

GEOMETRICAL STRUCTURE OF DUAL DESIGNS

Hülya Bayrak* and Soner Gönen†

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Abstract

The purpose of this study is to discuss some dual designs of balanced incomplete block designs and of partially balanced incomplete block designs. Furthermore, an attempt is made to give relations between the parameters of balanced and unbalanced incomplete block designs, and properties of finite projective geometries and related geometries.

Key Words: Balanced incomplete block design, Partially balanced incomplete block design, Dual design, Finite analytic projective geometry, Regular graph, Partial geometry.

1. Introduction and preliminaries

It will be shown that a geometrical configuration consisting of certain suitably chosen lines of the finite projective geometry $PG(m, p^n)$ and finite Euclid geometry $EG(m, p^n)$ may be interpreted as a balanced incomplete block design (BIBD) and a partially balanced incomplete block design (PBIBD).

A PBIBD $(r, k, \lambda_1, \lambda_2)$ is an arrangement of r objects (treatments or treatment combinations) into b sets (called blocks) such that:

A_1 : Each object is contained in exactly r sets.

A_2 : Each block contains k distinct objects.

A_3 : A pair of objects occur together either λ_1 times or λ_2 times. Those occurring together λ_1 times are called first associates, those occurring together λ_2 times are called second associates. When the numbers λ_i are equal we get a BIBD [6].

Bose [3] and Bose and Nair [5] have used finite geometries and Galois fields in the construction of incomplete block designs.

Bose has defined a strongly regular graph which is equivalent to the association scheme of a PBIBD with two associate classes [5].

*Gazi University, Department of Statistics, Ankara, Turkey. E-mail: hbayrak@gazi.edu.tr

†Gazi University, Department of Statistics, Ankara, Turkey. E-Mail: sgonen@gazi.edu.tr

In [4], Bose introduced the concept of a partial geometry (r, k, t) which is a system of undefined points and lines together with an incidence relation satisfying:

B_1 : Any two points are incident with not more than one line.

B_2 : Each point is incident with r lines.

B_3 : Each line is incident with k points.

B_4 : If the point P is not incident with the line L , there pass through P exactly t lines intersecting L .

From a partial geometry (r, k, t) , a PBIBD $(r, k, 1, 0)$ may be constructed where the objects and blocks are the points and lines, respectively, of the partial geometry and where two objects are first associates if corresponding points are collinear, second associates otherwise. It then follows that the number of points ν and the number of lines b in the geometry are given by $\nu = k[(r-1)(k-1) + 1]/t$, $b = r[(r-1)(k-1) + 1]/t$.

The graph G of a partial geometry $(r, k, 1)$ is a graph whose vertices correspond to the points of the geometry and two vertices are adjacent (1st associates) or non adjacent (2nd associates) according as the corresponding points lie or do not lie on a common line. This graph G is called a $(r, k, 1)$ graph. A graph G with ν vertices is said to be regular if each vertex is joined to n_1 other vertices, and unjoined to n_2 other vertices; clearly $\nu - 1 = n_1 + n_2$.

If further any two joined vertices of G , are both joined to exactly P_{11}^1 other vertices, and any two unjoined vertices are both joined to exactly P_{11}^2 other vertices, then the graph G is said to be strongly regular with parameters $n_1, n_2, P_{11}^1, P_{11}^2$.

A k -net, N , is a system of undefined points and lines, together with an incidence relation subject to the following axioms:

C_1 : N has at least one point.

C_2 : The lines of N are partitioned into k disjoint, nonempty "parallel classes" such that

- (a) each point of N is incident with exactly one line of each class,
- (b) given two lines belonging to distinct classes, there corresponds exactly one point of N which is incident with both lines.

Each line of N contains exactly n distinct points, where $n > 1$. Each point of N lies in exactly k classes, where $k > 1$. N has exactly kn distinct lines. These fall into k parallel classes of n lines each. Distinct lines of the same parallel class have no common points. Two lines of different classes have exactly one common point. N has exactly n^2 distinct point. A system N satisfying the above, we shall call a net of order n and degree k [7].

2. Dual Designs

The dual of a design is defined as a new design whose treatments and blocks are in correspondence with the blocks and treatments of the original design, and incidence is preserved (where a block and treatment are incident if the treatment is contained in the block and non-incident otherwise). For a BIBD, Fisher's inequality, $\nu \leq b$, must hold. In the dual design, blocks and treatments interchange roles, so in general Fisher's inequality cannot hold for both a BIBD and its dual. Consequently, the dual is not a balanced design. The only exception occurs when $\nu = b$, in which case the design is said to be a symmetrical BIBD, (SBIBD). It is easy to show that the dual of an SBIBD is also an SBIBD.

2.1. Theorem: *The dual of a BIBD is BIBD if and only if $\nu = b$, that is the design is symmetrical [11].*

Shrikhande proved that the duals of asymmetrical BIBD with $\lambda = 1$ or $\lambda = 2$ are PBIBD with associate classes [9].

The same results were re-established by Shrikhande and Bhagwandas using graph theory [10].

2.2. Theorem: [8] *If D is an asymmetrical BIBD with parameters $\nu, b, r, k, \lambda = 1$, then its dual D^* is a two associate-class PBIBD with the parameters,*

$$\nu^* = b, b^* = \nu, r^* = k, k^* = r, n_1^* = k(r - 1), n_2^* = b - 1 - n_1,$$

$$\lambda_1^* = 1, \lambda_2^* = 0, P_{11}^{1*} = r - 2 + (k - 1)^2, P_{11}^{2*} = k^2.$$

3. Geometric meaning of Dual Designs

We assume in this study that the varieties of the design are the points and the blocks of the design are the lines of a finite analytic geometry. In this way a one to one relationship can be formed between a SBIBD and a finite analytic projective geometry. We explain what has been said up to now with examples.

By Theorem 2.2, the dual of a BIBD with $\lambda = 1$ is a PBIBD. We illustrate this with an example.

The BIBD with parameters $\nu = 4, b = 6, k = 2, r = 3, \lambda = 1$ is EG(2, 2). The dual design of this BIBD is the PBIBD with parameters $\nu^* = 6, b^* = 4, r^* = 2, k^* = 3, \lambda_1^* = 1, \lambda_2^* = 0, n_1^* = 4, n_2^* = 1$ and $P_{11}^{1*} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}, P_{11}^{2*} = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$.

The geometrical structure of a dual design is a strongly regular graph. Then, the dual design obtained determines a pseudo singly linked block graph. Any two blocks (lines) of this design intersect in a unique treatment (point). Hence the association scheme of this design has been called a singly linked block (SLB).

It is interesting to note that a PBIBD can be obtained by omitting all the blocks containing any particular treatment from the BIBD.

For example, let us consider the PBIBD with parameter $\nu = 12$, $b = 9$, $r = 3$, $k = 4$, $\lambda_1 = 1$, $\lambda_2 = 0$, $n_1 = 9$, $n_2 = 2$ and $P_{ij}^1 = \begin{bmatrix} 6 & 2 \\ 2 & 0 \end{bmatrix}$, $P_{ij}^2 = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}$.

This design is the dual of the BIBD with parameters $\nu = 9$, $b = 12$, $r = 4$, $k = 3$ and $\lambda = 1$. The geometrical structure of this BIBD is a net [2]. The geometrical structure of the dual design is a pseudo singly linked block graph.

It is easily shown that the same design is obtained by omitting a block from the SBIBD with parameter $\nu = b = 13$, $r = k = 4$, $\lambda = 1$. The SBIBD is PG(2, 3) [1], but the dual design is not.

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