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Simplified Theory of Ringbom Stirling Machines

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Abstract

This paper presents a first order analysis of four types of overdriven free-displacer Stirling machines. All the presented types of machines can work as refrigerating machines, prime movers or heat exchange accelerators depending on parameters such as the hot to cold source temperatures ratio, the nondimensional mass of working gas in the machine, the displacer rod to displacer cross sectional area ratio, the corrected dead space to piston cylinder volume ratio and the displacer to piston cylinder volume ratio.

In its analytical form this theory holds for machines at low speed as it is assumed that the piston displacement can be neglected during the displacer movement duration. This analysis may be used to find the conditions and values giving either the best theoretical refrigerating cycle or the best theoretical prime mover cycle, the associated reference work, reference time, efficiency and heat quantities involved. A table gives the analytical expressions and the limiting values of the main parameters for the four different types of Ringbom machines considered.

The preliminary design of a Ringbom prime mover is then presented. The main parameters influences are predicted and the magnitude of work, rotational speed limit and efficiency are obtained.

Keywords: Stirling machine, Ringbom machine, free-displacer, first order analysis, ideal analysis, Schmidt analysis.

1. Introduction

The Stirling cycle is one of the most interesting practical thermodynamic cycles and up to now, numerous works have been devoted to its development, especially with the kinematic Stirling engines and the refrigerating machines (Organ 1992, Reader and Hooper 1983, Urieli and Berchowitz 1984, Walker 1980, Walker 1983, West 1986). But the low-cost simple free-displacer machines, also called Ringbom machines (Walker and Senft 1985, Senft 1993, Senft 1996), have been less studied though the Schmidt analysis could be easily applied to that kind of system in which the displacer movement is merely caused by the difference between working-gas and bounce-space pressures.

The presented model is extensively developed for a separate lower-volume guiding-rod free displacer engine (type I-*Figure 1*), and the results are extended to the three other main types of machines (*Figure 2*). They give pre-dictions of the main parameters influences.

2. Principle of Operation

In order to minimize the difficulties of treatment and, also, not to be unrealistic, the following assumptions have been used:

- constant mass of a perfect gas
- isothermal heat exchanges during compression and expansion
- perfect regeneration
- inelastic shocks of the displacer
- clearance volume (dead space) V_d at an overall temperature T_d
- uniform instantaneous gas pressure
- sinusoidal motion of the heavy working piston

- constant bounce space pressure.

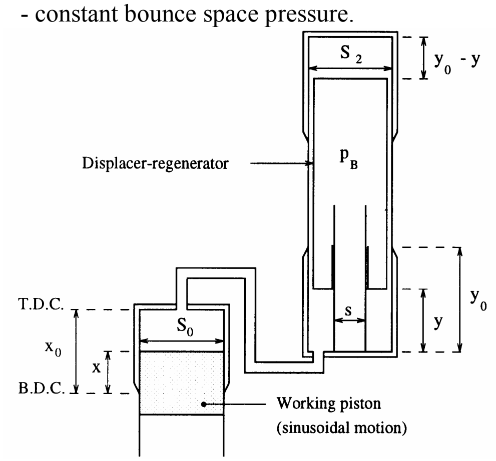

Figure 1. Type I of free-displacer Stirling machines.

The operation of the free-displacer Stirling engine is described with reference to *Figure 1*. The annular gap between displacer and cylinder wall is used as hot and cold heat exchanger and regenerator. Let us assume that the displacer is near the top of the displacer cylinder and that the piston is moving upward. The piston motion causes an increase of the working gas pressure which becomes greater than the bounce space pressure. Then the increasing gas pressure difference acting on the displacer rod area pushes the displacer down. The displacer moves the working gas from the cold to the hot volume, thus further increasing its pressure and accelerating the displacer downward until it reaches the bottom of the displacer cylinder. The piston moves downward (working stroke) increasing the cold volume and reducing the working gas pressure which becomes lower than the bounce space pressure. Then, the pressure difference induces an upward motion of the displacer which moves the hot gas to the cold space, reducing the working gas pressure and accelerating the displacer motion. The piston is then moving upward due to its sinusoidal motion (flywheel energy or elastic spring energy) and increases the working gas pressure once again. Then the cycle repeats.

3. Basic Equations

In order to generalize the results of the calculations we use dimensionless parameters and variables with the following reference parameters:

- temperature: T_C working cylinder temperature
- volume: $V_w = x_0 S_0$ working cylinder volume
- pressure: p_B bounce space mean pressure
- mass: $p_B V_W / r T_C$ working fluid mass contained in the working cylinder when at maximum volume and at mean pressure.

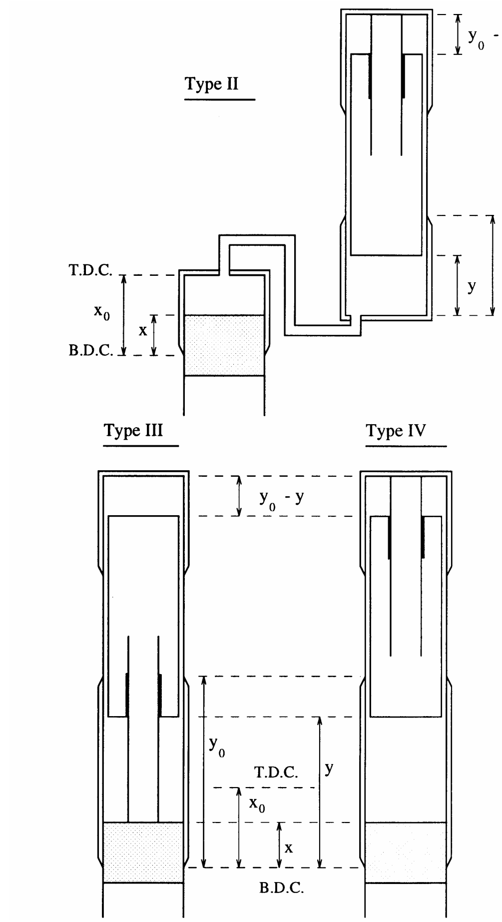

Figure 2. Other types of free-displacer Stirling machines.

The total mass of working gas enclosed in the cylinders is:

$$
\begin{aligned}
m &= \frac{p}{r}\left[\frac{(x_0 - x)S_0}{T_C} + \frac{(S_2 - S)y}{T_C} + \frac{S_2(y_0 - y)}{T_E} + \frac{V_d}{T_d}\right] \\
&= p\frac{x_0 S_0}{rT_C}\left[(1 - x^*) + \Sigma(1 - s^*)y^* + \Sigma(1 - y^*)\frac{1}{\tau} + \sigma\Sigma\right]
\end{aligned}
$$

where:

$$
x^* = \frac{x}{x_0};\quad y^* = \frac{y}{y_0};\quad \Sigma = \frac{y_0 S_2}{x_0 S_0};\quad V_w = x_0 S_0;
$$

$$
\tau = \frac{T_E}{T_C};\quad s^* = \frac{s}{S_2};\quad p^* = \frac{p}{p_B};\quad \sigma = \frac{T_C}{T_d}\frac{V_d}{y_0 S_2}.
$$

The dimensionless total mass of working gas is thus:

$$m^* = \frac{m}{\frac{p_B V_W}{r T_C}} = p^*\left[\left(1-x^*\right)+\Sigma\left(\sigma+\frac{1}{\tau}\right) + \Sigma\left(1-s^*-\frac{1}{\tau}\right)y^*\right] \quad (1)$$

and the dimensionless pressure is:

$$p^* = \frac{m^*}{1-x^*+\Sigma\left(\sigma+\frac{1}{\tau}\right)+\Sigma\left(1-s^*-\frac{1}{\tau}\right)y^*} \quad (2)$$

The dynamic equation of the displacer is given by:

$$M_D \frac{d^2y}{dt^2} = \left(p_B - p\right)s$$

Defining the reference time:

$$t_0 = \left(\frac{M_D y_0}{p_B s}\right)^{\frac{1}{2}}$$

the dimensionless expression of the displacer dynamic equation is:

$$\frac{d^2y^*}{dt^{*2}} = \left(2\pi\nu t_0\right)^2 \frac{d^2y^*}{d\alpha^2} = 1-p^* \quad (3)$$

where $t^* = \frac{t}{t_0}$, $\alpha = 2\pi\nu t$ is the rotation angle (in radian), and ν is the frequency of the piston motion. Let the working piston have a sinusoidal motion:

$$x^* = \frac{1}{2}\left(1+\cos\alpha\right) \quad (4)$$

4. Description of the Cycles

We consider a separate lower-volume guiding-rod free displacer machine (Type I).

From Eq. (2), we obtain the limiting pressure curves. When the displacer is down, that is for $y^* = 0$, we have:

$$p_0^*\left(x^*\right) = \frac{m^*}{1-x^*+\Sigma\left(\sigma+\frac{1}{\tau}\right)} \quad (5)$$

and when the displacer is up, that is for $y^* = 1$, we have:

$$p_1^*\left(x^*\right) = \frac{m^*}{1-x^*+\Sigma\left(1+\sigma-s^*\right)} \quad (6)$$

Moreover a value of x^* (with $0 < x^* < 1$) must exist which makes p^* equal to 1 to permit the displacer take-off. When the displacer is down, this leads:

$$x_0^* = 1-m^*+\Sigma\left(\sigma+\frac{1}{\tau}\right) \quad (7)$$

with the necessary condition:

$$\Sigma\left(\sigma+\frac{1}{\tau}\right) < m^* < 1+\Sigma\left(\sigma+\frac{1}{\tau}\right) \quad (8)$$

and when the displacer is in the upper position:

$$x_1^* = 1-m^*+\Sigma\left(1+\sigma-s^*\right) \quad (9)$$

with the necessary condition:

$$\Sigma\left(1+\sigma-s^*\right) < m^* < 1+\Sigma\left(1+\sigma-s^*\right) \quad (10)$$

Two cases should be distinguished, according to the relative values of s^* and τ.

4.1 First case (*Figure 3*)

The dimensionless guiding-rod area and the temperature ratio are related by:

$$1-s^* < \frac{1}{\tau} \quad (11)$$

It yields:

$$p_0^*\left(x^*\right) < p_1^*\left(x^*\right) \quad (12)$$

Relations (8), (10) and (11) give:

$$\Sigma\left(\sigma+\frac{1}{\tau}\right) < m^* < 1+\Sigma\left(1+\sigma-s^*\right) \quad (13)$$

which yields:

$$\frac{1}{\tau} < \frac{1}{\Sigma}+\left(1-s^*\right) \quad (14)$$

From relation (11) we have:

$$\tau_{max} = \frac{1}{\left(1-s^*\right)}$$

and from relation (14) we get:

$$\tau_{min} = \frac{1}{\frac{1}{\Sigma}+\left(1-s^*\right)}$$

Hence the possible temperature ratio range is:

$$\frac{1}{\frac{1}{\Sigma}+\left(1-s^*\right)} < \tau < \frac{1}{\left(1-s^*\right)} \quad (15)$$

We have a refrigerating machine or a heat pump (counterclockwise cycle); when pressure p_0^* passes under 1 (x^* decreasing) the displacer takes off and y* increases from 0; when pressure p_1^*

passes over 1 (x^* increasing) the displacer takes off from the upper position $y^*=1$.

- For $\tau < 1$, heat is taken from the cold volume and is rejected through the hot volume walls.
- For $\tau > 1$, we have a heat exchange accelerator. Though the work is positive, heat is taken from the hot volume and rejected through the cold volumes walls.

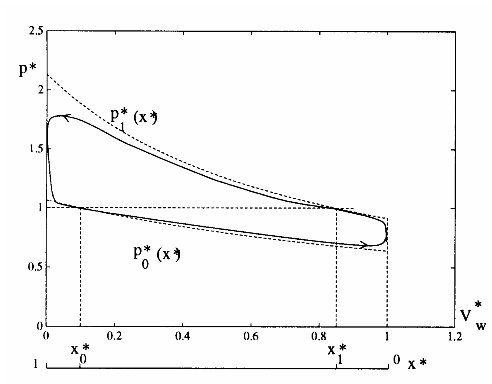

Figure 3. Refrigerating cycle.

It is possible to deduce some particular additional parameters such as:

$$p^*_{max} = \frac{m^*}{\Sigma\left(1+\sigma-s^*\right)};p^*_{min} = \frac{m^*}{1+\Sigma\left(\sigma+\frac{1}{\tau}\right)}$$

and the maximum pressure to minimum pressure ratio:

$$\Pi = \frac{p^*_{max}}{p^*_{min}} = \frac{1+\Sigma\left(\sigma+\frac{1}{\tau}\right)}{\Sigma\left(1+\sigma-s^*\right)} \tag{16}$$

4.2 Second case (*Figure 4*)

The dimensionless guiding-rod area and the temperature ratio are related by:

$$1-s^* > \frac{1}{\tau} \tag{17}$$

It yields:

$$p_1^*\left(x^*\right) < p_0^*\left(x^*\right) \tag{18}$$

Relations (8), (10) and (17) give:

$$\Sigma\left(1+\sigma-s^*\right) < m^* < 1+\Sigma\left(\sigma+\frac{1}{\tau}\right) \tag{19}$$

From relation (17) the minimum temperature ratio is:

$$\tau_{min} = \frac{1}{\left(1-s^*\right)}$$

Relation (19) gives a maximum temperature ratio depending on Σ value:

$$\tau_{max} = \frac{1}{1-s^*-\frac{1}{\Sigma}} = \frac{1}{\frac{1}{\tau_{min}}-\frac{1}{\Sigma}}$$

if

$$\Sigma > \frac{1}{1-s^*} = \tau_{min}$$

and

$$\tau_{max} \to \infty$$

if

$$\Sigma < \frac{1}{1-s^*} = \tau_{min}$$

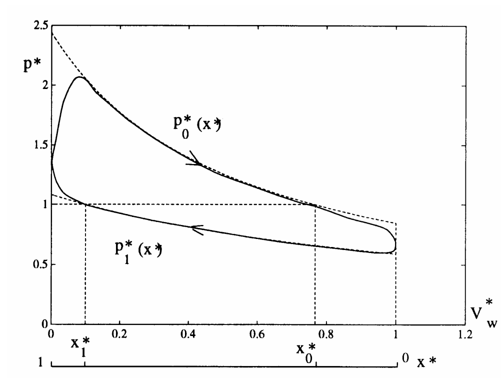

Figure 4. Prime mover cycle.

Hence the possible temperature ratio range is:

$$\frac{1}{1-s^*} < \tau < \frac{1}{1-s^*-\frac{1}{\Sigma}} \text{or} +\infty \tag{20}$$

We have a prime mover (clockwise cycle): when pressure p_1^* passes over 1 (x^* increasing) the displacer takes off from the upper position $y^*=1$; when pressure p_0^* passes under 1 (x^* decreasing) the displacer takes off from the lower position $y^*=0$. Heat is given to the 'expansion' space and rejected from the 'compression' space. The particular parameters are:

$$p^*_{max} = \frac{m^*}{\Sigma\left(\sigma+\frac{1}{\tau}\right)};p^*_{min} = \frac{m^*}{1+\Sigma\left(1+\sigma-s^*\right)}$$

and the maximum pressure to minimum pressure ratio:

$$\Pi = \frac{p^*_{max}}{p^*_{min}} = \frac{1+\Sigma\left(1+\sigma-s^*\right)}{\Sigma\left(\sigma+\frac{1}{\tau}\right)} \tag{21}$$

5. Analysis of the Prime Mover Cycle

5.1 Thermodynamic analysis

From the first law of thermodynamics and from the assumption of isothermal process in the 'expansion' volume the cycle averaged heat supplied to the engine is given by:

$$Q_E = \oint p dV_E$$

The dimensionless heat supplied is written as:

$$Q_E = \oint \frac{p}{p_B}\frac{dV_E}{V_W} = \oint p^* \frac{S_2 y_0}{V_W} d\left(1-y^*\right) = -\oint \Sigma p^* dy^* \quad (22)$$

At *very low speed* of rotation, let us assume that the piston displacement can be neglected during the displacer motion duration (*Figure 5*).

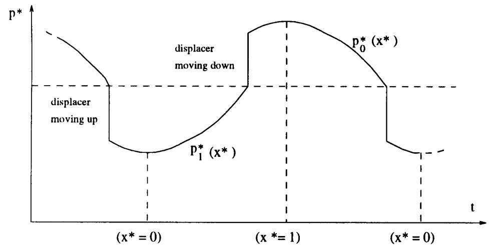

Figure 5. Pressure evolution of a prime mover (low r.p.m.).

From Eqs. (2), (7), (9), Eq. (22) can be rewritten as:

$$Q^*_{ls,E} = -\sum\left[\int_0^1 \frac{dy^*}{1+\frac{\Sigma}{m^*}\left(1-s^*-\frac{1}{\tau}\right)} - \int_0^1 \frac{dy^*}{1-\frac{\Sigma}{m^*}\left(1-s^*-\frac{1}{\tau}\right)\left(1-y^*\right)}\right] \quad (23)$$

which gives, after integration:

$$Q^*_{ls,E} = \frac{-m^*}{\left(1-s^*-\frac{1}{\tau}\right)} \ln\left|1-\left(\frac{\Sigma}{m^*}\left(1-s^*-\frac{1}{\tau}\right)\right)^2\right| \quad (24)$$

The dimensionless low speed cycle averaged work is given by:

$$W^*_{ls} = \frac{W_{ls}}{p_B V_W} = \int_{x_0^*}^{x_1^*}\left(p_1^* - p_0^*\right) \qquad dx^*$$

which becomes after integration taking account of Eq. (5), (6), (7), (9):

$$W^*_{ls} = m^* \ln\left|1-\left(\frac{\Sigma}{m^*}\left(1-s^*-\frac{1}{\tau}\right)\right)^2\right| \quad (25)$$

Hence the thermodynamic efficiency is expressed as:

$$\eta^*_{th,ls} = \frac{\left|W^*_{ls}\right|}{Q^*_{ls,E}} = 1-s^*-\frac{1}{\tau} \quad (26)$$

Defining:

$$A_0 = \frac{\Sigma}{m^*}\eta^*_{th,ls} = \frac{\Sigma}{m^*}\left(1-s^*-\frac{1}{\tau}\right) \quad (27)$$

we obtain the following expressions:

$$Q^*_{ls,E} = \frac{-m^*}{\eta_{th,ls}}\ln\left|1-A_0^2\right| \quad (28)$$

$$W^*_{ls,E} = m^* \ln\left|1-A_0^2\right| \quad (29)$$

with m^* in the range given by expression (19).

Maximum absolute value of W^*_{ls} and $\eta_{th,ls}$ occur with a displacer taking-off at $x_0^* = 0$ and $x_1^* = 1$ corresponding to:

$$\tau_{opt} = \tau_{max} \quad (30)$$

and

$$m^*_{opt} = 1+\Sigma\left(\sigma+\frac{1}{\tau_{max}}\right) = \Sigma\left(1+\sigma-s^*\right) = \frac{\frac{1}{\tau_{min}}+\sigma}{\frac{1}{\tau_{min}}-\frac{1}{\tau_{max}}} \quad (31)$$

It implies

$$A_{0,opt} = \frac{1}{m^*_{opt}} \quad (32)$$

$$\Sigma_{opt} = \frac{1}{\eta_{th,ls,max}} \quad (33)$$

where $\eta_{th,ls,max}$ is the maximum thermodynamic efficiency:

$$\eta_{th,ls,max} = 1-s^*-\frac{1}{\tau_{max}} = \frac{1}{\tau_{min}}-\frac{1}{\tau_{max}} \quad (34)$$

This leads to the following expressions for the maximum work:

$$W^*_{ls,max} = m^*_{opt}\ln\left|1-\frac{1}{m^{*2}_{opt}}\right| \quad (35)$$

and for the optimum heat supplied:

$$Q^*_{ls,E,opt} = \frac{-m^*_{opt}}{\eta_{th,ls,max}}\ln\left|1-\frac{1}{m^{*2}_{opt}}\right| \quad (36)$$

Note that, in the case of a prime-mover, having $W^*_{ls} = W^*_{ls,max}$ implies that τ reaches a maximum while s^*, σ, m^* and Σ reach a minimum. Minimum values of s^* and Σ are contradictory to a minimum value of reference time t_0. It seems that, as usual in technology, an equilibrium has to be found between opposite trends.

5.2 Displacer dynamics

As before we assume that the piston displacement is negligible during the displacer movement duration. For a displacer take-off from

$y^* = 0$ (displacer moving up) we obtain from Eq. (2), (3), (5) and (27):

$$\frac{d^2 y^*}{dt^{*2}} = 1 - p^*\left(y^*, x_0^*\right) = 1 - \frac{1}{1 + y^* A_0} \quad (37)$$

and for a take-off from $y^* = 1$ (displacer moving down)

$$\frac{d^2 y^*}{dt^{*2}} = 1 - p^*\left(y^*, x_1^*\right) = 1 - \frac{1}{1 - \left(1 - y^*\right) A_0} \quad (38)$$

Let us define $z^* = -\left(1 - y^*\right)$ and $Y = 1 + y^* A_0$ or $Y = 1 + z^* A_0$, then Eq. (37) and Eq. (38) have the same expression:

$$\frac{d^2 Y}{dt^{*2}} = A_0 \left(1 - \frac{1}{Y}\right) \quad (39)$$

which gives after integration:

$$\frac{dY}{dt^*} = \pm\left[2A_0\left(Y - \ln Y\right) + C\right]^{\frac{1}{2}} \quad (40)$$

At any take-off initial time the displacer speed is zero, so that $C = -2A_0$. Hence with a take-off from $y^* = 0$ the displacer velocity is:

$$\frac{dy^*}{dt^*} = \left(\frac{2}{A_0}\right)^{\frac{1}{2}} \left[y^* A_0 - \ln\left(1 + y^* A_0\right)\right]^{\frac{1}{2}} \quad (41)$$

At the end of the displacer stroke ($y^* = 1$) the dimensionless ultimate speed is:

$$v_r^* = \frac{v t_0}{y_0} = \left(\frac{2}{A_0}\right)^{\frac{1}{2}} \left[A_0 - \ln\left(1 + A_0\right)\right]^{\frac{1}{2}} \quad (42)$$

The dimensionless displacer rising time is:

$$t_r^* = \left(\frac{A_0}{2}\right)^{\frac{1}{2}} \int \frac{dy^*}{\left[y^* A_0 - \ln\left(1 + y^* A_0\right)\right]^{\frac{1}{2}}} \quad (43)$$

With a take-off from the upper position ($y^* = 1$) the dimensionless ultimate speed at the end of the falling stroke is:

$$v_f^* = -\left(\frac{2}{A_0}\right)^{\frac{1}{2}} \left[-A_0 - \ln\left(1 - A_0\right)\right]^{\frac{1}{2}} \quad (44)$$

The dimensionless displacer falling time is:

$$t_f^* = \left(\frac{A_0}{2}\right)^{\frac{1}{2}} \int_{-1}^{0} \frac{dz^*}{\left[z^* A_0 - \ln\left(1 + z^* A_0\right)\right]^{\frac{1}{2}}} \quad (45)$$

So the total dimensionless displacer motion duration is:

$$t_T^* = \left(\frac{A_0}{2}\right)^{\frac{1}{2}} \int_{-1}^{1} \frac{dy^*}{\left[y^* A_0 - \ln\left(1 + y^* A_0\right)\right]^{\frac{1}{2}}} \quad (46)$$

The dimensionless time t_T^* gives the order of magnitude of the dimensionless revolution time lower limit. *Figure 6* gives values of that time and of the dimensionless work W_{ls}^*/m^* versus A_0. We should remember that according to relations (19) and (27) A_0 is in the range:

$$0 \le \frac{\eta_{th,ls}}{\frac{1}{\Sigma} + \frac{1}{\tau} + \sigma} < A_0 < \frac{\eta_{th,ls}}{1 - s^* + \sigma} < 1$$

6. Preliminary Design of a Prime Mover

We shall use the lower part (crankcase and crankshaft) of an existing single cylinder S.I.

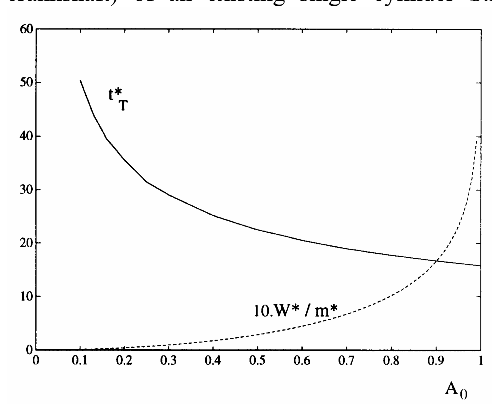

Figure 6. Dimensionless total displacer motion duration and work versus A_0.

engine as the working equipment of the Stirling engine. The working volume is $V_w = 50$ cm^3 and the stroke is $x_0 = 42$mm. The other fixed parameters (some of them being imposed by thermal or mechanical stresses limits) are:

- cold source temperature: $T_C = 300$K
- displacer mass: $M_D = 0.4$kg
- bounce space pressure: $p_B = 3.2 \; 10^5$ Pa
- nominal temperature ratio: $\tau = 3$
- minimum temperature ratio: $\tau_{min} = \frac{4}{3}$
- maximum temperature ratio: $\tau_{max} = 4$
- negligible clearance volume: $\sigma = 0$

We assume a sealed engine with a constant mass of gas m. From the previous data and from Eq. (20), (31) and (33) we find, for the optimal case where $x_0^* = 0$ and $x_0^* = 1$:

$$s^* = 0.25; m_{opt}^* = 1.5; \Sigma_{opt} = 2.$$

Nevertheless, as the displacer must be allowed to surely take-off, we need:

$$0 < x_0^* < x_1^* < 1$$

So it is more realistic to choose lower values of Σ and m^*, for instance:

$$m^* = 1.3; \Sigma = 1.6$$

Then Eq. (7), (9), (26), (27), (29) and (46) yield:

$$x_0^* = 0.233; \quad x_1^* = 0.900; \quad \eta_{th,ls} = 0.42;$$
$$A_0 = 0.513; \quad \left|W_{ls}^*\right| = 0.397; \quad t_T^* = 22.1$$

Furthermore if we assume $y_0 = 4$cm we obtain:

$$S_2 = 20\text{cm}^2; \quad s = 5\text{cm}^2; \quad t_0 = 10^{-2}\text{s}$$

From these results we deduce the lower revolution time limit: $t_{rev} > 22.1\ 10^{-2}$s.

TABLE I. Some Characteristics Relations for Ringbom Machines.

	TypeI	**TypeII**	**TypeIII** $(\Sigma > 1)$	**TypeIV** $(\Sigma > 1)$
$p^* =$	$m^*/[1 - x^* + \Sigma (1/\tau + \sigma) + \Sigma \left(1 - s^* - 1/\tau\right) y^*]$	$m^*/[1 - x^* + \Sigma[(1 - s^*)/\tau + \sigma - \Sigma ((1 - s^*)/\tau - 1) y^*]$	$m^*/[\Sigma(1/\tau + \sigma) - (1 - s^*) x^* + \Sigma (1 - s^* - 1/\tau) y^*]$	$m^*/[\Sigma[(1 - s^*)/\tau + \sigma] - x^* - \Sigma ((1 - s^*)/\tau - 1) y^*]$
$p_0^* =$	$m^*/[1 - x^* + \Sigma (1/\tau + \sigma)]$ $(y^* = 0)$	$m^*/[1 - x^* + \Sigma[(1 - s^*)/\tau + \sigma]]$ $(y^* = 0)$	$m^*/[\Sigma(1/\tau + \sigma) - x^*/\tau]$ $(y^* = x^*/\Sigma)$	$m^*/[\Sigma[(1 - s^*)/\tau + \sigma] - x^*(1 - s^*)/\tau]$ $(y^* = x^*/\Sigma)$
$p_1^* =$ $(y^* = 1)$	$m^*/[1 - x^* + \Sigma (1 - s^* + \sigma)]$	$m^*/[1 - x^* + \Sigma (1 + \sigma)]$	$m^*/[\Sigma(1 - s^* + \sigma) - x^*(1 - s^*)]$	$m^*/[\Sigma(1 + \sigma) - x^*]$
$x_0^* =$ $(p_0^* = 1)$	$1 - m^* + \Sigma (1/\tau + \sigma)$	$1 - m^* + \Sigma [(1 - s^*)/\tau + \sigma]$	$\Sigma + \tau\left(\Sigma\sigma - m^*\right)$	$\Sigma + \tau(\Sigma\sigma - m^*) /(1 - s^*)$
$x_1^* =$ $(p_1^* = 1)$	$1 - m^* + \Sigma (1 - s^* + \sigma)$	$1 - m^* + \Sigma (1 + \sigma)$	$\Sigma + (\Sigma\sigma - m^*) /(1 - s^*)$	$\Sigma + \left(\Sigma\sigma - m^*\right)$
ref.&heat exch.acc.	$p_1^*(x^*) > p_0^*(x^*)$ $x_1^* < x_0^*$	$p_1^*(x^*) < p_0^*(x^*)$ $x_1^* > x_0^*$	$p_1^*(x^*) > p_0^*(x^*)$ $x_1^* < x_0^*$	$p_1^*(x^*) < p_0^*(x^*)$ $x_1^* > x_0^*$
prime mover	$p_1^*(x^*) < p_0^*(x^*)$ $x_1^* > x_0^*$	$p_1^*(x^*) > p_0^*(x^*)$ $x_1^* < x_0^*$	$p_1^*(x^*) < p_0^*(x^*)$ $x_1^* > x_0^*$	$p_1^*(x^*) > p_0^*(x^*)$ $x_1^* < x_0^*$
$< \tau_{ref} <$ $\left(\text{if}\,\Sigma < 1/s^*\right)$	$1/(1 - s^* + 1/\Sigma) < \tau < 1$	$1 < \tau < \Sigma(1 - s^*) /(\Sigma - 1)$ or $+\infty$ if $\Sigma < 1$	$(\Sigma - 1)/[\Sigma(1 - s^*)] < \tau < 1$	$1 < \tau < \Sigma(1 - s^*)/(\Sigma - 1)$
$< \tau_{hea} <$	$1 < \tau < 1/(1 - s^*)$	$1 - s^* < \tau < 1$	$1 < \tau < 1/(1 - s^*)$	$1 - s^* < \tau < 1$
$< \tau_{prim} <$	$1/(1 - s^*) < \tau < 1/(1 - s^* + 1/\Sigma)$ or $+\infty$ if $\Sigma < 1/(1 - s^*)$	$\Sigma(1 - s^*)/(\Sigma + 1) < \tau < \left(1 - s^*\right)$	$1/(1 - s^*) < \tau < \Sigma/[(1 - s^*)(\Sigma - 1)]$	$(1 - s^*)(\Sigma - 1)/\Sigma < \tau < (1 - s^*)$
$< m_{ref}^*$ & $m_{hea}^* <$	$\Sigma(1/\tau + \sigma) < m^* < 1 + \Sigma(1 - s^* + \sigma)$	$\Sigma(1 + \sigma) < m^* < 1 + \Sigma [(1 - s^*)/\tau + \sigma]$	$\Sigma(1/\tau + \sigma) - 1/\tau < m^* < \Sigma(1 - s^* + \sigma)$	$\Sigma(1 + \sigma) - 1 < m^* < \Sigma [(1 - s^*)/\tau + \sigma]$
$< m_{prim} <$	$\Sigma(1 - s^* + \sigma) < m^* < 1 + \Sigma(1/\tau + \sigma)$	$[(1 - s^*)/\tau + \sigma] \Sigma < m^* < 1 + \Sigma(1 + \sigma)$	$\Sigma(1 - s^* + \sigma) - (1 - s^*) < m^* < \Sigma(1/\tau + \sigma)$	$[(1 - s^*)/\tau + \sigma] \Sigma - (1 - s^*)/\tau < m^* < \Sigma(1 + \sigma)$
$\eta_{th,ls} =$	$1 - s^* - 1/\tau$	$(1 - s^* - \tau) /(1 - s^*)$	$(1 - s^* - 1/\tau) /(1 - s^*)$	$(1 - s^* - \tau) /(1 - s^*)$
$W_{ls,prim}^* =$	$m^* \ln[1 - (\Sigma\eta_{th,ls}/m^*)^2]$	$m^* \ln[1 - (\Sigma\eta_{th,ls}/m^*)^2]$	$m^*\tau[\ln[(1 - \eta_{th,ls}) + \eta_{th,ls}\Sigma\sigma/m^*] - (1 - \eta_{th,ls}) \ln[(1 - \eta_{th,ls}) (1 - \eta_{th,ls}\Sigma\sigma/m^*)]]$	$m^*[\ln[(1 - \eta_{th,ls}) + \eta_{th,ls}\Sigma\sigma/m^*] + (1 - \eta_{th,ls}) \ln[(1 - \eta_{th,ls}\Sigma\sigma/m^*) (1 - \eta_{th,ls})]]$
$m_{opt}^* =$	$1 + (1/\tau_{max} + \sigma) /\eta_{th,ls,max}$	$(1 + \sigma) /\eta_{th,ls,max} - \sigma$	$(1/\tau_{max} + \sigma) /\eta_{th,ls,max}$	$(1 + \sigma) /\eta_{th,ls,max}$
$t_0 =$	$y_0[M_D / (p_B V_w \Sigma s^*)]^{1/2}$	$y_0[M_D / (p_B V_w \Sigma s^*)]^{1/2}$	$y_0[M_D / (p_B V_w \Sigma s^*)]^{1/2}$	$y_0[M_D / (p_B V_w \Sigma s^*)]^{1/2}$

7. Other Main Types of Free-Displacer Stirling Machines

The preceeding analysis could be easily repeated with the three remaining Ringbom configurations (*Figure 2*). The main results are gathered in TABLE I. They permit the evaluation of a Ringbom machine performance as a function of the main parameters s^*, Σ, σ, τ, y_0, m^*, M_D, p_B, V_w.

For instance power is an increasing function of absolute work and efficiency and a decreasing function of reference time. Maximum prime-mover absolute work imposes minimum m^*, Σ, σ

and s^* and maximum p_B, V_w and τ. Maximum prime-mover efficiency imposes minimum s^* and maximum temperature ratio τ. Minimum reference time impose minimum y_0, M_D and maximum p_B, V_w, Σ and s^*. Some of those parameters such as Σ and s^* have to match contradictory theoretical conditions. Finally technological conditions and practical considerations impose realistic values.

8. Conclusion

The Schmidt analysis is known for its simplicity. Obviously the associated assumptions depart from real phenomena but they are useful to get a simple model of different types of free-displacer machines.

All the presented types of free-displacer machines can work as refrigerating machines, prime movers or heat exchange accelerators, depending on parameters τ, m^*, s^*, Σ and σ. We are able to predict the main parameters influences and to get the magnitude of work, rotational speed limit and efficiency of a theoretical free-displacer machine provided that the main parameters are chosen. The values of those parameters are determined by the above study and by thermal stress, thermal exchange, mechanical stress or space considerations.

A more accurate design procedure should require a more refined and complicated model taking into account frictional works, temperature gradients, actual heat exchanges and imperfect regeneration.

Nomenclature

A_0	a parameter $=\left(\Sigma/m^*\right)\eta_{th,ls0}$
m	mass of gas
m_D	mass of the displacer
p	pressure
Q	cycle averaged heat exchanged
r	specific gas constant
s	guiding rod area
S	cylinder area
T	temperature
t	time
v	displacer velocity
V	volume
W	cycle averaged work
x	piston position
y	displacer position
α	angle of 'rotation'
η_{th}	thermodynamic efficiency
ν	frequency
π	3.14...
Π	maximum to minimum pressure ratio
Σ	displacer volume to piston volume ratio
σ	reduced dead volume
τ	temperature ratio

Subscripts:

0	with displacer down (p_0^*, x_0^*) or reference value (S_0, x_0, y_0, t_0, W_0, ...)
1	with displacer up (p_1^*, x_1^*)
2	in the displacer cylinder (S_2)
B	in the bounce space
C	in the 'compression' (heat rejected) space
d	in the 'dead' space (clearance volume)
E	in the 'expansion' (heat added) space
f	with displacer falling
hea	heat exchange accelerator
ls	with low rotational speed assumption
m	mean value
max	maximum value
min	minimum value
opt	optimum value
prim	prime mover
r	with displacer rising
ref	refrigerator
T	total value (t_T)
w	in the working volume (V_w)
*	dimensionless value (superscript)

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