z}_2, +)$, we obtain three complete hypergroups $H_3, H_4, H_5$, and by Proposition 3.4, it follows that $i.f.g.(H_i) = 1$, for $i \in \{3, 4, 5\}$.

($b$) Setting $G \simeq (\mathbb{Z}_3, +)$, we distinguish two hypergroups, denoted by $H_6, H_7$.

($b_1$) For $H_6$ represented here bellow

| $\circ$ | $e$ | $a_1$ | $a_2$ | $a_3$ |
|---|---|---|---|---|
| $e$ | $e$ | $a_1$ | $A_2$ | $A_2$ |
| $a_1$ | | $A_2$ | $e$ | $e$ |
| $a_2$ | | | $a_1$ | $a_1$ |
| $a_3$ | | | | $a_1$ |

where $A_0 = \{e\}, A_1 = \{a_1\}, A_2 = \{a_2, a_3\}$, we calculate that

$$\begin{array}{lll} \bar{\mu}(e) = 10/32, & \bar{\mu}(a_1) = 12/32, & \bar{\mu}(a_2) = \bar{\mu}(a_3) = 5/32, \\ \bar{\lambda}(e) = 17/32, & \bar{\lambda}(a_1) = 15/32, & \bar{\lambda}(a_2) = \bar{\lambda}(a_3) = 22/32. \end{array}$$

Therefore the associated join space ${}_0(H_6)$ is as follows:

| $\circ_{\bar{\mu}\wedge\bar{\lambda}}$ | $e$ | $a_1$ | $a_2$ | $a_3$ |
|---|---|---|---|---|
| $e$ | $e$ | $\{e, a_1\}$ | $B_1$ | $B_1$ |
| $a_1$ | | $a_1$ | $H$ | $H$ |
| $a_2$ | | | $A_2$ | $A_2$ |
| $a_3$ | | | | $A_2$ |

and thus we obtain that

$$\begin{array}{ll} \bar{\mu}_1(e) = \bar{\mu}_1(a_2) = \bar{\mu}_1(a_3) = 13/48, & \bar{\mu}_1(a_1) = 9/48, \\ \bar{\lambda}_1(e) = \bar{\lambda}_1(a_2) = \bar{\lambda}_1(a_3) = 9/48, & \bar{\lambda}_1(a_1) = 13/48. \end{array}$$

Therefore the associated join space ${}_1(H_6)$ is the total hypergroup. Then, for any $r \geq 2$, ${}_r(H_6) \simeq {}_1(H_6)$ and thereby $i.f.g.(H_6) = 2$.

($b_2$) Taking the complete hypergroup $H_7$

| $\circ$ | $e$ | $a_1$ | $a_2$ | $a_3$ |
|---|---|---|---|---|
| $e$ | $A_0$ | $A_0$ | $a_2$ | $a_3$ |
| $a_1$ | | $A_0$ | $a_2$ | $a_3$ |
| $a_2$ | | | $a_3$ | $A_0$ |
| $a_3$ | | | | $a_2$ |

with $A_0 = \{e, a_1\}, A_1 = \{a_2\}, A_2 = \{a_3\}$, one gets that

$$\bar{\mu}(e) = \bar{\mu}(a_1) = 3/16, \quad \bar{\mu}(a_2) = \bar{\mu}(a_3) = 5/16,$$
$$\bar{\lambda}(e) = \bar{\lambda}(a_1) = 10/16, \quad \bar{\lambda}(a_2) = \bar{\lambda}(a_3) = 8/16.$$

Therefore the associated join space ${}_0(H_7)$ is as follows:

| $\circ_{\bar{\mu}\wedge\bar{\lambda}}$ | $e$ | $a_1$ | $a_2$ | $a_3$ |
|---|---|---|---|---|
| $e$ | $A_0$ | $A_0$ | $H$ | $H$ |
| $a_1$ | | $A_0$ | $H$ | $H$ |
| $a_2$ | | | $\{a_2, a_3\}$ | $\{a_2, a_3\}$ |
| $a_3$ | | | | $\{a_2, a_3\}$ |

and

$$\bar{\mu}_1(e) = \bar{\mu}_1(a_1) = \bar{\mu}_1(a_2) = \bar{\mu}_1(a_3) = 4/16,$$
$$\bar{\lambda}_1(e) = \bar{\lambda}_1(a_1) = \bar{\lambda}_1(a_2) = \bar{\lambda}_1(a_3) = 2/16.$$

Therefore the associated join space ${}_1(H_7)$ is the total hypergroup and, for any $r \geq 2$, ${}_r(H_7) \simeq {}_1(H_7)$, so $i.f.g.(H_7) = 2$.

$(iii)$ We consider now the complete hypergroups of order 5.

$(a)$ There are five complete 1-hypergroups of order 5, denoted here by $H_8, \ldots, H_{12}$.

$(a_1)$ For the first one $H_8$ generated by the group $G \simeq (\mathbb{Z}_2, +)$, by Proposition 3.4, it follows that $i.f.g.(H_8) = 1$.

$(a_2)$ For $H_9$ represented by the table

| $\circ$ | $e$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ |
|---|---|---|---|---|---|
| $e$ | $e$ | $a_1$ | $A_2$ | $A_2$ | $A_2$ |
| $a_1$ | | $A_2$ | $e$ | $e$ | $e$ |
| $a_2$ | | | $a_1$ | $a_1$ | $a_1$ |
| $a_3$ | | | | $a_1$ | $a_1$ |
| $a_4$ | | | | | $a_1$ |

with $A_0 = \{e\}, A_1 = \{a_1\}, A_2 = \{a_2, a_3, a_4\}$, we find that

$$\bar{\mu}(e) = 21/75, \quad \bar{\mu}(a_1) = 33/75, \quad \bar{\mu}(a_2) = \bar{\mu}(a_3) = \bar{\mu}(a_4) = 7/75,$$
$$\bar{\lambda}(e) = 40/75, \quad \bar{\lambda}(a_1) = 28/75, \quad \bar{\lambda}(a_2) = \bar{\lambda}(a_3) = \bar{\lambda}(a_4) = 54/75.$$

Therefore the associated join space ${}_0(H_9)$ is as follows:

| $\circ_{\bar{\mu}\wedge\bar{\lambda}}$ | $e$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ |
|---|---|---|---|---|---|
| $e$ | $e$ | $\{e, a_1\}$ | $B_1$ | $B_1$ | $B_1$ |
| $a_1$ | | $a_1$ | $H$ | $H$ | $H$ |
| $a_2$ | | | $A_2$ | $A_2$ | $A_2$ |
| $a_3$ | | | | $A_2$ | $A_2$ |
| $a_4$ | | | | | $A_2$ |

then

$$\bar{\mu}_1(e) = 47/250, \quad \bar{\mu}_1(a_1) = 32/250, \quad \bar{\mu}_1(a_2) = \bar{\mu}_1(a_3) = \bar{\mu}_1(a_4) = 57/250,$$
$$\bar{\lambda}_1(e) = 40/250, \quad \bar{\lambda}_1(a_1) = 55/250, \quad \bar{\lambda}_1(a_2) = \bar{\lambda}_1(a_3) = \bar{\lambda}_1(a_4) = 30/250.$$

Then, for any $r \geq 1$, ${}_r(H_9) \simeq {}_0(H_9)$ and therefore $i.f.g.(H_9) = 1$.

$(a_3)$ Set the complete hypergroup $H_{10}$ as

| $\circ$ | $e$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ |
|---|---|---|---|---|---|
| $e$ | $e$ | $A_1$ | $A_1$ | $A_2$ | $A_2$ |
| $a_1$ | | $A_2$ | $A_2$ | $e$ | $e$ |
| $a_2$ | | | $A_2$ | $e$ | $e$ |
| $a_3$ | | | | $A_1$ | $A_1$ |
| $a_4$ | | | | | $A_1$ |

where $A_0 = \{e\}, A_1 = \{a_1, a_2\}, A_2 = \{a_3, a_4\}$. Then, for any $i \in \{1,2,3,4\}$, we calculate that $\bar{\mu}(e) = 9/25$, $\bar{\mu}(a_i) = 4/25$, $\bar{\lambda}(e) = 8/25$, $\bar{\lambda}(a_i) = 13/25$.

It result the following join space ${}_0(H_{10})$

| $\circ_{\bar{\mu}\wedge\bar{\lambda}}$ | $e$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ |
|---|---|---|---|---|---|
| $e$ | $e$ | $H$ | $H$ | $H$ | $H$ |
| $a_1$ | | $B_0$ | $B_0$ | $B_0$ | $B_0$ |
| $a_2$ | | | $B_0$ | $B_0$ | $B_0$ |
| $a_3$ | | | | $B_0$ | $B_0$ |
| $a_4$ | | | | | $B_0$ |

and, for any $i \in \{1,2,3,4\}$, $\bar{\mu}_1(e) = 13/125$, $\bar{\mu}_1(a_i) = 28/125$, $\bar{\lambda}_1(e) = 20/125$, $\bar{\lambda}_1(a_i) = 5/125$. Therefore, we have, for any $r \geq 1$, ${}_r(H_{10}) \simeq {}_0(H_{10})$ and $i.f.g.(H_{10}) = 1$.

$(a_4)$ Let us consider $H_{11}$ as

| $\circ$ | $e$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ |
|---|---|---|---|---|---|
| $e$ | $e$ | $a_1$ | $a_2$ | $A_3$ | $A_3$ |
| $a_1$ | | $a_2$ | $A_3$ | $e$ | $e$ |
| $a_2$ | | | $e$ | $a_1$ | $a_1$ |
| $a_3$ | | | | $a_2$ | $a_2$ |
| $a_4$ | | | | | $a_2$ |

where $A_0 = \{e\}, A_1 = \{a_1\}, A_2 = \{a_2\}, A_3 = \{a_3, a_4\}$ (i.e. the 4-tuple associated with $H$ is $[1,1,1,2]$) and $G \simeq (\mathbb{Z}_4, +)$, for which we calculate

$$\begin{array}{lll} \bar{\mu}(e) = \bar{\mu}(a_1) = 6/25, & \bar{\mu}(a_2) = 7/25, & \bar{\mu}(a_3) = \bar{\mu}(a_4) = 3/25, \\ \bar{\lambda}(e) = \bar{\lambda}(a_1) = 16/25, & \bar{\lambda}(a_2) = 15/25, & \bar{\lambda}(a_3) = \bar{\lambda}(a_4) = 19/25. \end{array}$$

Therefore the associated join space ${}_0(H_{11})$ is as follows:

| $\circ_{\bar{\mu}\wedge\bar{\lambda}}$ | $e$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ |
|---|---|---|---|---|---|
| $e$ | $\{e, a_1\}$ | $\{e, a_1\}$ | $\{e, a_1, a_2\}$ | $B_2$ | $B_2$ |
| $a_1$ | | $\{e, a_1\}$ | $\{e, a_1, a_2\}$ | $B_2$ | $B_2$ |
| $a_2$ | | | $a_2$ | $H$ | $H$ |
| $a_3$ | | | | $A_3$ | $A_3$ |
| $a_4$ | | | | | $A_3$ |

then

$$\begin{array}{lll} \bar{\mu}_1(e) = \bar{\mu}_1(a_1) = 92/375, & \bar{\mu}_1(a_2) = 47/375, & \bar{\mu}_1(a_3) = \bar{\mu}_1(a_4) = 72/375, \\ \bar{\lambda}_1(e) = \bar{\lambda}_1(a_1) = 45/375, & \bar{\lambda}_1(a_2) = 90/375, & \bar{\lambda}_1(a_3) = \bar{\lambda}_1(a_4) = 65/375. \end{array}$$

Therefore the associated join space ${}_1(H_{11})$ is as follows:

| $\circ_{\bar{\mu}_1\wedge\bar{\lambda}_1}$ | $e$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ |
|---|---|---|---|---|---|
| $e$ | $\{e, a_1\}$ | $\{e, a_1\}$ | $\{e, a_1, a_2\}$ | $H$ | $H$ |
| $a_1$ | | $\{e, a_1\}$ | $\{e, a_1, a_2\}$ | $H$ | $H$ |
| $a_2$ | | | $a_2$ | $\{a_2, a_3, a_4\}$ | $\{a_2, a_3, a_4\}$ |
| $a_3$ | | | | $A_3$ | $A_3$ |
| $a_4$ | | | | | $A_3$ |

for which we find that

$$\begin{array}{ll} \bar{\mu}_2(e) = \bar{\mu}_2(a_1) = \bar{\mu}_2(a_3) = \bar{\mu}_2(a_4) = 74/375, & \bar{\mu}_2(a_2) = 79/375, \\ \bar{\lambda}_2(e) = \bar{\lambda}_2(a_1) = \bar{\lambda}_2(a_3) = \bar{\lambda}_2(a_4) = 65/375, & \bar{\lambda}_2(a_2) = 60/375. \end{array}$$

Therefore the associated join space ${}_2(H_{11})$ is as follows:

| $\circ_{\bar{\mu}_2 \wedge \bar{\lambda}_2}$ | $e$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ |
|---|---|---|---|---|---|
| $e$ | $B_2$ | $B_2$ | $H$ | $B_2$ | $B_2$ |
| $a_1$ | | $B_2$ | $H$ | $B_2$ | $B_2$ |
| $a_2$ | | | $a_2$ | $H$ | $H$ |
| $a_3$ | | | | $B_2$ | $B_2$ |
| $a_4$ | | | | | $B_2$ |

then

$$\bar{\mu}_3(e) = \bar{\mu}_3(a_1) = \bar{\mu}_3(a_3) = \bar{\mu}_3(a_4) \neq \bar{\mu}_3(a_2),$$
$$\bar{\lambda}_3(e) = \bar{\lambda}_3(a_1) = \bar{\lambda}_3(a_3) = \bar{\lambda}_3(a_4) \neq \bar{\lambda}_3(a_2).$$

Then, for any $r \geq 3$, ${}_r(H_{11}) \simeq {}_2(H_{11})$ and therefore $i.f.g.(H_{11}) = 3$.

$(a_5)$ For the same 4-tuple $[1,1,1,2]$ associated with $H$, i.e. $A_0 = \{e\}, A_1 = \{a_1\}, A_2 = \{a_2\}, A_3 = \{a_3, a_4\}$, but with $G \simeq (K, \cdot)$ the Klein four-group, it results the following complete hypergroup $H_{12}$

| $\circ$ | $e$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ |
|---|---|---|---|---|---|
| $e$ | $e$ | $a_1$ | $a_2$ | $A_3$ | $A_3$ |
| $a_1$ | | $e$ | $A_3$ | $a_2$ | $a_2$ |
| $a_2$ | | | $e$ | $a_1$ | $a_1$ |
| $a_3$ | | | | $e$ | $e$ |
| $a_4$ | | | | | $e$ |

with

$$\bar{\mu}(e) = 7/25, \quad \bar{\mu}(a_1) = \bar{\mu}(a_2) = 6/25, \quad \bar{\mu}(a_3) = \bar{\mu}(a_4) = 3/25,$$
$$\bar{\lambda}(e) = 15/25, \quad \bar{\lambda}(a_1) = \bar{\lambda}(a_2) = 16/25, \quad \bar{\lambda}(a_3) = \bar{\lambda}(a_4) = 19/25.$$

Therefore the associated join space ${}_0(H_{12})$ is isomorphic with ${}_0(H_{11})$ and thereby we have that $i.f.g.(H_{12}) = 3$.

$(b)$The following complete hypergroups, denoted by $H_{13}, \ldots, H_{19}$, are not 1-hypergroups.

$(b_1)$ There exist three complete hypergroups of order 5 (which are not 1-hypergroups) such that $G \simeq (\mathbb{Z}_2, +)$, (corresponding to the 2-tuples $[2,3]$, $[3,2]$, and $[4,1]$); for each of them we obtain, by Proposition 3.4, that $i.f.g.(H_i) = 1$, with $i \in \{13, 14, 15\}$.

$(b_2)$ Let us consider $H_{16}$ as the following complete hypergroup

| $\circ$ | $e$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ |
|---|---|---|---|---|---|
| $e$ | $A_0$ | $A_0$ | $A_0$ | $a_3$ | $a_4$ |
| $a_1$ | | $A_0$ | $A_0$ | $a_3$ | $a_4$ |
| $a_2$ | | | $A_0$ | $a_3$ | $a_4$ |
| $a_3$ | | | | $a_4$ | $A_0$ |
| $a_4$ | | | | | $a_3$ |

where $A_0 = \{e, a_1, a_2\}, A_1 = \{a_3\}, A_2 = \{a_4\}$, for which we find that

$$\bar{\mu}(e) = \bar{\mu}(a_1) = \bar{\mu}(a_2) = 11/75, \quad \bar{\mu}(a_3) = \bar{\mu}(a_4) = 21/75,$$
$$\bar{\lambda}(e) = \bar{\lambda}(a_1) = \bar{\lambda}(a_2) = 42/75, \quad \bar{\lambda}(a_3) = \bar{\lambda}(a_4) = 32/75.$$

Therefore the associated join space ${}_0(H_{16})$ is as follows:

| $\circ_{\bar{\mu} \wedge \bar{\lambda}}$ | $e$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ |
|---|---|---|---|---|---|
| $e$ | $A_0$ | $A_0$ | $A_0$ | $H$ | $H$ |
| $a_1$ | | $A_0$ | $A_0$ | $H$ | $H$ |
| $a_2$ | | | $A_0$ | $H$ | $H$ |
| $a_3$ | | | | $\{a_3, a_4\}$ | $\{a_3, a_4\}$ |
| $a_4$ | | | | | $\{a_3, a_4\}$ |

and

$$\bar{\mu}_1(e) = \bar{\mu}_1(a_1) = \bar{\mu}_1(a_2) \neq \bar{\mu}_1(a_3) = \bar{\mu}_1(a_4),$$
$$\bar{\lambda}_1(e) = \bar{\lambda}_1(a_1) = \bar{\lambda}_1(a_2) \neq \bar{\lambda}_1(a_3) = \bar{\lambda}_1(a_4).$$

Then, for any $r \geq 1$, ${}_r(H_{16}) \simeq {}_0(H_{16})$ and therefore $i.f.g.(H_{16}) = 1$.

$(b_3)$ There exist two complete hypergroups $H_{17}$ and $H_{18}$ of order 5 generated by a group of order 4 and characterized by the 4-tuple $[2, 1, 1, 1]$. Setting $A_0 = \{e, a_1\}, A_1 = \{a_2\}, A_2 = \{a_3\}, A_3 = \{a_4\}$, if $G \simeq (\mathbb{Z}_4, +)$, then the hypergroup $H_{17}$ is the following one

| $\circ$ | $e$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ |
|---|---|---|---|---|---|
| $e$ | $A_0$ | $A_0$ | $a_2$ | $a_3$ | $a_4$ |
| $a_1$ | | $A_0$ | $a_2$ | $a_3$ | $a_4$ |
| $a_2$ | | | $a_3$ | $a_4$ | $A_0$ |
| $a_3$ | | | | $A_0$ | $a_2$ |
| $a_4$ | | | | | $a_3$ |

and if $G \simeq (K, \cdot)$ the Klein four-group, then the hypergroup $H_{18}$ is represented by the table

| $\circ$ | $e$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ |
|---|---|---|---|---|---|
| $e$ | $A_0$ | $A_0$ | $a_2$ | $a_3$ | $a_4$ |
| $a_1$ | | $A_0$ | $a_2$ | $a_3$ | $a_4$ |
| $a_2$ | | | $A_0$ | $a_4$ | $a_3$ |
| $a_3$ | | | | $A_0$ | $a_2$ |
| $a_4$ | | | | | $A_0$ |

In both cases one finds that

$$\bar{\mu}(e) = \bar{\mu}(a_1) = 7/50, \quad \bar{\mu}(a_2) = \bar{\mu}(a_3) = \bar{\mu}(a_4) = 12/50,$$
$$\bar{\lambda}(e) = \bar{\lambda}(a_1) = 36/50, \quad \bar{\lambda}(a_2) = \bar{\lambda}(a_3) = \bar{\lambda}(a_4) = 31/50.$$

Therefore the associated join space ${}_0(H_{17}) ={}_0 (H_{18})$ is as follows:

| $\circ_{\bar{\mu}\wedge\bar{\lambda}}$ | $e$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ |
|---|---|---|---|---|---|
| $e$ | $A_0$ | $A_0$ | $H$ | $H$ | $H$ |
| $a_1$ | | $A_0$ | $H$ | $H$ | $H$ |
| $a_2$ | | | $\{a_2, a_3, a_4\}$ | $\{a_2, a_3, a_4\}$ | $\{a_2, a_3, a_4\}$ |
| $a_3$ | | | | $\{a_2, a_3, a_4\}$ | $\{a_2, a_3, a_4\}$ |
| $a_4$ | | | | | $\{a_2, a_3, a_4\}$ |

and then

$$\bar{\mu}_1(e) = \bar{\mu}_1(a_1) \neq \bar{\mu}_1(a_2) = \bar{\mu}_1(a_3) = \bar{\mu}_1(a_4),$$
$$\bar{\lambda}_1(e) = \bar{\lambda}_1(a_1) \neq \bar{\lambda}_1(a_2) = \bar{\lambda}_1(a_3) = \bar{\lambda}_1(a_4).$$

It follows that, for any $r \geq 1$, ${}_r(H_i) \simeq {}_0(H_i)$, with $i \in \{17, 18\}$, and therefore $i.f.g.(H_{17}) = i.f.g.(H_{18}) = 1$.

$(b_4)$ For $H_{19}$

| $\circ$ | $e$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ |
|---|---|---|---|---|---|
| $e$ | $A_0$ | $A_0$ | $a_2$ | $A_2$ | $A_2$ |
| $a_1$ | | $A_0$ | $a_2$ | $A_2$ | $A_2$ |
| $a_2$ | | | $A_2$ | $A_0$ | $A_0$ |
| $a_3$ | | | | $a_2$ | $a_2$ |
| $a_4$ | | | | | $a_2$ |

where $A_0 = \{e, a_1\}, A_1 = \{a_2\}, A_2 = \{a_3, a_4\}$, one finds the following membership functions

$$\bar{\mu}(e) = \bar{\mu}(a_1) = 8/50, \quad \bar{\mu}(a_2) = 16/50, \quad \bar{\mu}(a_3) = \bar{\mu}(a_4) = 9/50,$$
$$\bar{\lambda}(e) = \bar{\lambda}(a_1) = 25/50, \quad \bar{\lambda}(a_2) = 17/50, \quad \bar{\lambda}(a_3) = \bar{\lambda}(a_4) = 24/50.$$

Therefore the associated join space ${}_0(H_{19})$ is as follows:

| $\circ_{\bar{\mu}\wedge\bar{\lambda}}$ | $e$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ |
|---|---|---|---|---|---|
| $e$ | $A_0$ | $A_0$ | $H$ | $B_2$ | $B_2$ |
| $a_1$ | | $A_0$ | $H$ | $B_2$ | $B_2$ |
| $a_2$ | | | $a_2$ | $\{a_2, a_3, a_4\}$ | $\{a_2, a_3, a_4\}$ |
| $a_3$ | | | | $A_2$ | $A_2$ |
| $a_4$ | | | | | $A_2$ |

and

$$\begin{array}{lll} \bar{\mu}_1(e) = \bar{\mu}_1(a_1) = 72/375, & \bar{\mu}_1(a_2) = 47/375, & \bar{\mu}_1(a_3) = \bar{\mu}_1(a_4) = 92/375, \\ \bar{\lambda}_1(e) = \bar{\lambda}_1(a_1) = 65/375, & \bar{\lambda}_1(a_2) = 90/375, & \bar{\lambda}_1(a_3) = \bar{\lambda}_1(a_4) = 45/375. \end{array}$$

It is clear that the associated join space ${}_1(H_{19})$ is isomorphic with ${}_1(H_{11})$ and thus we obtain that $i.f.g.(H_{19}) = 3$.

$(iv)$ Now we study the complete hypergroups of order 6. We denote the twenty one non-isomorphic complete hypergroups of order 6 by $H_{20}, H_{21}, \ldots, H_{40}$.

There are sixteen complete hypergroups of order 6 with the intuitionistic fuzzy grade equal to 1, listed in the sequel.

$(a_1)$ Let us consider the complete hypergroup $H_{20}$

| $\circ$ | $e$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ |
|---|---|---|---|---|---|---|
| $e$ | $e$ | $a_1$ | $a_2$ | $A_3$ | $A_3$ | $A_3$ |
| $a_1$ | | $a_2$ | $A_3$ | $e$ | $e$ | $e$ |
| $a_2$ | | | $e$ | $a_1$ | $a_1$ | $a_1$ |
| $a_3$ | | | | $a_2$ | $a_2$ | $a_2$ |
| $a_4$ | | | | | $a_2$ | $a_2$ |
| $a_5$ | | | | | | $a_2$ |

where $A_0 = \{e\}, A_1 = \{a_1\}, A_2 = \{a_2\}, A_3 = \{a_3, a_4, a_5\}$ and $G \simeq (\mathbb{Z}_4, +)$. In particular, $H_{20}$ is an 1-hypergroup. Then

$$\begin{array}{lll} \bar{\mu}(e) = \bar{\mu}(a_1) = 24/108, & \bar{\mu}(a_2) = 36/108, & \bar{\mu}(a_3) = \bar{\mu}(a_4) = \bar{\mu}(a_5) = 8/108, \\ \bar{\lambda}(e) = \bar{\lambda}(a_1) = 68/108, & \bar{\lambda}(a_2) = 56/108, & \bar{\lambda}(a_3) = \bar{\lambda}(a_4) = \bar{\lambda}(a_5) = 84/108. \end{array}$$

Therefore the associated join space ${}_0(H_{20})$ is as follows:

| $\circ_{\bar{\mu}\wedge\bar{\lambda}}$ | $e$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ |
|---|---|---|---|---|---|---|
| $e$ | $\{e, a_1\}$ | $\{e, a_1\}$ | $\{e, a_1, a_2\}$ | $B_2$ | $B_2$ | $B_2$ |
| $a_1$ | | $\{e, a_1\}$ | $\{e, a_1, a_2\}$ | $B_2$ | $B_2$ | $B_2$ |
| $a_2$ | | | $a_2$ | $H$ | $H$ | $H$ |
| $a_3$ | | | | $A_3$ | $A_3$ | $A_3$ |
| $a_4$ | | | | | $A_3$ | $A_3$ |
| $a_5$ | | | | | | $A_3$ |

We obtain that

$$\begin{array}{lll} \bar{\mu}_1(e) = \bar{\mu}_1(a_1) = 101/540, & \bar{\mu}_1(a_2) = 50/540, & \bar{\mu}_1(a_3) = \bar{\mu}_1(a_4) = \bar{\mu}_1(a_5) = 96/540, \\ \bar{\lambda}_1(e) = \bar{\lambda}_1(a_1) = 60/540, & \bar{\lambda}_1(a_2) = 111/540, & \bar{\lambda}_1(a_3) = \bar{\lambda}_1(a_4) = \bar{\lambda}_1(a_5) = 65/540. \end{array}$$

Then, for any $r \geq 1$, ${}_r(H_{20}) \simeq {}_0(H_{20})$ and therefore $i.f.g.(H_{20}) = 1$.

$(a_2)$ Let us see the complete hypergroup $H_{21}$

| $\circ$ | $e$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ |
|---|---|---|---|---|---|---|
| $e$ | $e$ | $a_1$ | $a_2$ | $A_3$ | $A_3$ | $A_3$ |
| $a_1$ | | $e$ | $A_3$ | $a_2$ | $a_2$ | $a_2$ |
| $a_2$ | | | $e$ | $a_1$ | $a_1$ | $a_1$ |
| $a_3$ | | | | $e$ | $e$ | $e$ |
| $a_4$ | | | | | $e$ | $e$ |
| $a_5$ | | | | | | $e$ |

with $G \simeq (K, \cdot)$ the Klein four-group, $A_0 = \{e\}, A_1 = \{a_1\}, A_2 = \{a_2\}, A_3 = \{a_3, a_4, a_5\}$. $H_{21}$ is an 1-hypergroup, too. Then

$$\begin{array}{lll} \bar{\mu}(e) = 36/108, & \bar{\mu}(a_1) = \bar{\mu}(a_2) = 24/108, & \bar{\mu}(a_3) = \bar{\mu}(a_4) = \bar{\mu}(a_5) = 8/108, \\ \bar{\lambda}(e) = 56/108, & \bar{\lambda}(a_1) = \bar{\lambda}(a_2) = 68/108, & \bar{\lambda}(a_3) = \bar{\lambda}(a_4) = \bar{\lambda}(a_5) = 84/108. \end{array}$$

It follows that the associated join space ${}_0(H_{21})$ is isomorphic to ${}_0(H_{20})$ and thereby we have that $i.f.g.(H_{21}) = 1$.

$(a_3)$ Setting now $G \simeq (\mathbb{Z}_2, +)$, it results five non-isomorphic complete hypergroups $H_{22}, \ldots, H_{26}$ corresponding to the 2-tuples $[1, 5]$, $[2, 4]$, $[3, 3]$, $[4, 2]$, $[5, 1]$. By Proposition 3.4, it follows immediately that $i.f.g.(H_i) = 1$, for $i \in \{22, \ldots, 26\}$.

$(a_4)$ For the complete hypergroup $H_{27}$, which is also an 1-hypergroup,

| $\circ$ | $e$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ |
|---|---|---|---|---|---|---|
| $e$ | $e$ | $a_5$ | $A_1$ | $A_1$ | $A_1$ | $A_1$ |
| $a_1$ | | $A_1$ | $e$ | $e$ | $e$ | $e$ |
| $a_2$ | | | $a_5$ | $a_5$ | $a_5$ | $a_5$ |
| $a_3$ | | | | $a_5$ | $a_5$ | $a_5$ |
| $a_4$ | | | | | $a_5$ | $a_5$ |
| $a_5$ | | | | | | $a_5$ |

where $A_0 = \{e\}, A_1 = \{a_1, a_2, a_3, a_4\}, A_2 = \{a_5\}$, we calculate that

$$\begin{array}{lll} \bar{\mu}(e) = 36/144, & \bar{\mu}(a_1) = \bar{\mu}(a_2) = \bar{\mu}(a_3) = \bar{\mu}(a_4) = 9/144, & \bar{\mu}(a_5) = 72/144, \\ \bar{\lambda}(e) = 81/144, & \bar{\lambda}(a_1) = \bar{\lambda}(a_2) = \bar{\lambda}(a_3) = \bar{\lambda}(a_4) = 108/144, & \bar{\lambda}(a_5) = 45/144. \end{array}$$

Therefore the associated join space ${}_0(H_{27})$ is as follows:

| $\circ_{\bar{\mu}\wedge\bar{\lambda}}$ | $e$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ |
|---|---|---|---|---|---|---|
| $e$ | $e$ | $B_5$ | $B_5$ | $B_5$ | $B_5$ | $\{e, a_5\}$ |
| $a_1$ | | $A_1$ | $A_1$ | $A_1$ | $A_1$ | $H$ |
| $a_2$ | | | $A_1$ | $A_1$ | $A_1$ | $H$ |
| $a_3$ | | | | $A_1$ | $A_1$ | $H$ |
| $a_4$ | | | | | $A_1$ | $H$ |
| $a_5$ | | | | | | $a_5$ |

We find

$$\begin{array}{lll} \bar{\mu}_1(e) = 74/540, & \bar{\mu}_1(a_1) = \bar{\mu}_1(a_2) = \bar{\mu}_1(a_3) = \bar{\mu}_1(a_4) = 104/540, & \bar{\mu}_1(a_5) = 50/540, \\ \bar{\lambda}_1(e) = 75/540, & \bar{\lambda}_1(a_1) = \bar{\lambda}_1(a_2) = \bar{\lambda}_1(a_3) = \bar{\lambda}_1(a_4) = 45/540, & \bar{\lambda}_1(a_5) = 99/540. \end{array}$$

Then, for any $r \geq 1$, ${}_r(H_{27}) \simeq {}_0(H_{27})$ and therefore $i.f.g.(H_{27}) = 1$.

$(a_5)$ For the complete hypergroup $H_{28}$ represented here bellow

| $\circ$ | $e$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ |
|---|---|---|---|---|---|---|
| $e$ | $e$ | $A_1$ | $A_1$ | $A_2$ | $A_2$ | $A_2$ |
| $a_1$ | | $A_2$ | $A_2$ | $e$ | $e$ | $e$ |
| $a_2$ | | | $A_2$ | $e$ | $e$ | $e$ |
| $a_3$ | | | | $A_1$ | $A_1$ | $A_1$ |
| $a_4$ | | | | | $A_1$ | $A_1$ |
| $a_5$ | | | | | | $A_1$ |

with $A_0 = \{e\}, A_1 = \{a_1, a_2\}, A_2 = \{a_3, a_4, a_5\}$ (in particular, $H_{28}$ is an 1-hypergroup), we obtain that

$$\begin{array}{lll}\bar{\mu}(e) = 78/216, & \bar{\mu}(a_1) = \bar{\mu}(a_2) = 39/216, & \bar{\mu}(a_3) = \bar{\mu}(a_4) = \bar{\mu}(a_5) = 20/216, \\ \bar{\lambda}(e) = 59/216, & \bar{\lambda}(a_1) = \bar{\lambda}(a_2) = 98/216, & \bar{\lambda}(a_3) = \bar{\lambda}(a_4) = \bar{\lambda}(a_5) = 117/216.\end{array}$$

Therefore the associated join space ${}_0(H_{28})$ is as follows:

| $\circ_{\bar{\mu}\wedge\bar{\lambda}}$ | $e$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ |
|---|---|---|---|---|---|---|
| $e$ | $e$ | $\{e, a_1, a_2\}$ | $\{e, a_1, a_2\}$ | $H$ | $H$ | $H$ |
| $a_1$ | | $A_1$ | $A_1$ | $B_0$ | $B_0$ | $B_0$ |
| $a_2$ | | | $A_1$ | $B_0$ | $B_0$ | $B_0$ |
| $a_3$ | | | | $A_2$ | $A_2$ | $A_2$ |
| $a_4$ | | | | | $A_2$ | $A_2$ |
| $a_5$ | | | | | | $A_2$ |

and

$$\begin{array}{lll}\bar{\mu}_1(e) = 50/540, & \bar{\mu}_1(a_1) = \bar{\mu}_1(a_2) = 101/540, & \bar{\mu}_1(a_3) = \bar{\mu}_1(a_4) = \bar{\mu}_1(a_5) = 96/540, \\ \bar{\lambda}_1(e) = 111/540, & \bar{\lambda}_1(a_1) = \bar{\lambda}_1(a_2) = 60/540, & \bar{\lambda}_1(a_3) = \bar{\lambda}_1(a_4) = \bar{\lambda}_1(a_5) = 65/540.\end{array}$$

Then, for any $r \geq 1$, ${}_r(H_{28}) \simeq {}_0(H_{28})$ and therefore $i.f.g.(H_{28}) = 1$.

$(a_6)$ Taking the complete hypergroup $H_{29}$

| $\circ$ | $e$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ |
|---|---|---|---|---|---|---|
| $e$ | $A_0$ | $A_0$ | $a_2$ | $A_2$ | $A_2$ | $A_2$ |
| $a_1$ | | $A_0$ | $a_2$ | $A_2$ | $A_2$ | $A_2$ |
| $a_2$ | | | $A_2$ | $A_0$ | $A_0$ | $A_0$ |
| $a_3$ | | | | $a_2$ | $a_2$ | $a_2$ |
| $a_4$ | | | | | $a_2$ | $a_2$ |
| $a_5$ | | | | | | $a_2$ |

with $A_0 = \{e, a_1\}, A_1 = \{a_2\}, A_2 = \{a_3, a_4, a_5\}$, we calculate that

$$\begin{array}{lll}\bar{\mu}(e) = \bar{\mu}(a_1) = 15/108, & \bar{\mu}(a_2) = 39/108, & \bar{\mu}(a_3) = \bar{\mu}(a_4) = \bar{\mu}(a_5) = 13/108, \\ \bar{\lambda}(e) = \bar{\lambda}(a_1) = 52/108, & \bar{\lambda}(a_2) = 28/108, & \bar{\lambda}(a_3) = \bar{\lambda}(a_4) = \bar{\lambda}(a_5) = 54/108.\end{array}$$

Therefore the associated join space ${}_0(H_{29})$ is as follows:

| $\circ_{\bar{\mu}\wedge\bar{\lambda}}$ | $e$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ |
|---|---|---|---|---|---|---|
| $e$ | $A_0$ | $A_0$ | $\{e, a_1, a_2\}$ | $B_2$ | $B_2$ | $B_2$ |
| $a_1$ | | $A_0$ | $\{e, a_1, a_2\}$ | $B_2$ | $B_2$ | $B_2$ |
| $a_2$ | | | $a_2$ | $H$ | $H$ | $H$ |
| $a_3$ | | | | $A_2$ | $A_2$ | $A_2$ |
| $a_4$ | | | | | $A_2$ | $A_2$ |
| $a_5$ | | | | | | $A_2$ |

We obtain that

$$\begin{array}{lll} \bar{\mu}_1(e) = \bar{\mu}_1(a_1) = 101/540, & \bar{\mu}_1(a_2) = 50/540, & \bar{\mu}_1(a_3) = \bar{\mu}_1(a_4) = \bar{\mu}_1(a_5) = 96/540, \\ \bar{\lambda}_1(e) = \bar{\lambda}_1(a_1) = 60/540, & \bar{\lambda}_1(a_2) = 111/540, & \bar{\lambda}_1(a_3) = \bar{\lambda}_1(a_4) = \bar{\lambda}_1(a_5) = 65/540. \end{array}$$

Then, for any $r \geq 1$, ${}_r(H_{29}) \simeq {}_0(H_{29})$ and therefore $i.f.g.(H_{29}) = 1$.

($a_7$) If we take the complete hypergroup $H_{30}$ as

| $\circ$ | $e$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ |
|---|---|---|---|---|---|---|
| $e$ | $A_0$ | $A_0$ | $A_1$ | $A_1$ | $A_2$ | $A_2$ |
| $a_1$ | | $A_0$ | $A_1$ | $A_1$ | $A_2$ | $A_2$ |
| $a_2$ | | | $A_2$ | $A_2$ | $A_0$ | $A_0$ |
| $a_3$ | | | | $A_2$ | $A_0$ | $A_0$ |
| $a_4$ | | | | | $A_1$ | $A_1$ |
| $a_5$ | | | | | | $A_1$ |

with $A_0 = \{e, a_1\}, A_1 = \{a_2, a_3\}, A_2 = \{a_4, a_5\}$, then it results that

$$\begin{array}{l} \bar{\mu}(e) = \bar{\mu}(a_1) = \bar{\mu}(a_2) = \bar{\mu}(a_3) = \bar{\mu}(a_4) = \bar{\mu}(a_5) = 6/36, \\ \bar{\lambda}(e) = \bar{\lambda}(a_1) = \bar{\lambda}(a_2) = \bar{\lambda}(a_3) = \bar{\lambda}(a_4) = \bar{\lambda}(a_5) = 12/36. \end{array}$$

Therefore ${}_0(H_{30})$ is a total hypergroup. Then, for any $r \geq 1$, ${}_r(H_{30}) \simeq {}_0(H_{30})$ and therefore $i.f.g.(H_{30}) = 1$.

($a_8$) Let us consider $H_{31}$ given by the following table

| $\circ$ | $e$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ |
|---|---|---|---|---|---|---|
| $e$ | $A_0$ | $A_0$ | $A_0$ | $a_3$ | $A_2$ | $A_2$ |
| $a_1$ | | $A_0$ | $A_0$ | $a_3$ | $A_2$ | $A_2$ |
| $a_2$ | | | $A_0$ | $a_3$ | $A_2$ | $A_2$ |
| $a_3$ | | | | $A_2$ | $A_0$ | $A_0$ |
| $a_4$ | | | | | $a_3$ | $a_3$ |
| $a_5$ | | | | | | $a_3$ |

with $A_0 = \{e, a_1, a_2\}, A_1 = \{a_3\}, A_2 = \{a_4, a_5\}$. We calculate that

$$\begin{array}{lll} \bar{\mu}(e) = \bar{\mu}(a_1) = \bar{\mu}(a_2) = 26/216, & \bar{\mu}(a_3) = 60/216, & \bar{\mu}(a_4) = \bar{\mu}(a_5) = 39/216, \\ \bar{\lambda}(e) = \bar{\lambda}(a_1) = \bar{\lambda}(a_2) = 99/216, & \bar{\lambda}(a_3) = 65/216, & \bar{\lambda}(a_4) = \bar{\lambda}(a_5) = 86/216. \end{array}$$

Therefore the associated join space ${}_0(H_{31})$ is as follows:

| $\circ_{\bar{\mu}\wedge\bar{\lambda}}$ | $e$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ |
|---|---|---|---|---|---|---|
| $e$ | $A_0$ | $A_0$ | $A_0$ | $H$ | $B_3$ | $B_3$ |
| $a_1$ | | $A_0$ | $A_0$ | $H$ | $B_3$ | $B_3$ |
| $a_2$ | | | $A_0$ | $H$ | $B_3$ | $B_3$ |
| $a_3$ | | | | $a_3$ | $\{a_3, a_4, a_5\}$ | $\{a_3, a_4, a_5\}$ |
| $a_4$ | | | | | $A_2$ | $A_2$ |
| $a_5$ | | | | | | $A_2$ |

Then

$$\begin{array}{lll} \bar{\mu}_1(e) = \bar{\mu}_1(a_1) = \bar{\mu}_1(a_2) = 96/540, & \bar{\mu}_1(a_3) = 50/540, & \bar{\mu}_1(a_4) = \bar{\mu}_1(a_5) = 101/540, \\ \bar{\lambda}_1(e) = \bar{\lambda}_1(a_1) = \bar{\lambda}_1(a_2) = 65/540, & \bar{\lambda}_1(a_3) = 111/540, & \bar{\lambda}_1(a_4) = \bar{\lambda}_1(a_5) = 60/540. \end{array}$$

Then, for any $r \geq 1$, ${}_r(H_{31}) \simeq {}_0(H_{31})$ and therefore $i.f.g.(H_{31}) = 1$.

$(a_9)$ Let us consider the following complete hypergroup $H_{32}$

| $\circ$ | $e$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ |
|---|---|---|---|---|---|---|
| $e$ | $A_0$ | $A_0$ | $A_0$ | $A_0$ | $a_4$ | $a_5$ |
| $a_1$ | | $A_0$ | $A_0$ | $A_0$ | $a_4$ | $a_5$ |
| $a_2$ | | | $A_0$ | $A_0$ | $a_4$ | $a_5$ |
| $a_3$ | | | | $A_0$ | $a_4$ | $a_5$ |
| $a_4$ | | | | | $a_5$ | $A_0$ |
| $a_5$ | | | | | | $a_4$ |

with $A_0 = \{e, a_1, a_2, a_3\}, A_1 = \{a_4\}, A_2 = \{a_5\}$. One gets that

$$\begin{array}{ll}\bar{\mu}(e) = \bar{\mu}(a_1) = \bar{\mu}(a_2) = \bar{\mu}(a_3) = 9/72, & \bar{\mu}(a_4) = \bar{\mu}(a_5) = 18/72,\\ \bar{\lambda}(e) = \bar{\lambda}(a_1) = \bar{\lambda}(a_2) = \bar{\lambda}(a_3) = 36/72, & \bar{\lambda}(a_4) = \bar{\lambda}(a_5) = 27/72.\end{array}$$

Therefore the associated join space ${}_0(H_{32})$ is as follows:

| $\circ_{\bar{\mu}\wedge\bar{\lambda}}$ | $e$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ |
|---|---|---|---|---|---|---|
| $e$ | $A_0$ | $A_0$ | $A_0$ | $A_0$ | $H$ | $H$ |
| $a_1$ | | $A_0$ | $A_0$ | $A_0$ | $H$ | $H$ |
| $a_2$ | | | $A_0$ | $A_0$ | $H$ | $H$ |
| $a_3$ | | | | $A_0$ | $H$ | $H$ |
| $a_4$ | | | | | $\{a_4, a_5\}$ | $\{a_4, a_5\}$ |
| $a_5$ | | | | | | $\{a_4, a_5\}$ |

with

$$\begin{array}{l}\bar{\mu}_1(e) = \bar{\mu}_1(a_1) = \bar{\mu}_1(a_2) = \bar{\mu}_1(a_3) \neq \bar{\mu}_1(a_4) = \bar{\mu}_1(a_5),\\ \bar{\lambda}_1(e) = \bar{\lambda}_1(a_1) = \bar{\lambda}_1(a_2) = \bar{\lambda}_1(a_3) \neq \bar{\lambda}_1(a_4) = \bar{\lambda}_1(a_5).\end{array}$$

Then, for any $r \geq 1$, ${}_r(H_{32}) \simeq {}_0(H_{32})$ and therefore $i.f.g.(H_{32}) = 1$.

$(a_{10})$ Let us consider $H_{33}$ given by the following table

| $\circ$ | $e$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ |
|---|---|---|---|---|---|---|
| $e$ | $e$ | $a_1$ | $A_2$ | $A_2$ | $A_3$ | $A_3$ |
| $a_1$ | | $e$ | $A_3$ | $A_3$ | $A_2$ | $A_2$ |
| $a_2$ | | | $e$ | $e$ | $a_1$ | $a_1$ |
| $a_3$ | | | | $e$ | $a_1$ | $a_1$ |
| $a_4$ | | | | | $e$ | $e$ |
| $a_5$ | | | | | | $e$ |

with $A_0 = \{e\}, A_1 = \{a_1\}, A_2 = \{a_2, a_3\}, A_3 = \{a_4, a_5\}$ and $G \simeq (K, \cdot)$ the Klein four-group. It is obvious that $H_{33}$ is an 1-hypergroup. It results that

$$\begin{array}{ll}\bar{\mu}(e) = \bar{\mu}(a_1) = 10/36, & \bar{\mu}(a_2) = \bar{\mu}(a_3) = \bar{\mu}(a_4) = \bar{\mu}(a_5) = 4/36,\\ \bar{\lambda}(e) = \bar{\lambda}(a_1) = 18/36, & \bar{\lambda}(a_2) = \bar{\lambda}(a_3) = \bar{\lambda}(a_4) = \bar{\lambda}(a_5) = 24/36.\end{array}$$

Therefore the associated join space ${}_0(H_{33})$ is as follows:

| $\circ_{\bar{\mu}\wedge\bar{\lambda}}$ | $e$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ |
|---|---|---|---|---|---|---|
| $e$ | $\{e, a_1\}$ | $\{e, a_1\}$ | $H$ | $H$ | $H$ | $H$ |
| $a_1$ | | $\{e, a_1\}$ | $H$ | $H$ | $H$ | $H$ |
| $a_2$ | | | $\{a_2, a_3, a_4, a_5\}$ | $\{a_2, a_3, a_4, a_5\}$ | $\{a_2, a_3, a_4, a_5\}$ | $\{a_2, a_3, a_4, a_5\}$ |
| $a_3$ | | | | $\{a_2, a_3, a_4, a_5\}$ | $\{a_2, a_3, a_4, a_5\}$ | $\{a_2, a_3, a_4, a_5\}$ |
| $a_4$ | | | | | $\{a_2, a_3, a_4, a_5\}$ | $\{a_2, a_3, a_4, a_5\}$ |
| $a_5$ | | | | | | $\{a_2, a_3, a_4, a_5\}$ |

and

$$\begin{array}{l}\bar{\mu}_1(e) = \bar{\mu}_1(a_1) \neq \bar{\mu}_1(a_2) = \bar{\mu}_1(a_3) = \bar{\mu}_1(a_4) = \bar{\mu}_1(a_5),\\ \bar{\lambda}_1(e) = \bar{\lambda}_1(a_1) \neq \bar{\lambda}_1(a_2) = \bar{\lambda}_1(a_3) = \bar{\lambda}_1(a_4) = \bar{\lambda}_1(a_5).\end{array}$$

Then, for any $r \geq 1$, ${}_r(H_{33}) \simeq {}_0(H_{33})$ and therefore $i.f.g.(H_{33}) = 1$.

$(a_{11})$ Let us consider the following complete hypergroup $H_{34}$

| $\circ$ | $e$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ |
|---|---|---|---|---|---|---|
| $e$ | $A_0$ | $A_0$ | $a_2$ | $a_3$ | $A_3$ | $A_3$ |
| $a_1$ | | $A_0$ | $a_2$ | $a_3$ | $A_3$ | $A_3$ |
| $a_2$ | | | $A_0$ | $A_3$ | $a_3$ | $a_3$ |
| $a_3$ | | | | $A_0$ | $a_2$ | $a_2$ |
| $a_4$ | | | | | $A_0$ | $A_0$ |
| $a_5$ | | | | | | $A_0$ |

with $A_0 = \{e, a_1\}, A_1 = \{a_2\}, A_2 = \{a_3\}, A_3 = \{a_4, a_5\}$ and $G \simeq (K, \cdot)$ the Klein four-group. We calculate that

$$\begin{array}{ll} \bar{\mu}(e) = \bar{\mu}(a_1) = \bar{\mu}(a_4) = \bar{\mu}(a_5) = 5/36, & \bar{\mu}(a_2) = \bar{\mu}(a_3) = 8/36, \\ \bar{\lambda}(e) = \bar{\lambda}(a_1) = \bar{\lambda}(a_4) = \bar{\lambda}(a_5) = 21/36, & \bar{\lambda}(a_2) = \bar{\lambda}(a_3) = 18/36. \end{array}$$

Therefore the associated join space ${}_0(H_{34})$ is as follows:

| $\circ_{\bar{\mu}\wedge\bar{\lambda}}$ | $e$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ |
|---|---|---|---|---|---|---|
| $e$ | $\{e, a_1, a_4, a_5\}$ | $\{e, a_1, a_4, a_5\}$ | $H$ | $H$ | $\{e, a_1, a_4, a_5\}$ | $\{e, a_1, a_4, a_5\}$ |
| $a_1$ | | $\{e, a_1, a_4, a_5\}$ | $H$ | $H$ | $\{e, a_1, a_4, a_5\}$ | $\{e, a_1, a_4, a_5\}$ |
| $a_2$ | | | $\{a_2, a_3\}$ | $\{a_2, a_3\}$ | $H$ | $H$ |
| $a_3$ | | | | $\{a_2, a_3\}$ | $H$ | $H$ |
| $a_4$ | | | | | $\{e, a_1, a_4, a_5\}$ | $\{e, a_1, a_4, a_5\}$ |
| $a_5$ | | | | | | $\{e, a_1, a_4, a_5\}$ |

then

$$\begin{array}{l} \bar{\mu}_1(e) = \bar{\mu}_1(a_1) = \bar{\mu}_1(a_4) = \bar{\mu}_1(a_5) \neq \bar{\mu}_1(a_2) = \bar{\mu}_1(a_3), \\ \bar{\lambda}_1(e) = \bar{\lambda}_1(a_1) = \bar{\lambda}_1(a_4) = \bar{\lambda}_1(a_5) \neq \bar{\lambda}_1(a_2) = \bar{\lambda}_1(a_3). \end{array}$$

Then, for any $r \geq 1$, ${}_r(H_{34}) \simeq {}_0(H_{34})$ and therefore $i.f.g.(H_{34}) = 1$.

$(a_{12})$ For the complete hypergroup $H_{35}$

| $\circ$ | $e$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ |
|---|---|---|---|---|---|---|
| $e$ | $A_0$ | $A_0$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ |
| $a_1$ | | $A_0$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ |
| $a_2$ | | | $a_3$ | $a_4$ | $a_5$ | $A_0$ |
| $a_3$ | | | | $a_5$ | $A_0$ | $a_2$ |
| $a_4$ | | | | | $a_2$ | $a_3$ |
| $a_5$ | | | | | | $a_4$ |

with $A_0 = \{e, a_1\}, A_1 = \{a_2\}, A_2 = \{a_3\}, A_3 = \{a_4\}, A_4 = \{a_5\}$, we calculate that

$$\begin{array}{ll} \bar{\mu}(e) = \bar{\mu}(a_1) = 4/36, & \bar{\mu}(a_2) = \bar{\mu}(a_3) = \bar{\mu}(a_4) = \bar{\mu}(a_5) = 7/36, \\ \bar{\lambda}(e) = \bar{\lambda}(a_1) = 28/36, & \bar{\lambda}(a_2) = \bar{\lambda}(a_3) = \bar{\lambda}(a_4) = \bar{\lambda}(a_5) = 25/36. \end{array}$$

We notice that ${}_0(H_{35})$ is isomorphic to ${}_0(H_{33})$ and therefore, for any $r \geq 1$, ${}_r(H_{35}) \simeq {}_0(H_{35})$ and so $i.f.g.(H_{35}) = 1$.

Now we present the complete hypergroups of order 6 which have the intuitionistic fuzzy grade equal to 2.

$(b_1)$ The complete hypergroup $H_{36}$ is the following one

| $\circ$ | $e$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ |
|---|---|---|---|---|---|---|
| $e$ | $A_0$ | $A_0$ | $A_0$ | $a_3$ | $a_4$ | $a_5$ |
| $a_1$ | | $A_0$ | $A_0$ | $a_3$ | $a_4$ | $a_5$ |
| $a_2$ | | | $A_0$ | $a_3$ | $a_4$ | $a_5$ |
| $a_3$ | | | | $a_4$ | $a_5$ | $A_0$ |
| $a_4$ | | | | | $A_0$ | $a_3$ |
| $a_5$ | | | | | | $a_4$ |

where $A_0 = \{e, a_1, a_2\}, A_1 = \{a_3\}, A_2 = \{a_4\}, A_3 = \{a_5\}$, and $G \simeq (\mathbb{Z}_4, +)$. Then

$$\bar{\mu}(e) = \bar{\mu}(a_1) = \bar{\mu}(a_2) = 4/36, \quad \bar{\mu}(a_3) = \bar{\mu}(a_4) = \bar{\mu}(a_5) = 8/36,$$
$$\bar{\lambda}(e) = \bar{\lambda}(a_1) = \bar{\lambda}(a_2) = 24/36, \quad \bar{\lambda}(a_3) = \bar{\lambda}(a_4) = \bar{\lambda}(a_5) = 20/36.$$

Therefore the associated join space ${}_0(H_{36})$ is as follows:

| $\circ_{\bar{\mu}\wedge\bar{\lambda}}$ | $e$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ |
|---|---|---|---|---|---|---|
| $e$ | $A_0$ | $A_0$ | $A_0$ | $H$ | $H$ | $H$ |
| $a_1$ | | $A_0$ | $A_0$ | $H$ | $H$ | $H$ |
| $a_2$ | | | $A_0$ | $H$ | $H$ | $H$ |
| $a_3$ | | | | $\{a_3, a_4, a_5\}$ | $\{a_3, a_4, a_5\}$ | $\{a_3, a_4, a_5\}$ |
| $a_4$ | | | | | $\{a_3, a_4, a_5\}$ | $\{a_3, a_4, a_5\}$ |
| $a_5$ | | | | | | $\{a_3, a_4, a_5\}$ |

We find that

$$\bar{\mu}_1(e) = \bar{\mu}_1(a_1) = \bar{\mu}_1(a_2) = \bar{\mu}_1(a_3) = \bar{\mu}_1(a_4) = \bar{\mu}_1(a_5) = 6/36,$$
$$\bar{\lambda}_1(e) = \bar{\lambda}_1(a_1) = \bar{\lambda}_1(a_2) = \bar{\lambda}_1(a_3) = \bar{\lambda}_1(a_4) = \bar{\lambda}_1(a_5) = 3/36.$$

It follows that ${}_1(H_{36})$ is a total hypergroup. Then, for any $r \geq 2$, ${}_r(H_{36}) = {}_1(H_{36})$ and therefore $i.f.g.(H_{36}) = 2$.

$(b_2)$ The complete hypergroup $H_{37}$ has the following table

| $\circ$ | $e$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ |
|---|---|---|---|---|---|---|
| $e$ | $A_0$ | $A_0$ | $A_0$ | $a_3$ | $a_4$ | $a_5$ |
| $a_1$ | | $A_0$ | $A_0$ | $a_3$ | $a_4$ | $a_5$ |
| $a_2$ | | | $A_0$ | $a_3$ | $a_4$ | $a_5$ |
| $a_3$ | | | | $A_0$ | $a_5$ | $a_4$ |
| $a_4$ | | | | | $A_0$ | $a_3$ |
| $a_5$ | | | | | | $A_0$ |

with $G \simeq (K, .)$ the Klein four-group, $A_0 = \{e, a_1, a_2\}, A_1 = \{a_3\}, A_2 = \{a_4\}, A_3 = \{a_5\}$. We obtain the same membership functions as in the previous case. So, $i.f.g.(H_{37}) = 2$.

$(b_3)$ Taking the complete hypergroup $H_{38}$ as the following 1-hypergroup

| $\circ$ | $e$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ |
|---|---|---|---|---|---|---|
| $e$ | $e$ | $a_1$ | $a_2$ | $a_3$ | $A_4$ | $A_4$ |
| $a_1$ | | $a_2$ | $a_3$ | $A_4$ | $e$ | $e$ |
| $a_2$ | | | $A_4$ | $e$ | $a_1$ | $a_1$ |
| $a_3$ | | | | $a_1$ | $a_2$ | $a_2$ |
| $a_4$ | | | | | $a_3$ | $a_3$ |
| $a_5$ | | | | | | $a_3$ |

where $A_0 = \{e\}, A_1 = \{a_1\}, A_2 = \{a_2\}, A_3 = \{a_3\}, A_4 = \{a_4, a_5\}$, then

$$\bar{\mu}(e) = \bar{\mu}(a_1) = \bar{\mu}(a_2) = 14/72, \quad \bar{\mu}(a_3) = 16/72, \quad \bar{\mu}(a_4) = \bar{\mu}(a_5) = 7/72,$$
$$\bar{\lambda}(e) = \bar{\lambda}(a_1) = \bar{\lambda}(a_2) = 51/72, \quad \bar{\lambda}(a_3) = 49/72, \quad \bar{\lambda}(a_4) = \bar{\lambda}(a_5) = 58/72.$$

Therefore the associated join space $_0(H_{38})$ is as follows:

| $\circ_{\bar{\mu}\wedge\bar{\lambda}}$ | $e$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ |
|---|---|---|---|---|---|---|
| $e$ | $\{e,a_1,a_2\}$ | $\{e,a_1,a_2\}$ | $\{e,a_1,a_2\}$ | $\{e,a_1,a_2,a_3\}$ | $B_3$ | $B_3$ |
| $a_1$ | | $\{e,a_1,a_2\}$ | $\{e,a_1,a_2\}$ | $\{e,a_1,a_2,a_3\}$ | $B_3$ | $B_3$ |
| $a_2$ | | | $\{e,a_1,a_2\}$ | $\{e,a_1,a_2,a_3\}$ | $B_3$ | $B_3$ |
| $a_3$ | | | | $a_3$ | $H$ | $H$ |
| $a_4$ | | | | | $A_4$ | $A_4$ |
| $a_5$ | | | | | | $A_4$ |

and we obtain that

$$\bar{\mu}_1(e)=\bar{\mu}_1(a_1)=\bar{\mu}_1(a_2)=227/1080,\quad \bar{\mu}_1(a_3)=95/1080,\quad \bar{\mu}_1(a_4)=\bar{\mu}_1(a_5)=152/1080,$$
$$\bar{\lambda}_1(e)=\bar{\lambda}_1(a_1)=\bar{\lambda}_1(a_2)=90/1080,\quad \bar{\lambda}_1(a_3)=222/1080,\quad \bar{\lambda}_1(a_4)=\bar{\lambda}_1(a_5)=165/1080.$$

Therefore the associated join space $_1(H_{38})$ is as follows:

| $\circ_{\bar{\mu}_1\wedge\bar{\lambda}_1}$ | $e$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ |
|---|---|---|---|---|---|---|
| $e$ | $\{e,a_1,a_2\}$ | $\{e,a_1,a_2\}$ | $\{e,a_1,a_2\}$ | $\{e,a_1,a_2,a_3\}$ | $H$ | $H$ |
| $a_1$ | | $\{e,a_1,a_2\}$ | $\{e,a_1,a_2\}$ | $\{e,a_1,a_2,a_3\}$ | $H$ | $H$ |
| $a_2$ | | | $\{e,a_1,a_2\}$ | $\{e,a_1,a_2,a_3\}$ | $H$ | $H$ |
| $a_3$ | | | | $a_3$ | $\{a_3,a_4,a_5\}$ | $\{a_3,a_4,a_5\}$ |
| $a_4$ | | | | | $A_4$ | $A_4$ |
| $a_5$ | | | | | | $A_4$ |

for which we find

$$\bar{\mu}_2(e)=\bar{\mu}_2(a_1)=\bar{\mu}_2(a_2)=39/216,\quad \bar{\mu}_2(a_3)=35/216,\quad \bar{\mu}_2(a_4)=\bar{\mu}_2(a_5)=32/216,$$
$$\bar{\lambda}_2(e)=\bar{\lambda}_2(a_1)=\bar{\lambda}_2(a_2)=26/216,\quad \bar{\lambda}_2(a_3)=30/216,\quad \bar{\lambda}_2(a_4)=\bar{\lambda}_2(a_5)=33/216.$$

Then, for any $r\geq 2$, $_r(H_{38})\simeq {}_1(H_{38})$ and therefore $i.f.g.(H_{38})=2$.

The last two complete hypergroups of order 6 have the intuitionistic fuzzy grade equal to 3.

$(c_1)$ For the 1-hypergroup $H_{39}$

| $\circ$ | $e$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ |
|---|---|---|---|---|---|---|
| $e$ | $e$ | $a_1$ | $A_2$ | $A_2$ | $A_3$ | $A_3$ |
| $a_1$ | | $A_2$ | $A_3$ | $A_3$ | $e$ | $e$ |
| $a_2$ | | | $e$ | $e$ | $a_1$ | $a_1$ |
| $a_3$ | | | | $e$ | $a_1$ | $a_1$ |
| $a_4$ | | | | | $A_2$ | $A_2$ |
| $a_5$ | | | | | | $A_2$ |

where $A_0=\{e\}, A_1=\{a_1\}, A_2=\{a_2,a_3\}, A_3=\{a_4,a_5\}$, and $G\simeq(\mathbb{Z}_4,+)$, we find that

$$\bar{\mu}(e)=18/72,\quad \bar{\mu}(a_1)=20/72,\quad \bar{\mu}(a_2)=\bar{\mu}(a_3)=9/72,\quad \bar{\mu}(a_4)=\bar{\mu}(a_5)=8/72,$$
$$\bar{\lambda}(e)=37/72,\quad \bar{\lambda}(a_1)=35/72,\quad \bar{\lambda}(a_2)=\bar{\lambda}(a_3)=46/72,\quad \bar{\lambda}(a_4)=\bar{\lambda}(a_5)=47/72.$$

Therefore the associated join space $_0(H_{39})$ is as follows:

| $\circ_{\bar{\mu}\wedge\bar{\lambda}}$ | $e$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ |
|---|---|---|---|---|---|---|
| $e$ | $e$ | $\{e,a_1\}$ | $\{e,a_2,a_3\}$ | $\{e,a_2,a_3\}$ | $B_1$ | $B_1$ |
| $a_1$ | | $a_1$ | $\{e,a_1,a_2,a_3\}$ | $\{e,a_1,a_2,a_3\}$ | $H$ | $H$ |
| $a_2$ | | | $A_2$ | $A_2$ | $\{a_2,a_3,a_4,a_5\}$ | $\{a_2,a_3,a_4,a_5\}$ |
| $a_3$ | | | | $A_2$ | $\{a_2,a_3,a_4,a_5\}$ | $\{a_2,a_3,a_4,a_5\}$ |
| $a_4$ | | | | | $A_3$ | $A_3$ |
| $a_5$ | | | | | | $A_3$ |

and

$$\begin{array}{lll}\bar{\mu}_1(e) = 87/540, & \bar{\mu}_1(a_1) = 55/540, & \bar{\mu}_1(a_2) = \bar{\mu}_1(a_3) = 117/540,\\ \bar{\lambda}_1(e) = 105/540, & \bar{\lambda}_1(a_1) = 137/540, & \bar{\lambda}_1(a_2) = \bar{\lambda}_1(a_3) = 75/540,\\ \bar{\mu}_1(a_4) = \bar{\mu}_1(a_5) = 82/540, & \bar{\lambda}_1(a_4) = \bar{\lambda}_1(a_5) = 110/540. & \end{array}$$

Therefore the associated join space $_1(H_{39})$ is as follows:

| $\circ_{\bar{\mu}_1 \wedge \bar{\lambda}_1}$ | $e$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ |
|---|---|---|---|---|---|---|
| $e$ | $e$ | $H$ | $B_1$ | $B_1$ | $\{e, a_4, a_5\}$ | $\{e, a_4, a_5\}$ |
| $a_1$ | | $a_1$ | $\{a_1, a_2, a_3\}$ | $\{a_1, a_2, a_3\}$ | $B_0$ | $B_0$ |
| $a_2$ | | | $A_2$ | $A_2$ | $\{a_2, a_3, a_4, a_5\}$ | $\{a_2, a_3, a_4, a_5\}$ |
| $a_3$ | | | | $A_2$ | $\{a_2, a_3, a_4, a_5\}$ | $\{a_2, a_3, a_4, a_5\}$ |
| $a_4$ | | | | | $A_3$ | $A_3$ |
| $a_5$ | | | | | | $A_3$ |

and

$$\begin{array}{ll}\bar{\mu}_2(e) = \bar{\mu}_2(a_1) = 52/540, & \bar{\mu}_2(a_2) = \bar{\mu}_2(a_3) = \bar{\mu}_2(a_4) = \bar{\mu}_2(a_5) = 109/540,\\ \bar{\lambda}_2(e) = \bar{\lambda}_2(a_1) = 137/540, & \bar{\lambda}_2(a_2) = \bar{\lambda}_2(a_3) = \bar{\lambda}_2(a_4) = \bar{\lambda}_2(a_5) = 80/540.\end{array}$$

Therefore the associated join space $_2(H_{39})$ is as follows:

| $\circ_{\bar{\mu}_2 \wedge \bar{\lambda}_2}$ | $e$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ |
|---|---|---|---|---|---|---|
| $e$ | $\{e, a_1\}$ | $\{e, a_1\}$ | $H$ | $H$ | $H$ | $H$ |
| $a_1$ | | $\{e, a_1\}$ | $H$ | $H$ | $H$ | $H$ |
| $a_2$ | | | $\{a_2, a_3, a_4, a_5\}$ | $\{a_2, a_3, a_4, a_5\}$ | $\{a_2, a_3, a_4, a_5\}$ | $\{a_2, a_3, a_4, a_5\}$ |
| $a_3$ | | | | $\{a_2, a_3, a_4, a_5\}$ | $\{a_2, a_3, a_4, a_5\}$ | $\{a_2, a_3, a_4, a_5\}$ |
| $a_4$ | | | | | $\{a_2, a_3, a_4, a_5\}$ | $\{a_2, a_3, a_4, a_5\}$ |
| $a_5$ | | | | | | $\{a_2, a_3, a_4, a_5\}$ |

for which we calculate

$$\begin{array}{l}\bar{\mu}_3(e) = \bar{\mu}_3(a_1) \neq \bar{\mu}_3(a_3) = \bar{\mu}_3(a_4) = \bar{\mu}_3(a_4) = \bar{\mu}_3(a_5),\\ \bar{\lambda}_3(e) = \bar{\lambda}_3(a_1) \neq \bar{\lambda}_3(a_2) = \bar{\lambda}_3(a_3) = \bar{\lambda}_3(a_4) = \bar{\lambda}_3(a_5).\end{array}$$

Then, for any $r \geq 3$, $_r(H_{39}) \simeq {}_2(H_{39})$ and therefore $i.f.g.(H_{39}) = 3$.

$(c_2)$ Let us consider $H_{40}$ as the following complete hypergroup

| $\circ$ | $e$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ |
|---|---|---|---|---|---|---|
| $e$ | $A_0$ | $A_0$ | $a_2$ | $a_3$ | $A_3$ | $A_3$ |
| $a_1$ | | $A_0$ | $a_2$ | $a_3$ | $A_3$ | $A_3$ |
| $a_2$ | | | $a_3$ | $A_3$ | $A_0$ | $A_0$ |
| $a_3$ | | | | $A_0$ | $a_2$ | $a_2$ |
| $a_4$ | | | | | $a_3$ | $a_3$ |
| $a_5$ | | | | | | $a_3$ |

with $A_0 = \{e, a_1\}, A_1 = \{a_2\}, A_2 = \{a_3\}, A_3 = \{a_4, a_5\}$, and $G \simeq (\mathbb{Z}_4, +)$ (the 4-tuple associated with $H_{40}$ is $[2, 1, 1, 2]$). Then, we obtain the following membership functions

$$\begin{array}{lll}\bar{\mu}(e) = \bar{\mu}(a_1) = 9/72, & \bar{\mu}(a_2) = 16/72, & \bar{\mu}(a_3) = 18/72,\\ \bar{\lambda}(e) = \bar{\lambda}(a_1) = 44/72, & \bar{\lambda}(a_2) = 37/72, & \bar{\lambda}(a_3) = 35/72,\\ \bar{\mu}(a_4) = \bar{\mu}(a_5) = 10/72, & \bar{\lambda}(a_4) = \bar{\lambda}(a_5) = 43/72. & \end{array}$$

It is clear that the associated join space $_0(H_{40})$ is isomorphic to the join space $_0(H_{39})$ and therefore $i.f.g.(H_{40}) = 3$.

$\square$

Making a short comparison with the fuzzy grade of the same hypergroups, we notice that there are no complete hypergroups of order less than or equal to 6 with the fuzzy grade equal to 3, instead there are 5 such hypergroups with the intuitionistic fuzzy grade equal to 3. Moreover, for the complete hypergroups of order 3 or 4, the fuzzy grade coincides with the intuitionistic fuzzy grade.

## 5. Conclusions and future work

In this paper, we have presented the join spaces and the membership functions of the intuitionistic fuzzy sets associated with all forty non-isomorphic complete hypergroups of order less than or equal to 6, determining their intuitionistic fuzzy grades. A similar work has been done by Cristea [14], regarding the fuzzy grades of the same hypergroups.

The fuzzy grade of a complete hypergroup $H$ constructed from a group $G$ does not depend on the group $G$, but only on the $m$-decomposition of $n = |H|$. More exactly, if $G_1$ and $G_2$ are non-isomorphic groups of the same order $m$, and $H_1$ and $H_2$ are the correspondent complete hypergroups of order $n$, then $f.g.(H_1) = f.g.(H_2)$. This is an immediate consequence of Theorem 2.3 [14]. In this paper, we noticed that the intuitionistic fuzzy grade of a complete hypergroup does not have the same property. For example, let $H$ be a complete hypergroup of order 6 such that $[1, 1, 2, 2]$ is the 4-tuple associated with it. Therefore, there exist two non-isomorphic hypergroups of such type: the hypergroup denoted in this article with $H_{39}$ (obtained with the group $G \simeq (\mathbb{Z}_4, +)$) and the hypergroup $H_{33}$ (obtained with the group $G \simeq (K, \cdot)$ the Klein four group). We have obtained that $i.f.g.(H_{39}) = 3$ and $i.f.g.(H_{33}) = 1$. Thereby the intuitionistic fuzzy grade of a complete hypergroup depends also on the group $G$. It seems interesting to find conditions connected with the group $G$ ( with $|G| = m$) such that $i.f.g.(H)$ depends only on the $m$-decomposition of $n = |H|$. This theme will be discussed in a future work.

## References


[1] Atanassov, K.T. *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems, **20** (1986), 87-96.

[2] Atanassov, K.T. *New operations defined over the intuitionistic fuzzy sets*, Fuzzy Sets and Systems, **61**(1994), 137-142.

[3] Corsini, P. *Prolegomena of hypergroup theory*, Aviani Editore, 1993.

[4] Corsini, P. *Join spaces, power sets, fuzzy sets*, in: Proc. Fifth Int. Cong. A.H.A. 1993, Iasi, Romania, Hadronic Press (1994), 45-52.

[5] Corsini, P. *A new connection between hypergroups and fuzzy sets*, Southeast Asian Bull. Math., **27**(2003), 221-229.

[6] Corsini, P., Cristea, I. *Fuzzy grade of i.p.s. hypergroups of order less than or equal to* 6, PU. M. A., **14**(2003), no.4, 275-288.

[7] Corsini, P., Cristea, I. *Fuzzy grade of i.p.s. hypergroups of order* 7, Iran. J. Fuzzy Syst., **1**(2004), 15-32.

[8] Corsini, P., Cristea, I. *Fuzzy sets and non complete 1-hypergroups*, An. St. Univ. Ovidius Constanta Ser. Mat., **13**(2005), no.1, 27-53.

[9] Corsini, P., Leoreanu, V. *Join spaces associated with fuzzy sets*, J. Combin. Inform. Syst. Sci., **20**(1995), no. 1-4, 293-303.

[10] Corsini, P., Leoreanu, V. *Applications of hyperstructure theory*, Advances in Mathematics, Kluwer Academic Publishers, Dordercht, 2003.

[11] Corsini, P., Leoreanu-Fotea, V. *On the grade of a sequence of fuzzy sets and join spaces determined by a hypergraph*, Southeast Asian Bull. Math., **34**(2010), no.2, 231-242.

[12] Corsini, P., Leoreanu, V., Iranmanesh, A. *On the sequence of hypergroups and membership functions determined by a hypergraph*, J. Mult.-Valued Logic Soft Comput., **14**(2008), no.6, 565-577.

[13] Corsini, P., Mahjoob, R. *Multivalued functions, fuzzy subsets and join spaces*, Ratio Math., **20**(2010), 1-41.

[14] Cristea, I. *Complete hypergroups,* 1*-hypergroups and fuzzy sets*, An. St. Univ. Ovidius Constanta Ser. Mat., **10**(2002), no. 2, 25-37.
[15] Cristea, I. *About the fuzzy grade of the direct product of two hypergroupoids*, Iran. J. Fuzzy Syst., **7**(2010), no.2, 95-108.
[16] Cristea, I., Darafsheh, M.R. *A decomposition of the cyclic groups of order $p$ and $p^2$, where $p$ is a prime number*, J. Intell. Fuzzy Syst., **21**(2010), no.4, 255-259.
[17] Cristea, I., Darafsheh, M.R. *About a decomposition of the group $\mathbb{Z}_{pq}$, where $p$ and $q$ are distinct odd primes*, European J. Combin., **31**(2010), no. 7, 1880-1886.
[18] Cristea, I., Davvaz, B. *Atanassov's intuitionistic fuzzy grade of hypergroups*, Inform. Sci., **180**(2010), 1506-1517.
[19] Davvaz, B. *Fuzzy $H_v$-groups*, Fuzzy Sets and Systems, **101** (1999), 191-195.
[20] Davvaz, B. *Fuzzy $H_v$-sumodules*, Fuzzy Sets and Systems, **117**(2001), 477-484.
[21] Davvaz, B., Corsini, P., Leoreanu-Fotea, V. *Atanassov's intuitionistic (S,T)-fuzzy $n$-ary sub-hypergroups and their properties*, Inform. Sci., **179**(2009), 654-666.
[22] Davvaz, B., Leoreanu-Fotea, V. *Intuitionistic fuzzy $n$-ary hypergroups*, J. Mult.-Valued Logic Soft Comput., **16**(1-2) (2010), 87-104.
[23] Davvaz, B., Leoreanu-Fotea, V. *Hyperring theory and Applications*, International Academic Press, USA, 2007.
[24] Davvaz, B., Hassani Sadrabadi, E., Cristea, I. *Atanassov's intuitionistic fuzzy grade of i.p.s. hypergroups of order* 7, J. Mult.-Valued Logic Soft Comput., **20**(2013), no.5-6, 467-506.
[25] Davvaz, B., Hassani Sadrabadi, E., Cristea, I. *Atanassov's intuitionistic fuzzy grade of i.p.s. hypergroups of order less than or equal to* 6, Iran. J. Fuzzy Syst., **9**(2012), no.4, 71-97.
[26] Migliorato, R. *On the complete hypergroups*, Riv. Mat. Pura Appl., **14** (1994), 21-31.
[27] Prenowitz, W., Jantosciak, J. *Geometries and join spaces*, J. Reind und Angew Math., **257**(1972), 100-128.
[28] Rosenfeld, A. *Fuzzy groups*, J. Math. Anal. Appl., **35**(1971), 512-517.
[29] Ştefănescu, M., Cristea, I. *On the fuzzy grade of hypergroups*, Fuzzy Sets and Systems, **159**(2008), 1097-1106.
[30] Vougiouklis, T. *Hyperstructures and their representations*, Hadronic Press, Inc, 115 Palm Harbor, USA. 1994.
[31] Zadeh, L.A. *Fuzzy sets*, Inform. Control, **8**(1965), 338-353.