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ON LEFT IDEALS OF PRIME RINGS WITH GENERALIZED DERIVATIONS

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Abstract

In this paper the author considers a prime ring R with characteristic different from two and extends some well known results concerning derivations of prime rings to the generalized derivation $f : R \to R$ associated with a derivation d of R and a nonzero left ideal U of R which is semiprime as a ring.

Keywords: Prime ring, Derivation, Generalized derivation, Homomorphism, Anti-homomorphism.

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1. Introduction

Throughout this paper, R will be a prime ring with characteristic different from two and I a nonzero left ideal of R which is semiprime as a ring, Z the multiplicative center of R, $Q_r(R)$ the right Martindale ring of quotients, C the extended centroid and $R_C = RC$ the central closure. For any $x, y \in R$, the symbol $[x, y]$ will represent the commutator $xy - yx$. An additive mapping $f : R \to R$ is called a generalized derivation if there exists a derivation $d : R \to R$ such that

$$f(xy) = f(x)y + xd(y)$$

for all $x, y \in R$ The concept of generalized derivation includes the concept of derivation. Moreover, a generalized derivation with $d = 0$ includes the concept of left multiplier, that is an additive map satisfying $f(xy) = f(x)y$, for all $x, y \in R$.

The study of the commutativity of prime rings with derivations was initiated by E. C. Posner [10]. Over the last two decades, a lot of work has been done on this subject. Recently, M. Bresar defined a generalized derivation in [5]. Many authors have investigated the properties of prime or semiprime rings with generalized derivations. In the present paper our objective is to generalize some results obtained in [2], [3], [4], [7] and [9] for generalized derivations and a left ideal of a prime ring R which is semiprime as a ring.

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2. Results

We begin by recalling the following two results.

2.1. Lemma. [6, Lemma 1] *Let R be a prime ring and U a nonzero left ideal of R which is semiprime as a ring. If $Ua = 0$ ($aU = 0$) for $a \in R$, then $a = 0$.*

2.2. Lemma. [8, Lemma 2] *Let $f : R \to R_C$ be an additive map satisfying $f(xy) = f(x)y$, for all $x, y \in R$. Then there exists $q \in Q_r(R_C)$ such that $f(x) = qx$, for all $x \in R$.*

Now we have:

2.3. Lemma. *Let R be a prime ring and U a nonzero left ideal of R which is semiprime as a ring. If d is a derivation of R such that $d(U) = 0$, then $d = 0$.*

Proof. For all $x \in U,\ r \in R$, we get

$$0 = d(rx) = d(r)x,$$

and so,

$$d(R)U = 0.$$

By Lemma 2.1, we obtain that $d = 0$. □

The following two theorems are generalization of [3, Theorem 3] and [4, Theorem 1], respectively.

2.4. Theorem. *Let R be a prime ring, U a nonzero left ideal of R which is semiprime as a ring and f a generalized derivation of R. If U is noncommutative and $f([x, y]) = 0$, for all $x, y \in U$, then there exists $q \in Q_r(R_C)$ such that $f(x) = qx$, for all $x \in R$.*

Proof. Substitute yx for y in $f([x, y]) = 0$, giving

$$0 = f([x, yx]) = f([x, y]x) = f([x, y])x + [x, y]d(x),$$

and so,

$$[x, y]d(x) = 0, \text{ for all } x, y \in U.$$

Hence $0 = [x, ry]d(x) = r[x, y]d(x) + [x, r]yd(x)$. Since the first summand is zero, it is clear that

$$[x, r]yd(x) = 0, \text{ for all } x, y \in U, r \in R.$$

Writing $sy,\ s \in R$, in place of y in this equation, we get

$$[x, r]syd(x) = 0, \text{ for all } x, y \in U, r, s \in R.$$

Since R is a prime ring, we have

$$[x, r] = 0 \text{ or } Ud(x) = 0, \text{ for all } x \in U, r \in R.$$

By Lemma 2.1, we get either $x \in Z$ or $d(x) = 0$ for all $x \in U$. Let $A = \{x \in U \mid x \in Z\}$ and $B = \{x \in U \mid d(x) = 0\}$. Then A and B are two additive subgroups of $(U, +)$ such that $U = A \cup B$. However, a group cannot be the union of proper subgroups. Hence either $U = A$ or $U = B$. If $U = A$ then $U \subset Z$, and so U is commutative, which contradicts the hypothesis. So, we must have $d(x) = 0$, for all $x \in U$. By Lemma 2.3, we get $d = 0$. Hence, there exists $q \in Q_r(R_C)$ such that $f(x) = qx$, for all $x \in R$, by Lemma 2.2. □

2.5. Theorem. *Let R be a prime ring, U a nonzero left ideal of R which is semiprime as a ring and f a generalized derivation of R. If U is noncommutative and $f([x, y]) = \pm[x, y]$, for all $x, y \in U$, then there exists $q \in Q_r(R_C)$ such that $f(x) = qx$, for all $x \in R$.*

Proof. Assume that $f([x,y]) = \pm[x,y]$, for all $x,y \in U$. Replacing y by yx in this equation, we have

$$[x,y]d(x) = 0, \text{ for all } x,y \in U.$$

Using the same argument as in the proof of Theorem 2.4, we get $d = 0$ and so, there exists $q \in Q_r(R_C)$ such that $f(x) = qx$, for all $x \in R$ by Lemma 2.2. □

2.6. Corollary. *Let R be a prime ring, U a nonzero left ideal of R which is semiprime as a ring and f a generalized derivation of R. If U is noncommutative and $f(xy) = \pm xy$, for all $x,y \in U$, then there exists $q \in Q_r(R_C)$ such that $f(x) = qx$, for all $x \in R$.*

2.7. Theorem. *Let R be a prime ring, U a nonzero left ideal of R which is semiprime as a ring and f a generalized derivation of R. If f acts as a homomorphism or anti-homomorphism on U, then there exists $q \in Q_r(R_C)$ such that $f(x) = qx$, for all $x \in R$.*

Proof. Assume that f acts as a homomorphism on U. Then

$$f(xy) = f(x)f(y) = f(x)y + xd(y), \text{ for all } x,y \in U. \tag{2.1}$$

Replacing x by zx, $z \in U$, in the second equality in (2.1), we have

$$f(xz)f(y) = f(xz)y + xzd(y) = f(x)f(z)y + xzd(y)$$

since f is a homomorphism. On the other hand, we have

$$\begin{aligned} f(xz)f(y) &= f(x)f(z)f(y) = f(x)f(zy) = f(x)(f(z)y + zd(y)) \\ &= f(x)f(z)y + f(x)zd(y), \end{aligned}$$

on replacing y by z in (2.1). Hence

$$f(x)f(z)y + f(x)zd(y) = f(x)f(z)y + xzd(y),$$

so

$$(f(x) - x)zd(y) = 0, \text{ for all } x,y,z \in U.$$

Replacing z by rz, $r \in R$, in the above equation, we arrive at

$$(f(x) - x)rzd(y) = 0, \text{ for all } x,y,z \in U, \ r \in R.$$

Since R is a prime ring, we have either f is the identity map on U, or $Ud(U) = 0$.

Suppose that $f(x) = x$, for all $x \in U$. Then

$$\begin{aligned} xy &= f(xy) \\ &= f(x)y + xd(y) \\ &= xy + xd(y) \end{aligned}$$

and so,

$$xd(y) = 0, \text{ for all } x,y \in U.$$

Hence, we conclude that $d = 0$ by Lemma 2.1. Thus, there exists $q \in Q_r(R_C)$ such that $f(x) = qx$, for all $x \in R$ by Lemma 2.2.

Now assume that f acts as an anti-homomorphism on U. Then

$$f(xy) = f(y)f(x) = f(x)y + xd(y), \text{ for all } x,y \in U. \tag{2.2}$$

Replacing x by xy in (2.2), we get

$$\begin{aligned} f(y)f(xy) &= f(xy)y + xyd(y), \text{ hence} \\ f(y)f(x)y + f(y)xd(y) &= f(y)f(x)y + xyd(y), \end{aligned}$$

and so,

(2.3) $\quad f(y)xd(y) = xyd(y), \text{ for all } x, y \in U.$

Replacing x by rx, $r \in R$, in (2.3), to get

$$f(y)rxd(y) = rxyd(y) = rf(y)xd(y).$$

That is,

(2.4) $\quad [f(y), r]xd(y) = 0, \text{ for all } x, y \in U, \ r \in R.$

Again writing x as sx, $s \in R$, we have either $[f(y), r] = 0$ or $Ud(y) = 0$, for all $y \in U$, $r \in R$. According to Brauer's Trick and Lemma 2.1, we conclude that $f(U) \subset Z$ or $d(U) = 0$. In the second case, the proof is complete. The first case gives that f acts as a homomorphism on U. Thus, there exists $q \in Q_r(R_C)$ such that $f(x) = qx$, for all $x \in R$. □

2.8. Theorem. *Let R be a prime ring with characteristic different from two, U a nonzero left ideal of R which is semiprime as a ring, and f a generalized derivation of R. If U is noncommutative and $[x, f(x)] = 0$, for all $x \in U$, then there exists $q \in Q_r(R_C)$ such that $f(x) = qx$, for all $x \in R$.*

Proof. A linearization of $[x, f(x)] = 0$ gives

(2.5) $\quad [x, y]d(x) + y[x, d(x)] = 0, \text{ for all } x, y \in U.$

Writing yz instead of y in (2.5), and using this equation, we obtain that

(2.6) $\quad [x, y]zd(x) = 0, \text{ for all } x, y, z \in U.$

Replacing z by rz, $r \in R$, in (2.6), we get

$$[x, y] = 0 \ \text{ or } \ Ud(x) = 0, \text{ for all } x, y \in U.$$

By Lemma 2.1, we have either $[x, y] = 0$ or $d(x) = 0$, for all $x \in U$. By a standard argument one of these must be held for all $x \in U$. The first result cannot hold since U is noncommutative, so the second possibility gives $d(U) = 0$, and hence $d = 0$. Therefore, the proof may be completed by using Lemma 2.2. □

2.9. Theorem. *Let R be a prime ring with characteristic different from two, U a nonzero left ideal of R which is semiprime as a ring, and f a generalized derivation of R. If U is noncommutative, $d(Z) \neq 0$ and $[f(x), f(y)] = [x, y]$, for all $x, y \in U$, then there exists $q \in Q_r(R_C)$ such that $f(x) = qx$, for all $x \in R$.*

Proof. Taking yx instead of y in the hypothesis, we get

$$\begin{aligned} &[x, yx] = [f(x), f(yx)], \text{ whence} \\ &[x, y]x = [f(x), f(y)x + yd(x)] \\ &\qquad = [f(x), f(y)]x + f(y)[f(x), x] + [f(x), y]d(x) + y[f(x), d(x)], \end{aligned}$$

and so,

(2.7) $\quad f(y)[f(x), x] + [f(x), y]d(x) + y[f(x), d(x)] = 0, \text{ for all } x, y \in U.$

Replacing y by $cy = yc$, where $c \in Z$, and using (2.7), we arrive at

$$yd(c)[f(x), x] = 0, \text{ for all } x, y \in U.$$

Since $0 \neq d(c) \in Z$ and U is a nonzero left ideal of R, we have

$$[f(x), x] = 0, \text{ for all } x \in U.$$

The proof is now completed using Theorem 2.8. □

2.10. Theorem. *Let R be a prime ring with characteristic different from two, U a nonzero left ideal of R which is semiprime as a ring, and f a generalized derivation of R. If U is noncommutative and $f(U) \subseteq Z$, then there exists $q \in Q_r(R_C)$ such that $f(x) = qx$, for all $x \in R$.*

Proof. For all $r \in R$, we get

$$\begin{aligned} 0 &= [f(xy), y] = [f(x)y + xd(y), y] \\ &= [x, y]d(y) + x[d(y), y]. \end{aligned}$$

Expanding this equation we conclude that

$$yxd(y) = xd(y)y, \text{ for all } x, y \in U. \tag{2.8}$$

Writing xz instead of x in (2.8), and using this equality, we get

$$yxzd(y) = xzd(y)y = xyzd(y).$$

That is

$$[x, y]zd(y) = 0, \text{ for all } x, y, z \in U.$$

Taking rz, $r \in R$ in place of z in the above equation, and using the fact that R is prime, we conclude that $[x, y] = 0$ or $d(y) = 0$, for all $x, y \in U$. By the standard argument, we have either that U is commutative or $d = 0$. Since U is noncommutative, the proof is complete. □

2.11. Theorem. *Let R be a prime ring with characteristic different from two, U a nonzero left ideal of R which is semiprime as a ring, f a generalized derivation of R and $a \in R$. If U is noncommutative, $d(Z) \neq 0$ and $[a, f(x)] \in Z$ for all $x \in U$, then $a \in Z$.*

Proof. Since $d(Z) \neq 0$, there exists $c \in Z$ such that $d(c) \neq 0$. Furthermore, since d is a derivation, it is clear that $d(c) \in Z$. Replacing x by $xc = cx$ in the hypothesis, we have

$$\begin{aligned} Z \ni [a, f(xc)] &= [a, f(x)c + xd(c)] \\ &= [a, f(x)]c + [a, x]d(c). \end{aligned}$$

Since the first term lies in Z, we get

$$[a, x]d(c) \in Z, \text{ for all } x \in U.$$

Thus, we obtain that $[a, x] \in Z$, for all $x \in U$, and so

$$[[a, x], r] = 0, \text{ for all } x \in U, r \in R. \tag{2.9}$$

Taking x^2 instead of x and using (2.9), we have

$$0 = [[a, x]x + x[a, x], r] = 2[[a, x]x, r], \text{ for all } x \in U, \ r \in R.$$

Since $\operatorname{char} R \neq 2$ and $[a, x] \in Z$, we arrive at

$$[a, x][x, r] = 0, \text{ for all } x \in U, \ r \in R,$$

and so,

$$[a, x] = 0 \text{ or } [x, r] = 0, \text{ for all } x \in U, \ r \in R.$$

Let $A = \{x \in U \mid [a, x] = 0\}$ and $B = \{x \in U \mid x \in Z\}$. Then A and B are two additive subgroups of $(U, +)$ such that $U = A \cup B$. By Brauer's Trick, either $U = A$ or $U = B$. Since U is noncommutative, we have $U = A$. Hence $[a, U] = 0$, and so $a \in Z$. □

2.12. Corollary. *Let R be a prime ring with characteristic different from two, U a nonzero left ideal of R which is semiprime as a ring and f a generalized derivation of R. If U is noncommutative, $d(Z) \neq 0$ and $[f(U), f(U)] \subseteq Z$, then there exists $q \in Q_r(R_C)$ such that $f(x) = qx$, for all $x \in R$.*

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