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GLOBAL EXISTENCE AND STABILITY RESULTS FOR HADAMARD–VOLTERRA–STIELTJES INTEGRAL EQUATIONS

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Abstract. In this paper we prove the existence and stability of solutions of a class of Hadamard–Volterra–Stieltjes integral equations in the Banach space of continuous and bounded functions on unbounded interval. That result is proved under rather general hypotheses. The main tools used in our considerations are the concept of measure of noncompactness in conjunction with the Darbo and Mönch fixed point theorems.

1. Introduction

The theory of integral operators and integral equations is an important part of nonlinear analysis. It is caused by the fact that this theory is frequently applicable in other branches of mathematics and mathematical physics, engineering, economics, biology as well in describing problems connected with real world see [6, 7, 8, 21, 28].

Fractional calculus is a generalization of the ordinary differentiation and integration to arbitrary non-integer order, it has developed up to the present day [9, 15, 17, 18, 19]. Differential and Integral equations of fractional order are one of the most useful mathematical tools in both pure and applied analysis, and various theoretical results have been obtained, see the papers of Abbas *et al.* [1, 2, 4, 5]. There are many results in nonlinear functional analysis which contain conditions with the Kuratowski measure of noncompactness, an axiomatic approach to the notion of a measure of noncompactness was introduced by Banas and Goebel. There were constructed many examples of axiomatic measures of noncompactness in Banach spaces, which are expressed by explicit formulas. This approach to the concept of a measure of noncompactness has found many applications in the theory of differential and integral equations in abstract spaces or in the fixed point theory [11, 12, 13, 14, 16]. The paper is devoted to the study of a class of integral equations of Hadamard–Volterra–Stieltjes type. That class comprises a lot of particular cases

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of fractional integral equations which can be encountered in research papers and monographs concerning the theory of integral equations and their applications to real world problems see [10, 19, 24, 26].

Consider the following integral equation

$$u(x,y)=\mu(x,y)+\frac{1}{\Gamma(r_1)\Gamma(r_2)}$$
$$\times\int_1^x\int_1^y\left(\ln\frac{x}{s}\right)^{r_1-1}\left(\ln\frac{y}{t}\right)^{r_2-1}\frac{f(t,s,u(t,s))}{st}d_th_1(y,t)d_sh_2(x,s);\ (x,y)\in J,\quad (1)$$

where $J=[1,+\infty)\times[1,b]$, $r_1,r_2>0$, $\mu:J\to\mathbb{R}$ is continuous and bounded function, $f:J\times\mathbb{R}\to\mathbb{R}$ is continuous function, $h_1:[1,b]\times[1,b]\to\mathbb{R}$, $h_2:[1,+\infty)\times[1,+\infty)\to\mathbb{R}$ are given functions, and $\Gamma(\cdot)$ is the Euler gamma function.

Using the technique associated with measures of noncompactness and fixed point theorems we show that Eq. (1) has solutions being continuous and bounded functions on the interval $[1,+\infty)\times[1,b]$. Moreover, the choice of suitable measures of noncompactness allows us to assert that those solutions are asymptotic stable in certain sense which will be defined in the sequel.

2. Preliminaries

This section is devoted to collect some definitions and auxiliary results which will be needed in further considerations.

At the beginning we present some basic facts concerning measures of noncompactness. Assume that $(X,\|\cdot\|)$ is an infinite dimensional Banach space with zero element θ. Denote by $B(x,r)$ the closed ball centered at x and radius r. We write B_r to denote the ball $B(\theta,r)$. If A is a subset of X then the symbols $\overline{A}$, $\mathrm{Conv}A$ stand for the closure and closed convex hull of A, respectively. Moreover, we denote by $\mathtt{M}_X$ the family of all nonempty and bounded subsets of X and by $\mathtt{N}_X$ its subfamily consisting of all relatively compact sets.

We accept the following definition of the concept of a measure of noncompactness.

Definition 1. [11, 14, 17] *A mapping $\psi:\mathtt{M}_X\to\mathbb{R}_+$ is said to be a measure of noncompactness in X if it satisfies the following conditions*

1. *The family ker $\psi=\{A\in\mathtt{M}_X:\psi(A)=0\}$ is nonempty and ker $\psi\subset\mathtt{N}_X$.*
2. $A\subset B\Rightarrow\psi(A)\le\psi(B)$.
3. $\psi(A)=\psi(\overline{A})$.
4. $\psi(A)=\psi(Conv\ A)$.
5. $\psi(\lambda A+(1-\lambda)B)\le\lambda\psi(A)+(1-\lambda)\psi(B)$ *for* $\lambda\in[0,1]$.
6. *If (A_n) is a sequence of closed sets from $\mathtt{M}_X$ such that $A_{n+1}\subset A_n$ $(n=0,1,2,\ldots)$ and if $\lim_{n\longrightarrow\infty}\psi(A_n)=0$, then the intersection set $A_\infty=\bigcap_{n=0}^{\infty}A_n$ is nonempty.*

The family ker ψ described in 1. is said to be the kernel *of the measure of noncompactness ψ.*

Remark 2. *Observe that the intersection set A_∞ from 6. is a member of the family ker ψ. In fact, since $\psi(A_\infty) \le \psi(A_n)$ for any n, we infer that $\psi(A_\infty) = 0$.*

In what follows we will work in the Banach space BC consisting of all real functions defined, continuous and bounded on $J = [1, +\infty) \times [1, b]$. this space is furnished with the standard norm

$$\|u\| = \sup\{|u(x, y)|;\ (x, y) \in J\}.$$

Denote by $L^1(J, \mathbb{R})$ the Banach space of function $u : J \to \mathbb{R}$ that are Lebesgue integrable with norm

$$\|u\|_{L^1} = \int\int_J |u(x, y)|dydx.$$

In order to define a measure of noncompactness in the space BC, let us fix a nonempty bounded subset Y of the space BC . For $u \in Y$, $T \ge 1$, $\epsilon_1, \epsilon_2 > 0$, $(x_1, y_1), (x_2, y_2) \in [1, T] \times [1, b]$ such that $|x_2 - x_1| \le \epsilon_1$ and $|y_2 - y_1| \le \epsilon_2$.

We denote by $\omega^T(u, \epsilon_1, \epsilon_2)$ the modulus of continuity of the function u on the interval $[1, T] \times [1, b]$ i.e

$$\begin{aligned}
\omega^T(u, \epsilon_1, \epsilon_2) &= \sup\{|u(x_2, y_2) - u(x_1, y_1)|; : (x_1, y_1), (x_2, y_2) \in [1, T] \times [1, b]\} \\
\omega^T(Y, \epsilon_1, \epsilon_2) &= \sup\{\omega^T(u, \epsilon_1, \epsilon_2); : u \in Y\} \\
\omega_0^T(Y) &= \lim_{\epsilon_1, \epsilon_2 \to 0} \omega^T(Y, \epsilon_1, \epsilon_2) \\
\omega_0(Y) &= \lim_{T \to \infty} \omega_0^T(Y).
\end{aligned}$$

If (t, s) is a fixed number from J, let us denote $Y(t, s) = \{u(t, s);\ u \in Y\}$ and

$$diam : Y(t, s) = \sup\left\{|u(t, s) - v(t, s)|;\ u, v \in Y\right\}.$$

Finally, consider the function ψ defined on the family $\mathtt{M}_{BC}$ by the formula

$$\psi(Y) = \omega_0(Y) + \lim_{t \longmapsto \infty} \sup : diam : Y(t, s). \tag{2}$$

Following the ideas from [12], we can show that the function ψ is a measure of noncompactness in the space BC. The kernel *ker* : ψ consists of nonempty and bounded sets Y such that functions from Y are locally equicontinuous on J and the thickness of the bundle formed by functions from Y tends to zero at infinity. This property will permit us to characterize solutions of the integral equation considered in the next section.

Now, let us assume that Ω is a nonempty subset of the space BC and F is an operator on Ω with values in BC. Consider the following equation

$$u(x, y) = (Fu)(x, y);\ (x, y) \in J. \tag{3}$$

Definition 3. [4, 5, 11, 12] *The solution $u = u(x, y)$ of Eq. (3) is said to be globally attractive if for each solution $v = v(x, y)$ of Eq. (3) we have that*

$$\lim_{x \longmapsto \infty} \big(u(x, y) - v(x, y)\big) = 0.$$

In the case when this limit is uniform i.e when for each $\epsilon > 0$ there exists $T > 0$ such that

$$\big|u(x,y) - v(x,y)\big| < \epsilon.$$

For $x \geq T$, we will say that solutions of Eq. (3) are uniformly globally attractive.

Let a nondegenerate interval $I = [a,b] \times [c,d] \subset \mathbb{R}^2$ be given. We consider a real function $p : I \longrightarrow \mathbb{R}$ defined on I. For a given subinterval $\overline{I} = [a_1, b_1] \times [c_1, d_1] \subset I$, $a \leq a_1 \leq b_1 \leq b$, $c \leq c_1 \leq d_1 \leq d$ we set

$$m_p(\overline{I}) = p(b_1, d_1) - p(b_1, c_1) - p(a_1, d_1) + p(a_1, c_1);$$

Let us de define

$$\bigvee_I(p) = \sup \sum_i |m_p(\overline{I}_i)|.$$

Where the supremum is taken over all finite systems of nonoverlapping intervals $\overline{I}_i \subset I$
(i.e. for the interiors $\overline{I}_i^{\circ}$ of the intervals $\overline{I}_i$ we assume that $\overline{I}_i^{\circ} \cap \overline{I}_j^{\circ} = \emptyset$ whenever $i \neq j$).
If $p : I = [a,b] \times [c,d] \longrightarrow \mathbb{R}$ and $\gamma \in [c,d]$ is fixed, then we denote the usual variation of the function $p(s,\gamma)$ in the interval $[a,b]$ by $\bigvee_a^b p(\cdot,\gamma)$.Similarly for $\bigvee_c^d p(\alpha,\cdot)$ where $\alpha \in [a,b]$ is fixed.

Definition 4. [10] *The real function $p : I \longrightarrow \mathbb{R}$ is of bounded variation on I if $\bigvee_I(p) < \infty$.*

Lemma 5. [10] *Let $p : I = [a,b] \times [c,d] \longrightarrow \mathbb{R}$ be given such that $\bigvee_I(p) < \infty$, $\bigvee_a^b p(\cdot,\gamma_0) < \infty$ for some $\gamma_0 \in [c,d]$. Then $\bigvee_a^b p(\cdot,\gamma) < \infty$ for all $\gamma \in [c,d]$ and*

$$\bigvee_a^b p(\cdot,\gamma) \leq \bigvee_I(p) + \bigvee_a^b p(\cdot,\gamma_0).$$

Let f and g be a functions defined on the interval $[a,b]$, the Stieltjes integral of the function f with respect to the function g is designated by

$$\int_a^b f(x)dg(x),$$

It is clear that the Riemann integral is a special case of the Stieltjes integral, obtained by setting $g(x) = x$. For more properties of the Stieltjes integral see [10, 25].

The Stieltjes integral exists under several conditions, we will restrict ourselves to only one theorem in this direction.

Theorem 6. [25] *The integral*

$$\int_a^b f(x)dg(x),$$

exists if the function f is continuous on $[a,b]$ and g is of finite variation on $[a,b]$, and we have

$$\left|\int_a^b f(x)dg(x)\right| \leq \sup_{x\in[a,b]} |f(x)| \bigvee_a^b (g).$$

Lemma 7. [25] *If the function f is continuous on $[a,b]$ and if the function g has a Riemann integrable derivative $g^{'}$ at every point of $[a,b]$, then*

$$\int_a^b f(x)dg(x) = \int_a^b f(x)g^{'}(x)dx.$$

For $u \in L^1(J,\mathbb{R})$, we consider the Hadamard–Stieltjes fractional integral of order $r=(r_1,r_2)$ of the form

$$({}^{HS}I^r u)(x,y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left(\ln\frac{x}{s}\right)^{r_1-1} \left(\ln\frac{y}{t}\right)^{r_2-1} \frac{u(s,t)}{st} d_t g_1(y,t) d_s g_2(x,s);$$

where $g_1 : [1,b]\times[1,b] \longrightarrow \mathbb{R}$, $g_2 : [1,+\infty)\times[1,+\infty) \longrightarrow \mathbb{R}$, and the symbols d_s, d_t indicate the integration with respect to s, t respectively.

Theorem 8. ***(Darbo)***[16]
Let Ω be a nonempty, bounded, closed and convex subset of the Banach space X and let $F : \Omega \longrightarrow \Omega$ be a continuous mapping. Assume that there exists a constant $k\in[0,1)$ such that $\psi(FA) \leq k\psi(A)$ for any nonempty subset A of Ω. Then F has a fixed point in the set Ω.

Remark 9. *Let us denote by Fix F the set of all fixed points of the operator F which belong to Ω. It can be shown that the set Fix F belongs to the family ker ψ.*

Theorem 10. ***(Mönch)***[17, 22]
Let D be a bounded, closed and convex subset of the Banach space X such that $0\in D$, and let $F : D \longrightarrow D$ be a continuous mapping. If the implication

$$V = \overline{Conv}F(V) \quad or \quad V = F(V)\cup\{0\} \Rightarrow \psi(V) = 0$$

holds for every subset V of D. Then F has a fixed point.

3. Main results

In this section we give two results for (1). The first one relies on the Darbo fixed point theorem and the second one on the Mönch fixed point theorem. (1) will be considered under the following assumptions :

(H_1)**:** The function f is continuous and there exists a continuous and bounded function $g : [1,+\infty)\times[1,b] \longrightarrow \mathbb{R}_+$ such that

$$|f(x,y,u_1) - f(x,y,u_2)| \leq \frac{g(x,y)|u_1-u_2|}{|u_1|+|u_2|+1};\ (x,y)\in J;: u_1,u_2\in\mathbb{R}.$$

(H_2)**:** The function $t \mapsto h_1(y,t)$ is continuous and of bounded variation on $[1,b]$ for each fixed $y \in [1,b]$, and as the function $s \mapsto h_2(x,s)$ is continuous and of bounded variation on $[1,+\infty)$ for each $x \in [1,+\infty)$.

(H_3) **:** There exist a constant $\lambda, \eta > 0$ such that

$$\sup_{x\geq 1; 1\leq y\leq b} \left| \int_1^x \int_1^y \left(\ln\frac{x}{s}\right)^{r_1-1} |f(s,t,0)| d_t \bigvee_{\alpha=1}^{\alpha=t} h_1(y,\alpha) d_s \bigvee_{\beta=1}^{\beta=s} h_1(x,\beta) \right| \leq \lambda.$$

And

$$\sup_{x\geq 1} \left| \int_1^x \left(\ln\frac{x}{s}\right)^{r_1-1} d_s \bigvee_{\beta=1}^{\beta=s} h_1(x,\beta) \right| \leq \eta.$$

With

$$k = \frac{\eta \|g\| \ln b}{\Gamma(r_1)\Gamma(r_2)} \bigvee_{\alpha=1}^{\alpha=b} h_2(y,\alpha) < 1.$$

Remark 11. *In view of the assumption (H_1) we infer that for each $u \in X$*

$$|f(x,y,u)| \leq \frac{g(x,y)|u|}{|u|+1} + |f(x,y,0)|;\ (x,y) \in J;\ u \in \mathbb{R}.$$

Theorem 12. *Under assumptions $(H_1)-(H_3)$ the integral equation (1) has at least one solution $u = u(x,y)$. Moreover, solutions of (1) are globally attractive.*

Proof. Consider the operator F defined on the space BC in the following way :

$$\begin{aligned}(Fu)(x,y) &= \mu(x,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \\ &\quad \times \int_1^x \int_1^y \left(\ln\frac{x}{s}\right)^{r_1-1} \left(\ln\frac{y}{t}\right)^{r_2-1} \frac{f(t,s,u(t,s))}{st} d_t h_1(y,t) d_s h_2(x,s);\end{aligned}$$

Observe that in view of our assumptions, for any function $u \in BC$ the function Fu is continuous on J. Next, let us take an arbitrary function $u \in BC$. Using our assumptions, for a fixed $(x,y) \in J$ we have

$$\begin{aligned}
|Fu(x,y)| &\leq |\mu(x,y)| \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left(\ln \frac{x}{s}\right)^{r_1-1} \left(\ln \frac{y}{t}\right)^{r_2-1} \frac{|f(t,s,u(s,t))|}{st} d_t h_1(y,t) d_s h_2(x,s) \\
&\leq |\mu(x,y)| + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left(\ln \frac{x}{s}\right)^{r_1-1} \left(\ln \frac{y}{t}\right)^{r_2-1} \\
&\quad \times \left[\frac{g(s,t)|u(s,t)|}{|u(s,t)|+1} + |f(s,t,0)|\right] d_t h_1(y,t) d_s h_2(x,s) \\
&\leq \|\mu\| + \frac{\ln b}{\Gamma(r_1)\Gamma(r_2)} \left[\frac{\|g\|.\|u\|}{\|u\|+1} \bigvee_{\alpha=1}^{\alpha=b} h_2(y,\alpha) \int_1^x \left(\ln \frac{x}{s}\right)^{r_1-1} d_s \bigvee_{\beta=1}^{\beta=s} h_1(x,\beta)\right. \\
&\quad \left. + \int_1^x \int_1^y \left(\ln \frac{x}{s}\right)^{r_1-1} |f(s,t,0)| d_t \bigvee_{\alpha=1}^{\alpha=t} h_2(y,\alpha) d_s \bigvee_{\beta=1}^{\beta=s} h_1(x,\beta)\right] \\
&\leq \|\mu\| + \frac{\ln b}{\Gamma(r_1)\Gamma(r_2)} \left[\frac{\|g\|.\|u\|}{\|u\|+1} \bigvee_{\alpha=1}^{\alpha=b} h_2(y,\alpha) \int_1^x \left(\ln \frac{x}{s}\right)^{r_1-1} d_s \bigvee_{\beta=1}^{\beta=s} h_1(x,\beta)\right. \\
&\quad \left. + \int_1^x \int_1^y \left(\ln \frac{x}{s}\right)^{r_1-1} |f(s,t,0)| d_t \bigvee_{\alpha=1}^{\alpha=t} h_2(y,\alpha) d_s \bigvee_{\beta=1}^{\beta=s} h_1(x,\beta)\right] \\
&\leq \|\mu\| + \frac{\ln b}{\Gamma(r_1)\Gamma(r_2)} \left[\|g\| \bigvee_{\alpha=1}^{\alpha=b} h_2(y,\alpha)\eta + \lambda\right].
\end{aligned}$$

Hence we obtain

$$\|Fu\| \leq \|\mu\| + \frac{\ln b}{\Gamma(r_1)\Gamma(r_2)} \left[\|g\| \bigvee_{\alpha=1}^{\alpha=b} h_2(y,\alpha)\eta + \lambda\right].$$

Thus, we infer that the function Fu is bounded on J. Then $Fu \in BC$.
We take

$$r = \|\mu\| + \frac{\ln b}{\Gamma(r_1)\Gamma(r_2)} \left[\|g\| \bigvee_{\alpha=1}^{\alpha=b} h_2(y,\alpha)\eta + \lambda\right].$$

We deduce that the operator F transforms the ball B_r into itself.

Further, let $(u_n) \subset B_r$ such that $u_n \to u$ we get

$$
\begin{aligned}
&|Fu_n(x,y)-Fu(x,y)|\\
&\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^x\int_1^y\left(\ln\frac{x}{s}\right)^{r_1-1}\left(\ln\frac{y}{t}\right)^{r_2-1}\frac{|f(t,s,u_n(s,t))-f(t,s,u(s,t))|}{st}\\
&\times d_th_1(y,t)d_sh_2(x,s)\\
&\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^x\int_1^y\left(\ln\frac{x}{s}\right)^{r_1-1}\left(\ln\frac{y}{t}\right)^{r_2-1}\frac{g(t,s)|u_n(s,t)-u(s,t)|}{|u_n(s,t)|+|u(s,t)|+1}\\
&\times d_th_1(y,t)d_sh_2(x,s)\\
&\leq \frac{\|g\|\ln b\|u_n-u\|}{\Gamma(r_1)\Gamma(r_2)(\|u_n\|+\|u\|+1)}\bigvee_{\alpha=1}^{\alpha=b}h_2(y,\alpha)\int_1^x\left(\ln\frac{x}{s}\right)^{r_1-1}d_s\bigvee_{\beta=1}^{\beta=s}h_1(x,\beta).
\end{aligned}
$$

Consequently

$$
\|Fu_n-Fu\|\leq\frac{\eta\|g\|\ln b\|u_n-u\|}{\Gamma(r_1)\Gamma(r_2)}\bigvee_{\alpha=1}^{\alpha=b}h_2(y,\alpha).
$$

Then when $n\to\infty$ we obtain $Fu_n\to Fu$ so F is continuous on B_r.
Now, we take a nonempty $Y\subset B_r$, for $T\geq 1$, $(x_1,y_1),(x_2,y_2)\in[1,T]\times[1,b]$ with $|x_2-x_1|\leq\epsilon_1$ and $|y_2-y_1|\leq\epsilon_2$; for each $\epsilon_1,\epsilon_2>0$. Fix arbitrarily u in Y we have

$$
\begin{aligned}
&|Fu(x_2,y_2)-Fu(x_1,y_1)|\\
&=\Bigg|\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{x_2}\int_1^{y_2}\left(\ln\frac{x_2}{s}\right)^{r_1-1}\left(\ln\frac{y_2}{t}\right)^{r_2-1}\frac{f(t,s,u(s,t))}{st}d_th_1(y_2,t)d_sh_2(x_2,s)\\
&-\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{x_1}\int_1^{y_1}\left(\ln\frac{x_1}{s}\right)^{r_1-1}\left(\ln\frac{y_1}{t}\right)^{r_2-1}\frac{f(t,s,u(s,t))}{st}d_th_1(y_1,t)d_sh_2(x_1,s)\Bigg|\\
&\leq\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{x_2}\int_1^{y_2}\left(\ln\frac{x_2}{s}\right)^{r_1-1}\left(\ln\frac{y_2}{t}\right)^{r_2-1}\\
&\times\sup_{(x_1,y_1),(x_2,y_2)\in[1,T]\times[1,b]}|f(t,s,u(x_2,y_2))-f(t,s,u(x_1,y_1))|\,d_th_1(y_2,t)d_sh_2(x_2,s)\\
&\leq\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{x_2}\int_1^{y_2}\left(\ln\frac{x_2}{s}\right)^{r_1-1}\left(\ln\frac{y_2}{t}\right)^{r_2-1}g(s,t)\\
&\times\sup_{(x_1,y_1),(x_2,y_2)\in[1,T]\times[1,b]}|u(x_2,y_2)-u(x_1,y_1)|\,d_th_1(y_2,t)d_sh_2(x_2,s)\\
&\leq\frac{\|g\|\ln b}{\Gamma(r_1)\Gamma(r_2)}\bigvee_{\alpha=1}^{\alpha=b}h_2(y,\alpha)\sup_{(x_1,y_1),(x_2,y_2)\in[1,T]\times[1,b]}|u(x_2,y_2)-u(x_1,y_1)|\\
&\times\int_1^{x_2}\left(\ln\frac{x_2}{s}\right)^{r_1-1}d_sh_2(x_2,s)\\
&\leq\frac{\eta\|g\|\ln b}{\Gamma(r_1)\Gamma(r_2)}\bigvee_{\alpha=1}^{\alpha=b}h_2(y,\alpha)\sup_{(x_1,y_1),(x_2,y_2)\in[1,T]\times[1,b]}|u(x_2,y_2)-u(x_1,y_1)|\,.
\end{aligned}
$$

Thus

$$\omega_0(FY) \leq k\omega_0(Y). \tag{4}$$

Further, for $u, v \in Y$ and an arbitrary fixed $(x, y) \in [1, T] \times [1, b]$ we obtain

$$\begin{aligned}
&|Fu(x,y) - Fv(x,y)| \\
&= \Bigg| \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left(\ln \frac{x}{s}\right)^{r_1-1} \left(\ln \frac{y}{t}\right)^{r_2-1} \frac{f(t,s,u(s,t))}{st} d_t h_1(y,t) d_s h_2(x,s) \\
&- \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left(\ln \frac{x}{s}\right)^{r_1-1} \left(\ln \frac{y}{t}\right)^{r_2-1} \frac{f(t,s,v(s,t))}{st} d_t h_1(y,t) d_s h_2(x,s) \Bigg| \\
&\leq \frac{\ln b}{\Gamma(r_1)\Gamma(r_2)} \bigvee_{\alpha=1}^{\alpha=b} h_2(y,\alpha) \int_1^x \int_1^y \left(\ln \frac{x}{s}\right)^{r_1-1} |f(t,s,u(s,t)) - f(t,s,v(s,t))| \, d_s h_2(x,s) \\
&\leq \frac{\ln b}{\Gamma(r_1)\Gamma(r_2)} \bigvee_{\alpha=1}^{\alpha=b} h_2(y,\alpha) \int_1^x \int_1^y \left(\ln \frac{x}{s}\right)^{r_1-1} g(t,s) \frac{|u(s,t) - v(s,t)|}{|u(t,s)| + |v(t,s)| + 1} d_s h_2(x,s) \\
&\leq \frac{\eta \ln b \|g\|}{\Gamma(r_1)\Gamma(r_2)} \bigvee_{\alpha=1}^{\alpha=b} h_2(y,\alpha) \sup_{u,v \in Y} |u(s,t) - v(s,t)|.
\end{aligned}$$

Then

$$\lim_{x \longrightarrow \infty} \sup \, diam\,(FY)(x,y) \leq k \lim_{x \longrightarrow \infty} \sup \, diam\, Y(x,y). \tag{5}$$

Observe, that linking (4), (5) and the definition of the measure of noncompactness ψ given by the formula (2), we obtain

$$\psi(FY) \leq k\psi(Y).$$

Finally, in view of the Darbo fixed point theorem we deduce that F has at least one fixed point in B_r which is a solution of Eq. (1). Moreover, taking into account the fact that the set $Fix\ F \in ker\ \psi$ and the characterization of sets belonging to $ker\ \psi$ (Remark 9) we conclude that all solutions of Eq. (1) are globally attractive in the sense of definition (3). $\square$

Now we will give another result using Mönch's fixed point Theorem. Eq. (1) will be considered under the following assumptions :

(C_1): The function f is continuous and there exists a continuous and bounded function $g : [1, +\infty) \times [1, b] \longrightarrow \mathbb{R}_+$ such that

$$|f(x,y,u_1) - f(x,y,u_2)| \leq \frac{g(x,y)|u_1 - u_2|}{|u_1| + |u_2| + 1}; \ (x,y) \in J; : u_1, u_2 \in \mathbb{R}.$$

(C_2): The function $t \mapsto h_1(y,t)$ is continuous and of bounded variation on $[1, b]$ for each fixed $y \in [1, b]$, and as the function $s \mapsto h_2(x,s)$ is continuous and of bounded variation on $[1, +\infty)$ for each $x \in [1, +\infty)$.

(C_3) **:** There exist a constant $\lambda, \eta > 0$ such that

$$\sup_{x\geq 1; 1\leq y\leq b}\left|\int_1^x\int_1^y \big(\ln\frac{x}{s}\big)^{r_1-1}|f(s,t,0)|d_t\bigvee_{\alpha=1}^{\alpha=t}h_1(y,\alpha)d_s\bigvee_{\beta=1}^{\beta=s}h_1(x,\beta)\right|\leq\lambda,$$

and

$$\sup_{x\geq 1}\left|\int_1^x \big(\ln\frac{x}{s}\big)^{r_1-1}d_s\bigvee_{\beta=1}^{\beta=s}h_1(x,\beta)\right|\leq\eta$$

with

$$k=\frac{\eta\|g\|\ln b}{\Gamma(r_1)\Gamma(r_2)}\bigvee_{\alpha=1}^{\alpha=b}h_2(y,\alpha)<1.$$

Theorem 13. *Under assumptions $(C_1)-(C_3)$ equation (1) has at least one solution $u = u(x, y)$ in the space X. Moreover, solutions of (1) are globally attractive.*

Proof. We have $F : B_r \to B_r$ continuous, let $Y \subset B_r$ with $Y = F(Y) \cup \{0\}$. Then for all u in Y, there exist v in Y such that $u = Fv$.
For $T \geq 1$, $(x_1, y_1), (x_2, y_2) \in [1, T] \times [1, b]$ such that $|x_2 - x_1| \leq \epsilon_1$, $|y_2 - y_1| \leq \epsilon_2$, $\epsilon_1, \epsilon_2 > 0$ and $u, v \in Y$ we get

$$\begin{aligned}
&|u(x_2,y_2)-u(x_1,y_1)|\\
&=\Bigg|\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{x_2}\int_1^{y_2}\left(\ln\frac{x_2}{s}\right)^{r_1-1}\left(\ln\frac{y_2}{t}\right)^{r_2-1}\frac{f(t,s,v(s,t))}{st}d_th_1(y_2,t)d_sh_2(x_2,s)\\
&-\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{x_1}\int_1^{y_1}\left(\ln\frac{x_1}{s}\right)^{r_1-1}\left(\ln\frac{y_1}{t}\right)^{r_2-1}\frac{f(t,s,v(s,t))}{st}d_th_1(y_1,t)d_sh_2(x_1,s)\Bigg|\\
&\leq\frac{\ln b}{\Gamma(r_1)\Gamma(r_2)}\int_1^{x_2}\int_1^{y_2}\left(\ln\frac{x_2}{s}\right)^{r_1-1}g(s,t)\left|v(x_2,y_2)-v(x_1,y_1)\right|d_th_1(y_2,t)d_sh_2(x_2,s)\\
&\leq\frac{\ln b.\|g\|}{\Gamma(r_1)\Gamma(r_2)}\bigvee_{\alpha=1}^{\alpha=b}h_2(y,\alpha)\sup_{v\in Y}|v(x_2,y_2)-v(x_1,y_1)|\int_1^{x_2}\left(\ln\frac{x_2}{s}\right)^{r_1-1}d_sh_2(x_2,s)\\
&\leq\frac{\eta\ln b.\|g\|}{\Gamma(r_1)\Gamma(r_2)}\bigvee_{\alpha=1}^{\alpha=b}h_2(y,\alpha)\sup_{v\in Y}|v(x_2,y_2)-v(x_1,y_1)|.
\end{aligned}$$

In view of our assumptions, we have

$$\sup_{u\in Y}|u(x_2,y_2)-u(x_1,y_1)|\leq\frac{\eta\ln b.\|g\|}{\Gamma(r_1)\Gamma(r_2)}\bigvee_{\alpha=1}^{\alpha=b}h_2(y,\alpha)\sup_{v\in Y}|v(x_2,y_2)-v(x_1,y_1)|.$$

Then

$$\omega_0(Y)\leq k\omega_0(Y). \tag{6}$$

Next, let $u, v, w, z \in Y$ such that $u = Fv$ and $w = Fz$, for $x, y \in J$ we have

$$\begin{aligned}
&|u(x,y) - w(x,y)| \\
&= \bigg| \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left(\ln \frac{x}{s}\right)^{r_1-1} \left(\ln \frac{y}{t}\right)^{r_2-1} \frac{f(t,s,v(s,t))}{st} d_t h_1(y_2,t) d_s h_2(x_2,s) \\
&- \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left(\ln \frac{x}{s}\right)^{r_1-1} \left(\ln \frac{y}{t}\right)^{r_2-1} \frac{f(t,s,z(s,t))}{st} d_t h_1(y_2,t) d_s h_2(x_2,s) \bigg| \\
&\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left(\ln \frac{x}{s}\right)^{r_1-1} \left(\ln \frac{y}{t}\right)^{r_2-1} |f(t,s,v(s,t)) - f(t,s,z(s,t))| \\
&\times d_t h_1(y_2,t) d_s h_2(x_2,s) \\
&\leq \frac{\ln b}{\Gamma(r_1)\Gamma(r_2)} \bigvee_{\alpha=1}^{\alpha=b} h_2(y,\alpha) \int_1^x \left(\ln \frac{x}{s}\right)^{r_1-1} g(t,s)\, |v(s,t) - z(s,t)|\, d_t h_1(y_2,t) d_s h_2(x_2,s).
\end{aligned}$$

We obtain

$$|u(x,y) - w(x,y)| \leq \frac{\eta \ln b. \|g\|}{\Gamma(r_1)\Gamma(r_2)} \bigvee_{\alpha=1}^{\alpha=b} h_2(y,\alpha) |v(s,t) - z(s,t)|.$$

Then

$$\lim_{x \longrightarrow \infty} \sup\, diam\,(Y)(x,y) \leq k \lim_{x \longrightarrow \infty} \sup\, diam\, Y(x,y). \tag{7}$$

From the estimates (6) and (7) we infer that

$$\psi(Y) \leq k\psi(Y).$$

Since $k < 1$, we obtain $\psi(Y) = 0$. Combining the above result and Theorem 10 we complete the proof. □

4. Example

We consider the following Hadamard–Volterra–Stieltjes integral equation

$$\begin{aligned}
u(x,y) &= \frac{\ln(x+y+1)}{1+e^{x+y}} \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left(\ln \frac{x}{s}\right)^{r_1-1} \left(\ln \frac{y}{t}\right)^{r_2-1} \frac{se^{-s-t}}{t|u(s,t)| + st} \frac{d_t \ln([y]+t) d_s e^{-[x]-s}}{st}
\end{aligned} \tag{8}$$

where $(x,y) \in J = [1, +\infty) \times [1, \pi]$, and $r_1, r_2 > 0$.

Set

$$\mu(x,y) = \frac{\ln(x+y+1)}{1+e^{x+y}}; \quad h_1(y,t) = \ln([y]+t); \quad h_2(x,s) = e^{-[x]-s}$$

and

$$f(s,t,u(s,t)) = \frac{se^{-s-t}}{t|u(s,t)| + st};: (s,t) \in J.$$

The symbol $[y]$ indicates the integer value of y. It is clear that equation (8) can be written as equation (1). Let us show that conditions $(H_1)-(H_3)$ hold. The function $t \mapsto \ln([y]+t)$ is continuous and nondecreasing for each fixed $y_0 \in [1,\pi]$, so it is of bounded variation on $[1,\pi]\times[1,\pi]$, and

$$\bigvee_{t=1}^{t=\pi} h_2(y,t) \leq \ln(3+\pi) - \ln 2.$$

The function $s \mapsto e^{-[x]-s}$ is continuous and decreasing for each fixed $x_0 \in [1,+\infty)$, and

$$\lim_{s\to+\infty} e^{-[x]-s} = 0.$$

So it is of bounded variation on $[1,+\infty)\times[1,+\infty)$, and for each fixed $x_0 \in [1,+\infty)$

$$d_s h_1(x_0,s) = -e^{-[x_0]-s}ds.$$

$$\begin{aligned}
|f(s,t,u_1)-f(s,t,u_2)| &= \left|\frac{se^{-s-t}}{t|u_1|+st} - \frac{se^{-s-t}}{t|u_2|+st}\right| \\
&= \left|\frac{tse^{-s-t}(|u_1|-|u_2|)}{(t|u_1|+st)(t|u_2|+st)}\right| \\
&\leq \frac{tse^{-s-t}|u_1-u_2|}{|u_1|+|u_2|+1}.
\end{aligned}$$

So $\quad g(s,t) = tse^{-s-t}$; $\quad f(s,t,0) = \dfrac{e^{-s-t}}{t}$ and for a fixed $x_0 \in [1,+\infty)$ we have :

$$\begin{aligned}
&\left|\int_1^{x_0}\int_1^{y}\left(\ln\frac{x_0}{s}\right)^{r_1-1}|f(s,t,0)|d_t\ln([y]+t)d_s e^{-[x_0]-s}\right| \\
&= \left|\int_1^{x_0}\int_1^{y}\left(\ln\frac{x_0}{s}\right)^{r_1-1}\frac{e^{-s-t}}{t}d_t\ln([y]+t)d_s e^{-[x_0]-s}\right| \\
&\leq \left|\ln\pi[\ln(3+\pi)-\ln 2]\int_1^{x_0} -\left(\ln\frac{x_0}{s}\right)^{r_1-1}e^{-s-[x_0]}e^{-[x_0]-s}ds\right| \\
&\leq \left|\ln\pi[\ln(3+\pi)-\ln 2]\ln x_0\int_1^{x_0} e^{-2s-2[x_0]}ds\right| \\
&\leq \left|\ln\pi[\ln(3+\pi)-\ln 2]\ln x_0\frac{1}{2}\left[e^{-2s-2[x_0]}\right]_1^{x_0}\right| \\
&\leq \frac{1}{2}\left|\ln\pi[\ln(3+\pi)-\ln 2]\ln x_0\left[e^{-2x_0-2[x_0]}-e^{-2[x_0]-2}\right]\right|.
\end{aligned}$$

The function

$$\varphi(x) = \ln\pi\left[\ln(3+\pi)-\ln 2\right]\ln x\left[e^{-2x-[x]}-e^{-2[x]-2}\right],$$

is bounded on $[1,+\infty)$. Then there exist $\lambda>0$ such that

$$\sup_{x\geq 1;1\leq y\leq \pi}\left|\int_1^x\int_1^y\left(\ln\frac{x}{s}\right)^{r_1-1}|f(s,t,o)|d_t\ln([y]+t)d_se^{-[x]-s}\right|\leq\lambda.$$

We have also

$$\begin{aligned}\left|\int_1^{x_0}\left(\ln\frac{x_0}{s}\right)^{r_1-1}d_se^{-[x_0]-s}\right| &\leq \left|\ln x_0\int_1^{x_0}e^{-[x_0]-s}ds\right|\\ &\leq\left|\ln x_0\left[e^{-[x_0]-x_0}-e^{-[x_0]-1}\right]\right|.\end{aligned}$$

The function

$$\psi(x)=\ln x\left[e^{-[x]-x}-e^{-[x]-1}\right],$$

is bounded on $[1,+\infty)$. Then there exist $\eta>0$ such that

$$\sup_{x\geq 1}\left|\int_1^x\left(\ln\frac{x}{s}\right)^{r_1-1}d_se^{-[x]-s}\right|\leq\eta.$$

It follows that

$$k=\frac{\eta\ln b\|g\|}{\Gamma(r_1)\Gamma(r_2)}\bigvee_{\alpha=1}^{\alpha=b}h_2(y,\alpha)=\frac{\eta e^{-2}\ln\pi}{\Gamma(r_1)\Gamma(r_2)}\left[\ln(3+\pi)-\ln 2\right]<1.$$

Thus, from Theorem 12 the Eq. (8) has at least solution in BC and solutions of Eq. (8) are globally attractive.

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