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Feckly reduced rings

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Abstract

Let R be a ring with identity and $J(R)$ denote the Jacobson radical of R. In this paper, we introduce a new class of rings called feckly reduced rings. The ring R is called *feckly reduced* if $R/J(R)$ is a reduced ring. We investigate relations between feckly reduced rings and other classes of rings. We obtain some characterizations of being a feckly reduced ring. It is proved that a ring R is feckly reduced if and only if every cyclic projective R-module has a feckly reduced endomorphism ring. Among others we show that every left Artinian ring is feckly reduced if and only if it is 2-primal, R is feckly reduced if and only if $T(R, R)$ is feckly reduced if and only if $R[x]/ < x^2 >$ is feckly reduced.

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1. Introduction

Throughout this paper, all rings are associative with identity unless otherwise stated. A ring is *reduced* if it has no nonzero nilpotent elements. It is well known that the structure of rings with Jacobson radical zero is easy to handle with, namely Artinian rings with Jacobson radical zero are direct sums of matrix rings. For any ring R, the ring $R/J(R)$ has zero Jacobson radical. Therefore it will be useful to study the rings with Jacobson radical zero. Some properties of rings are common with a ring R and $R/J(R)$, such as being Dedekind finite, stably finite, right (left) quasi-duo, and having stable range one. Invertible elements in $R/J(R)$ have invertible preimages in R and vice versa. Also, R and $R/J(R)$ have the same simple modules. By this motivation we introduce

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a class of rings, namely, feckly reduced rings. We supply some examples to show that there is no implication between the classes of reduced rings and feckly reduced rings. We show that a ring R is feckly reduced if and only if every cyclic projective R-module has a feckly reduced endomorphism ring. Apart from this, we obtain a characterization of feckly reduced rings in terms of its Jacobson radical. On the other hand, we prove that being a feckly reduced ring is not Morita invariant. In addition to these, we study trivial extensions and Dorroh extensions of feckly reduced rings.

Throughout this paper, $\mathbb{Z}$ and $\mathbb{Q}$ denote the ring of integers and the ring of rational numbers and for a positive integer n, $\mathbb{Z}_n$ is the ring of integers modulo n. We write $R[x]$, $R[[x]]$, $N(R)$ and $J(R)$ for the polynomial ring, the power series ring over a ring R, the set of all nilpotent elements and the Jacobson radical of R, respectively.

2. Feckly Reduced Rings

In this section, we introduce the concept of a feckly reduced ring. We show that there is no implication between the classes of reduced rings and feckly reduced rings.

2.1. Definition. A ring R is called *feckly reduced* if $R/J(R)$ is a reduced ring.

Note that feckly reduced rings need not be reduced and reduced rings may not be feckly reduced as the following examples show.

2.2. Example. Let F be a field. Consider the ring $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$. Then $J(R) = \begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix}$ and $R/J(R) \cong \begin{bmatrix} F & 0 \\ 0 & F \end{bmatrix}$. Since $R/J(R)$ is a reduced ring, R is feckly reduced but it is not reduced.

2.3. Example. Let R denote the localization of $\mathbb{Z}$ at $3\mathbb{Z}$, that is, $R = \{\frac{m}{n} \mid m, n \in \mathbb{Z}, 3 \nmid n\}$. Let Q denote the set of quaternions over the ring R, that is, a free R-module with basis $1, i, j, k$. Then Q is a noncommutative domain, and so it is reduced. On the other hand, $J(Q) = 3Q$, and $Q/J(Q)$ is isomorphic to 2×2 full matrix ring over $\mathbb{Z}_3$ via an isomorphism f defined by $f\big((a_0/b_0)1 + (a_1/b_1)i + (a_2/b_2)j + (a_3/b_3)k + 3Q\big) = \begin{bmatrix} a_0b_0^{-1} + a_1b_1^{-1} - a_2b_2^{-1} & a_1b_1^{-1} + a_2b_2^{-1} - a_3b_3^{-1} \\ a_1b_1^{-1} + a_2b_2^{-1} + a_3b_3^{-1} & a_0b_0^{-1} - a_1b_1^{-1} + a_2b_2^{-1} \end{bmatrix}$ for any $(a_0/b_0)1+(a_1/b_1)i+(a_2/b_2)j + (a_3/b_3)k + 3Q \in Q/3Q$ where the entries of the matrix are read modulo the ideal (3) of $\mathbb{Z}$. Hence $Q/J(Q)$ has a nonzero nilpotent element. Therefore Q is not feckly reduced.

Note that obviously, being a reduced ring and a feckly reduced ring coincide when the ring is semisimple.

2.4. Remark. Let R be a ring with $R/J(R)$ semisimple. By Weddernburn-Artin Theorem, $R/J(R)$ is isomorphic to $A_1 \times \cdots \times A_n$ where A_i is isomorphic to the ring of all $(m_i \times m_i)$-matrices over division rings D_i $(i = 1, \cdots, n)$. If the aforementioned matrix rings' types are $m_i \times m_i$ with $m_i \geq 2$, then $R/J(R)$ is not reduced. Therefore R is not feckly reduced. If $m_i = 1$ for all i, then this is not true. For example, let R denote the localization of $\mathbb{Z}$ at $3\mathbb{Z}$, i.e., $R = \{x/y \in \mathbb{Q} : 3 \nmid y\}$. Then $J(R) = \{x/y \in R : 3 \mid x\}$, and so $R/J(R)$ is a semisimple reduced ring, also $R/J(R)$ is isomorphic to $\mathbb{Z}_3$. Therefore R is feckly reduced.

Let $J^{\#}(R)$ denote the subset $\{x \in R \mid \exists\, n \in \mathbb{N}$ such that $x^n \in J(R)\}$ of R. It is obvious that $J(R) \subseteq J^{\#}(R)$, but the following example shows that the reverse inclusion does not hold in general.

2.5. Example. Let R denote the ring $M_2(\mathbb{Z}_2)$. Then

$$J^{\#}(R) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\},$$

while $J(R) = 0$.

We now give a characterization of feckly reduced rings in terms of its Jacobson radical.

2.6. Proposition. *A ring R is feckly reduced if and only if $J(R) = J^{\#}(R)$.*

Proof. Let R be a feckly reduced ring. We always have $J(R) \subseteq J^{\#}(R)$. For the converse inclusion, if $x \in J^{\#}(R)$, then $x^n \in J(R)$ for some $n \geq 1$ and so $x \in J(R)$. Thus $J(R) = J^{\#}(R)$. For the sufficiency, let $x \in R$ such that $x^n \in J(R)$ for some positive integer n. Then $x \in J^{\#}(R)$. Since $J(R) = J^{\#}(R)$, $x \in J(R)$ and so R is feckly reduced. □

By 2.6. Proposition, we can say that commutative rings and local rings are feckly reduced. The following result is an easy consequence of 2.6. Proposition.

2.7. Corollary. *Let R be a feckly reduced ring. Then all nilpotent elements of R belong to $J(R)$.*

In a ring R, $N(R) \subset J(R)$ is not an adequate condition in order that R being feckly reduced as is seen from 2.3. Example.

2.8. Lemma. *Let R be a ring with $N(R) = J(R)$. Then it is feckly reduced.*

Proof. Since $R/N(R)$ does not have any nonzero nilpotent elements, $R/J(R)$ is reduced. □

3. Examples

The purpose of this section is to supply several examples of feckly reduced rings. We see that feckly reduced rings are abundant.

3.1. Example. Let $N_2(R)$ be the set of all nilpotent elements of index two of a ring R. Assume that $J(R)$ contains $N_2(R)$. By [2, Corollary 4], we have the following.

(1) If R is a semiperfect ring, then it is feckly reduced.
(2) If R is a right or left self-injective ring, then it is feckly reduced.
(3) If R is an I-ring, i.e., every non-nil right ideal of R contains a nonzero idempotent, then it is feckly reduced.

3.2. Proposition. Every semi-abelian π-regular ring is feckly reduced.

Proof. Let R be a semi-abelian π-regular ring. According to [1, Corollary 3.13], $J(R) = N(R)$, and so R is feckly reduced by 2.8. Lemma. □

Recall that a left ideal L of a ring R is called *GW-ideal* if for any $a \in L$, there exists a positive integer n such that $a^n R \subseteq L$ and the ring R is called *left WQD* if every maximal left ideal of R is a GW-ideal.

3.3. Example. Every left WQD ring is feckly reduced by [12, Theorem 2.7].

3.4. Proposition. *Every locally finite abelian ring is feckly reduced.*

Proof. Let R be a locally finite abelian ring. Due to [4, Proposition 2.5], we have $N(R) = J(R)$. Then 2.8. Lemma completes the proof. □

Recall that a ring R is called *semicommutative* if for any $a, b \in R$, $ab = 0$ implies $aRb = 0$. Let R be a left morphic ring, that is, for any $a \in R$ there exists $b \in R$ such that $Ra = l(b)$ and $l(a) = Rb$. Then $J(R) = Z(R_R)$ ([8]).

3.5. Theorem. *Every semicommutative left and right morphic ring is feckly reduced.*

Proof. Let R be a semicommutative left and right morphic ring. By [8, Theorem 24], R being right morphic implies that it is left p-injective. We first note that R is right duo. In fact, for any $a \in R$, in view of left p-injectivity $aR = rl(a)$. By semicommutativity, $l(a)$ is a two sided ideal and so is $rl(a) = aR$. Because of this fact, every right ideal of R is also a left ideal. On the other hand, again by [8, Theorem 24], R being left morphic implies that $Z(R_R) = J(R)$. To complete the proof it is enough to show that $a^2 \in J(R)$ implies $a \in J(R)$. Otherwise, since $Z(R_R) = J(R)$, $r(a^2)$ is essential in R but $r(a)$ is neither essential in R nor in $r(a^2)$. There exists a right ideal $K \leq r(a^2)$ such that $r(a) \oplus K$ is essential in $r(a^2)$. Since K is also a left ideal, $aK \leq K$. Hence $a(aK) = 0$ since $K \leq r(a^2)$, and then $aK \leq r(a) \cap K$. It follows that $K \leq r(a) \cap K = 0$. Thus $K = 0$ and $r(a)$ is essential in R. This is the required contradiction. □

3.6. Theorem. *Every semicommutative left morphic ring with ACC on right annihilators is feckly reduced.*

Proof. Let R be a semicommutative left morphic ring with ACC on right annihilators. Then R is right p-injective and so it is left duo. Also we have $Z({}_RR) = J(R)$ by [8, Theorem 31]. The rest is similar to the proof of 3.5. Theorem. □

A ring R with involution $*$ is called a *$*$-ring*. An element p in a $*$-ring R is called a *projection* if $p^2 = p = p^*$. A $*$-ring R is said to be *$*$-clean* if each of its elements is the sum of a unit and a projection, and R is called *strongly $*$-clean* if each of its elements is the sum of a unit and a projection that commute with each other. If the preceding projection is unique, we call R *uniquely strongly $*$-clean*.

We call a $*$-ring R *strongly nil-$*$-clean* if every element of R is the sum of a nilpotent element and a projection that commute with each other.

3.7. Theorem. *Let R be a strongly nil-$*$-clean ring. Then*

(1) *Every idempotent in R is a projection.*
(2) *$N(R)$ forms an ideal.*
(3) *$R/N(R)$ is Boolean.*
(4) *$N(R) = J(R)$.*
(5) *R is feckly reduced.*

Proof. Let $e^2 = e \in R$. There exist a projection p and a nilpotent v in R such that $e = p+v$ and $pv = vp$. Then it is easily proved that e is also projection, that is $e = e^* = e^2$ and e is central. For any $x \in R$, there exist an idempotent $g \in R$ and a nilpotent $v \in N(R)$ such that $x = g + v$. Thus $x^2 = g + (2g + v)v$, and so $x - x^2 = (-2g + 1 - v)v \in R$ is nilpotent. Write $(x - x^2)^m = 0$, and so $x^m \in x^{m+1}R$ and $x^m = x^{m+1}y = yx^{m+1}$. Clearly, $xy = yx$ and $x^n y^n$ is an idempotent. This shows that R is strongly π-regular. It is well known that $N(R)$ forms an ideal of R. Hence $N(R) \subseteq J(R)$ since $J(R)$ contains all nil left or nil right ideals. Further, $x - x^2 \in N(R)$, and so $R/N(R)$ is Boolean. Let $x \in J(R)$. There exists an idempotent $e \in R$ such that $x - e \in N(R) \subseteq J(R)$. Hence $e \in J(R)$. Thus $e = 0$ and so $x \in N(R)$. It follows that $J(R) = N(R)$. Therefore R is feckly reduced. □

Recall that R is called a *gsr-ring* [10] if for any $x \in R$, there exists some integer $n(x) \geq 2$ such that $xRx = x^{n(x)}Rx^{n(x)}$.

3.8. Proposition. *Every gsr-ring is feckly reduced.*

Proof. Let R be a gsr-ring and $x \in R$ with $x^2 \in J(R)$. Then $xRx = x^{n(x)}Rx^{n(x)}$ for some integer $n(x)$ with $n(x) \geq 2$. This implies that $xRx = x^2Rx^2$. Hence $xRx \subseteq J(R)$, and so $(RxR)^2 \subseteq J(R)$. Also $J(R)$ is a semiprime ideal of R by [6, Ex. 10.20]. It follows that $RxR \subseteq J(R)$, thus $x \in J(R)$. This completes the proof. $\square$

4. Further Results

A ring R is said to be *right continuous* [11] if (1) every right ideal of R isomorphic to a direct summand of R is a direct summand of R and (2) every complement right ideal of R is a direct summand of R. Thus if R is right continuous, then $J(R) = Z(R_R)$ and $R/Z(R_R)$ is von Neumann regular.

4.1. Theorem. *Let R be a ring with $J(R) = Z(R_R)$. If R is reduced, then it is feckly reduced.*

Proof. To complete the proof it is enough to show that $x^2 \in J(R)$ implies $x \in J(R)$. Let $x \in R$ with $x^2 \in J(R) = Z(R_R)$ and so $r(x^2)$ is an essential right ideal of R. Let $t \in r(x^2)$. So $x^2t = 0$. Since R is reduced, we have $xt = 0$. Hence $t \in r(x)$. It follows that $r(x) = r(x^2)$ and $r(x)$ is an essential right ideal of R and so $x \in Z(R_R) = J(R)$. This completes the proof. $\square$

An ideal of a feckly reduced ring need not be feckly reduced, as the following example shows.

4.2. Example. Let F be a field and R the ring $\left\{ \begin{bmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & e \end{bmatrix} : a, b, c, d, e \in F \right\}$ and I an ideal $\left\{ \begin{bmatrix} 0 & b & c \\ 0 & 0 & d \\ 0 & 0 & e \end{bmatrix} : b, c, d, e \in F \right\}$ of R. Then it can be shown $J(R) = \left\{ \begin{bmatrix} 0 & b & c \\ 0 & 0 & d \\ 0 & 0 & 0 \end{bmatrix} : b, c, d \in F \right\}$ and $J(I) = \left\{ \begin{bmatrix} 0 & 0 & c \\ 0 & 0 & d \\ 0 & 0 & 0 \end{bmatrix} : c, d \in F \right\}$. Since F is a field, $R/J(R)$ is reduced, and so R is feckly reduced. On the other hand, consider the element $x = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in I$. Since x is nilpotent, we have $x \in J^{\#}(I)$, but $x \notin J(I)$. Hence $J^{\#}(I) \neq J(I)$. By 2.6. Proposition, I is not feckly reduced.

4.3. Theorem. *Let I be an ideal of a ring R with $I \subseteq J(R)$. Then R is feckly reduced if and only if R/I is feckly reduced.*

Proof. Let $\overline{R} = R/I$. Since $I \subseteq J(R)$, $J(\overline{R}) = J(R)/I$. Suppose that R is feckly reduced. Since $\overline{R}/J(\overline{R}) \cong R/J(R)$, $\overline{R}$ is feckly reduced. Conversely, assume that $\overline{R}$ is feckly reduced and $a \in R$ with $a^2 \in J(R)$. Then $\overline{a}^2 \in J(R)/I = J(\overline{R})$ and so $\overline{a} \in J(\overline{R})$. Hence $a \in J(R)$, as desired. $\square$

4.4. Theorem. *If R is feckly reduced, then eRe is also feckly reduced for any $e^2 = e \in R$.*

Proof. Assume that $a \in eRe$ with $\overline{a}^2 = \overline{0}$. Then $a^2 \in J(eRe) = eJ(R)e \subseteq J(R)$ and so $a \in J(R)$. Thus $eae = a \in eJ(R)e$, so $\overline{a} = \overline{0}$ in $eRe/J(eRe)$. $\square$

4.5. Corollary. *Let M be a module with its endomorphism ring feckly reduced. Then every direct summand of M has a feckly reduced endomorphism ring.*

We now give a characterization of feckly reduced rings.

4.6. Theorem. *A ring R is feckly reduced if and only if every cyclic projective R-module has a feckly reduced endomorphism ring.*

Proof. Let R be a feckly reduced ring and mR a projective R-module. Then mR is isomorphic to a direct summand I of R as an R-module. 4.5. Corollary implies that the endomorphism ring of mR is feckly reduced. The sufficiency is clear due to $R \cong \mathrm{End}_R(R)$. □

4.7. Proposition. *Let M_1 and M_2 be R-modules for a ring R. If M_1 and M_2 have feckly reduced endomorphism rings and $Hom(M_1, M_2) = 0$, then $M = M_1 \oplus M_2$ has a feckly reduced endomorphism ring.*

Proof. Let $S_i = \mathrm{End}_R(M_i)$ for $i = 1, 2$ and $S = \mathrm{End}_R(M)$. We may write S as $\begin{bmatrix} S_1 & Hom(M_2, M_1) \\ 0 & S_2 \end{bmatrix}$. Then $J(S) = \begin{bmatrix} J(S_1) & Hom(M_2, M_1) \\ 0 & J(S_2) \end{bmatrix}$. Thus $S/J(S) \cong S_1/J(S_1) \times S_2/J(S_2)$. By assumption, S_1 and S_2 are feckly reduced. This implies that S is also feckly reduced. □

Note that every field is feckly reduced and every matrix ring over any field contains nilpotent elements. Therefore feckly reduced property is not Morita invariant. Also, the full matrix ring $M_n(R)$ over a ring R is never feckly reduced for all $n \geq 2$ because of $M_n(R)/J(M_n(R)) = M_n(R)/M_n(J(R)) \cong M_n(R/J(R))$.

If R is feckly reduced, then it need not be abelian, semicommutative, symmetric, reversible, and reduced (see 2.2. Example). In this direction we have the following.

4.8. Proposition. *Every feckly reduced ring is directly finite.*

Proof. Let R be a feckly reduced ring and $x, y \in R$ with $xy = 1$. Then yx is an idempotent. Since all nilpotents belong to $J(R)$, $yxy - yxyyx = y - y^2x \in J(R)$. Multiplying the latter from the left by x, $xy - xy^2x = 1 - yx \in J(R)$. Hence $yx = 1$. □

Recall that a ring R is called *2-primal* if $P(R) = N(R)$ where $P(R)$ is the prime radical of R.

4.9. Proposition. *Let R be a left Artinian ring. Then R is feckly reduced if and only if it is 2-primal.*

Proof. By [5, p.449], we have $P(R) = J(R)$. If R is feckly reduced, then $J(R) = N(R)$, and so it is 2-primal. If R is 2-primal, then $N(R) = P(R)$, and so it is feckly reduced due to 2.8. Lemma. This completes the proof. □

Note that direct products of reduced ring is again reduced.

4.10. Proposition. *Let $\{R_i\}_{i\in I}$ be a class of rings for an index set I. Then $\prod_{i\in I} R_i$ is feckly reduced if and only if for each $i \in I$, R_i is feckly reduced.*

Proof. If R_i is feckly reduced for each $i \in I$, then $\prod_{i\in I} R_i$ is a feckly reduced ring since $\prod_{i\in I} R_i/J(\prod_{i\in I} R_i) \cong \prod_{i\in I} \big(R_i/J(R_i)\big)$. Suppose that $\prod_{i\in I} R_i$ is feckly reduced and let $a_i \in R_i$ with $a_i^2 \in J(R_i)$ for $i \in I$. Then $(0, \dots, a_i^2, \dots, 0) = (0, \dots, a_i, \dots, 0)^2 \in J(\prod_{i\in I} R_i) = \prod_{i\in I} J(R_i)$, and so $(0, \dots, a_i, \dots, 0) \in J(\prod_{i\in I} R_i)$. Hence $a_i \in J(R_i)$, as asserted. □

Let S and T be any rings, M an S-T-bimodule and R the formal triangular matrix ring $\begin{bmatrix} S & M \\ 0 & T \end{bmatrix}$. It is well-known that $J(R) = \begin{bmatrix} J(S) & M \\ 0 & J(T) \end{bmatrix}$ and $R/J(R) \cong S/J(S) \times T/J(T)$.

4.11. Proposition. *Let $R = \begin{bmatrix} S & M \\ 0 & T \end{bmatrix}$. Then R is feckly reduced if and only if S and T are feckly reduced.*

Proof. The necessity is obvious from 4.10. Proposition. Assume that S and T are feckly reduced. Then $S/J(S)$ and $T/J(T)$ are feckly reduced, by the remark above, $R/J(R) \cong S/J(S) \times T/J(T)$. Since a direct product of reduced rings is again reduced, $R/J(R)$ is reduced and so R is feckly reduced. □

For a ring R, let $R \propto R$ denote the ring $\left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \mid a, b \in R \right\}$. Then $J(R \propto R) = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \mid a \in J(R),\ b \in R \right\}$.

4.12. Theorem. *Let R be a ring. Then $R \propto R$ is feckly reduced if and only if R is feckly reduced.*

Proof. Let R be a feckly reduced ring and $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}^2 = \begin{bmatrix} a^2 & ab+ba \\ 0 & a^2 \end{bmatrix} \in J(R \propto R)$. By the remark above, $a^2 \in J(R)$ and so $a \in J(R)$. Hence $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \in J(R \propto R)$. Assume that $R \propto R$ is feckly reduced and let $a \in R$ with $a^2 \in J(R)$. Then $\begin{bmatrix} a^2 & 0 \\ 0 & a^2 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}^2 \in J(R \propto R)$. Therefore, $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \in J(R \propto R)$ and so $a \in J(R)$, as asserted. □

For a ring R, let $T(R,R) = \{(a,b) \mid a, b \in R\}$ with the addition componentwise and multiplication defined by $(a_1, b_1)(a_2, b_2) = (a_1a_2, a_1b_2 + b_1a_2)$. Then $T(R,R)$ is a ring which is called the *trivial extension of R by R*. Clearly, $T(R,R)$ is isomorphic to the ring $R \propto R$ and $T(R,R)$ is also isomorphic to the ring $R[x]/ < x^2 >$. Hence by 4.12. Theorem, we have the following.

4.13. Corollary. *The following conditions are equivalent for a ring R.*

(1) *R is feckly reduced.*
(2) *$T(R,R)$ is feckly reduced.*
(3) *$R[x]/ < x^2 >$ is feckly reduced.*

4.14. Theorem. *Let R be a ring. Then the following are equivalent.*

(1) *R is feckly reduced.*
(2) *$T_n(R)$ is feckly reduced for all $n \in \mathbb{N}$.*

Proof. Let $I = \{[a_{ij}] \in T_n(R) : a_{ii} = 0, i = 1, 2, \ldots, n\}$. Then $I \subseteq J(T_n(R))$ and $T_n(R)/I \cong \bigoplus_{i=1}^{n} R_i$ where each $R_i = R$. So by 4.3. Theorem and 4.10. Proposition, we have (1) $\Leftrightarrow$ (2). Therefore the proof is completed. □

Let R be a ring and V an R-R-bimodule which is a general ring (possibly with no unity) in which $(vw)r = v(wr)$, $(vr)w = v(rw)$ and $(rv)w = r(vw)$ hold for all $v, w \in V$ and $r \in R$. Then *ideal-extension* (it is also called *Dorroh extension*) $I(R; V)$ of R by

V is defined to be the additive abelian group $I(R;V) = R \oplus V$ with multiplication $(r,v)(s,w) = (rs, rw + vs + vw)$.

4.15. Proposition. *Suppose that for any $v \in V$ there exists $w \in V$ such that $v+w+vw = 0$. Then the following are equivalent for a ring R.*

(1) *R is feckly reduced.*
(2) *An ideal-extension $S = I(R;V)$ is feckly reduced.*

Proof. (1) $\Rightarrow$ (2) Let $s = (r,v) \in S$ with $s^2 = (r^2, rv+vr+v^2) \in J(S)$. It is easy to verify that $r^2 \in J(R)$ and so $r \in J(R)$ by (1). Note that $(0,V) \subseteq J(S)$ by hypothesis. Since $s = (r,v) = (r,0)+(0,v)$, it suffices to show that $(r,0) \in J(S)$. For any $(x,y) \in S$, $(1,0)-(r,0)(x,y) = (1-rx, -ry) \in U(S)$ because $(1-rx,-ry) = (1-rx,0)(1,(1-rx)^{-1}(-ry))$ and $(1,(1-rx)^{-1}(-ry)) = (1,0)+(0,(1-rx)^{-1}(-ry)) \in U(S)$ by $(0,V) \subseteq J(S)$. Thus $s = (r,v) \in J(S)$.

(2) $\Rightarrow$ (1) Suppose that S is feckly reduced and let $a \in R$ with $a^2 \in J(R)$. Then $(a,0)^2 = (a^2,0) \in S$. By the preceding discussion, $(a^2,0) \in J(S)$ and so $(a,0) \in J(S)$ by (2). Therefore $a \in J(R)$, as desired. □

4.16. Example. Let R be a feckly reduced ring, n a positive integer and $S = \{[a_{ij}] \in T_n(R) \mid a_{11} = \cdots = a_{nn}\}$. If $V = \{[a_{ij}] \in T_n(R) \mid a_{11} = \cdots = a_{nn} = 0\}$, then $S \cong I(R;V)$. Since $V \subseteq J(S)$, S is feckly reduced by 4.15. Proposition and noncommutative if $n \geq 3$.

If R is a ring and $\sigma : R \to R$ is a ring homomorphism, let $R[[x,\sigma]]$ denote the ring of skew formal power series over R; that is all formal power series in x with coefficients from R with multiplication defined by $xr = \sigma(r)x$ for all $r \in R$. In particular, $R[[x]] = R[[x,1_R]]$ is the ring of formal power series over R. Note that $J(R[[x,\sigma]]) = J(R) + < x >$. Since $R[[x,\sigma]] \cong I(R; < x >)$ where $< x >$ is the ideal generated by x, 4.15. Proposition gives the next result.

4.17. Corollary. *Let R be a ring and $\sigma : R \to R$ a ring homomorphism. Then the following are equivalent.*

(1) *R is feckly reduced.*
(2) *$R[[x,\sigma]]$ is feckly reduced.*

4.18. Remark. Let R be a ring. Then the ring $R[[x]]$ of formal power series is feckly reduced if and only if R is feckly reduced.

We now investigate some relations between clean rings, exchange rings and feckly reduced rings.

4.19. Proposition. *Every clean ring is exchange. The converse holds for feckly reduced rings.*

Proof. By [7], it is known that every clean ring is exchange. Let R be a feckly reduced exchange ring. Then $R/J(R)$ is exchange and abelian. Hence it is clean. On the other hand, since R is exchange, by [7], idempotents lift modulo $J(R)$. Therefore R is clean. □

Recall that a ring R is called *J-clean (nil clean)* if for every $a \in R$, there exist $e^2 = e \in R$ and $b \in J(R)(b \in N(R))$ such that $a = e + b$.

4.20. Theorem. *Consider the following conditions for a ring R.*

(1) *R is an abelian exchange ring.*
(2) *R is a J-clean ring.*
(3) *R is a feckly reduced ring.*

Then (1) $\Rightarrow$ (3) and (2) $\Rightarrow$ (3). The converse statements hold if R is nil-clean.

Proof. (1) ⇒ (3) Let R be an abelian exchange ring. Since R is exchange, $R/J(R)$ is also exchange and idempotents lift modulo $J(R)$. Then $R/J(R)$ is abelian. The rest follows from [14, Corollary 3.12].
(2) ⇒ (3) Clear from [9].
The converse statements hold by noting that every nil-clean ring is clean and abelian, and every clean ring is exchange. □

The converse statements (3) ⇒ (1) and (3) ⇒ (2) do not hold in general.

4.21. Examples. (1) Let F be a field. Then $M_2(F)$ is an exchange ring which is not feckly reduced.
(2) Consider the ring $\mathbb{Z}$ of integers. For $3 \in \mathbb{Z}$, there is no any idempotent e of $3\mathbb{Z}$ such that $1 - e \in 2\mathbb{Z}$. Therefore $\mathbb{Z}$ is not exchange. But clearly, it is feckly reduced.
(3) Let R be the ring $\{m/n \in \mathbb{Q} : gcd(m, n) = 1,\ 2 \nmid n,\ 3 \nmid n\}$. Then $R/J(R) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3$. This implies that R is feckly reduced. On the other hand, $4 \in R$ can not be written as the sum of an idempotent and a unit in R. Hence R is not clean.
(4) The ring $\mathbb{Z}_5$ is feckly reduced but not J-clean.

4.22. Proposition. *Every semiregular feckly reduced ring is clean.*

Proof. Let R be a semiregular feckly reduced ring. Then $R/J(R)$ is strongly regular. Hence it is clean and idempotents of R lift modulo $J(R)$. This implies that R is clean. □

4.23. Proposition. *Every right (left) quasi-duo ring is feckly reduced.*

Proof. Let R be a right (left) quasi-duo ring and $a \in R$ with $a^2 \in J(R)$. Every factor ring of a right (left) quasi-duo ring is again right (left) quasi-duo and by [13, Lemma 2.3] every nilpotent element of a right (left) quasi-duo ring is in Jacobson radical. Accordingly, we have $a \in J(R)$. Therefore R is feckly reduced. □

On the contrary of 4.23. Proposition, there is a feckly reduced ring which is not right quasi-duo, for example, consider the Hamilton quaternion over the field of real numbers and let R denote this ring. Since R is a division ring, we have $J(R) = 0$, and so $J(R[x]) = 0$ due to $J(R[x]) \subseteq J(R)[x]$. Also $R[x]$ is a domain and so it is reduced. This implies that $R[x]$ is a feckly reduced ring. On the other hand, consider the maximal right ideal $I = (1+ix)R[x]$ of $R[x]$. If I were a left ideal, then $\big((1+ix)k + k(1+ix)\big)(2k)^{-1} = 1 \in I$, this is a contradiction. Therefore $R[x]$ is not right quasi-duo. Nevertheless, for exchange rings these notions are equivalent as the following theorem shows.

4.24. Theorem. *Let R be an exchange ring. Then the following are equivalent.*

(1) *R is feckly reduced.*
(2) *$N(R) \subseteq J(R)$.*
(3) *$N_2(R) \subseteq J(R)$.*
(4) *R is right quasi-duo.*

Proof. (1) ⇒ (2) From 2.7. Corollary. (2) ⇒ (3) Clear. (3) ⇔ (4) From [3, Proposition 2.3]. (3) ⇒ (1) Since R is exchange, $R/J(R)$ is also exchange. Then $R/J(R)$ is reduced by [2, Theorem 2]. □

We say that B is a *subring* of a ring A if $\emptyset \neq B \subseteq A$ and for any $x, y \in B$, $x - y, xy \in B$ and $1_A \in B$. Let A be a ring and B a subring of A and $R[A, B]$ denote the set $\{(a_1, a_2, \cdots, a_n, b, b, \cdots) : a_i \in A, b \in B, n \geq 1, 1 \leq i \leq n\}$. Then $R[A, B]$ is a ring under the componentwise addition and multiplication. Also $J(R[A, B]) = R[J(A), J(A) \cap J(B)]$.

4.25. Proposition. *Consider the following conditions for a ring A and a subring B of A.*

(1) *A and B are feckly reduced.*
(2) *$R[A,B]$ is feckly reduced.*
(3) *A is feckly reduced and $N(B) \subseteq J(B)$.*

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

Proof. (1) $\Rightarrow$ (2) Let $(a_1, \cdots, a_n, b, b, \cdots) \in R[A,B]$ with $(a_1, \cdots, a_n, b, b, \cdots)^2 \in J(R[A,B])$ for some $n \geq 1$. Then $(a_1^2, \cdots, a_n^2, b^2, b^2, \cdots) \in J(R[A,B])$. This implies that $a_i^2, b^2 \in J(A)$ for $i = 1, \ldots, n$ and $b^2 \in J(B)$. By assumption, $a_i, b \in J(A)$ for $i = 1, \ldots, n$ and $b \in J(B)$. Therefore $(a_1, \cdots, a_n, b, b, \cdots) \in J(R[A,B])$.

(2) $\Rightarrow$ (3) Let $a \in A$ with $a^2 \in J(A)$. Then $(a, 0, 0, \cdots)^2 = (a^2, 0, 0, \cdots) \in J(R[A,B])$. By (1), we have $(a, 0, 0, \cdots) \in J(R[A,B])$, and so $a \in J(A)$. Therefore A is feckly reduced. In order to show $N(B) \subseteq J(B)$, let $b \in B$ with $b^n = 0$ for some positive integer n. Then $(0, b, b, \cdots)^n = (0, 0, 0, \cdots) \in J(R[A,B])$. Since $R[A,B]$ is feckly reduced, $(0, b, b, \cdots) \in J(R[A,B])$. Hence $b \in J(B)$, as desired. □

The following result is an immediate consequence of 4.24. Theorem and 4.25. Proposition.

4.26. Corollary. *Let B be a subring of a ring A. If B is an exchange ring, then the following are equivalent.*

(1) *$R[A,B]$ is feckly reduced.*
(2) *A and B are feckly reduced.*

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