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INTUITIONISTIC TEXTURES REVISITED

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Abstract

The authors consider difunctions between double-set (intuitionistic) textures, and place the study of double-set textures and ditopological double-set textures in a categorical setting.

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1. Introduction

The notion of intuitionistic fuzzy set was introduced by K. T. Atanassov [2], and a considerable amount of work has been done in this direction, particularly in the area of applications. An intuitionistic fuzzy subset of a crisp set X is a generalization of Zadeh's notion of fuzzy subset of X [23], but it has been noted recently that intuitionistic sets can be represented as $\mathbb{L}$–sets for a suitable Hutton algebra $\mathbb{L}$. This was first stated explicitly by G.-J. Wang and Y. Y. He [22], and in the same issue of Fuzzy Sets and Systems, L. M. Brown and M. Diker announced that intuitionistic sets, a specialization of the notion of intuitionistic fuzzy sets introduced by D. Çoker in [12], may be represented as **3**-sets. The proof of this result, given in [7], was indirect and we express this relation more directly in the present paper.

In the meantime J. G. Garcia and S. E. Rodabaugh [16] have investigated in some detail the order-theoretic and categoric relations between intuitionistic fuzzy sets and topologies on the one hand and $\mathbb{L}$-sets and topologies on the other. They point out that the term "intuitionistic" is a misnomer in this context, and in parallel with their use of the term double-fuzzy set we have adopted the term "double-set" for the intuitonistic set of Doğan Çoker, and will therefore speak of double topology, double-set texture and so on throughout the remainder of this paper.

The layout of the paper is as follows. In Section 2 we recall double-set textures, and go on to investigate difunctions between such textures. The concrete nature and relative

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simplicity of double-set textures enables us to give a very specific representation of difunctions between them, and we look in detail at images and co-images, inverse (co)images and complementation. We show in particular that the complemented difunctions are of a very special type, a result which has important categorical implications. We also characterize injectivity and surjectivity of difunctions.

Section 3 gives the explicit isomorphism between double-sets and **3**-sets mentioned above, and discusses various subcategories of known categories of textures involved in the study of double-set textures. Finally, Section 4 is devoted to constructs of ditopological double-set textures, and their relationship with the constructs **3-Top**, **Top** and **Bitop**.

We note that our general reference for concepts from category theory is [1], while the reader is referred to [17] for notions from lattice theory not defined here. Basic definitions and background from the theory of ditopological texture spaces may be found in [3–11, 20]. Papers on intuitionistic (fuzzy) sets and topologies include [2, 12–16] and [18].

2. Double-Set Textures

Given a base set X, a *double-subset of* X (intuitionistic subset of X in the sense of Çoker [12]) may be represented by an ordered pair (A, B) of subsets A, B of X satisfying $A \cap B = \emptyset$. We denote by $\mathbb{D}_X$ the set of all double-subsets of X ordered by $(A, B) \subseteq (C, D) \iff A \subseteq C$ and $D \subseteq B$. It is known that $\mathbb{D}_X$ is a Hutton algebra, that is a complete, completely distributive lattice with order reversing involution $\prime : \mathbb{D}_X \to \mathbb{D}_X$ defined by $(A, B)' = (B, A)$.

In [5] it was shown that we may associate a complemented simple texture with $\mathbb{D}_X$. With a slight change of notation this is the complemented texture $(D_X, \mathcal{D}_X, \delta_X)$, where

$$D_X = (X \times \{1\}) \cup (X \times \{\tfrac{1}{2}\}) = X \times \{1, \tfrac{1}{2}\},$$
$$\mathcal{D}_X = \{(A \times \{1\}) \cup ((X \setminus B) \times \{\tfrac{1}{2}\}) \mid (A, B) \in \mathbb{D}_X\} \text{ and}$$
$$\delta_X((A \times \{1\}) \cup ((X \setminus B) \times \{\tfrac{1}{2}\})) = (B \times \{1\}) \cup ((X \setminus A) \times \{\tfrac{1}{2}\}).$$

Here the mapping $(A, B) \mapsto (A \times \{1\}) \cup ((X \setminus B) \times \{\frac{1}{2}\})$ from $\mathbb{D}_X$ to $\mathcal{D}_X$, which we will denote by β_X, or just by β if there is no chance of confusion, is a Hutton algebra isomorphism.

A general method of associating a complemented simple texture with a Hutton algebra was given in [7]. Precisely, if $\mathbb{L}$ is a Hutton algebra and M the set of molecules of $\mathbb{L}$, then the associated complemented simple texture is $(M, \mathcal{M}, \mu)$, where $\mathcal{M} = \{\widehat{\alpha} \mid \alpha \in \mathbb{L}\}$, $\widehat{\alpha} = \{m \in M \mid m \leq \alpha\}$ and $\mu(\widehat{\alpha}) = \widehat{\alpha'}$, $\prime$ being the order reversing involution on $\mathbb{L}$. We have:

2.1. Proposition. *The complemented texture associated with the Hutton algebra* $\mathbb{D}_X$ *as in* [7] *is isomorphic to* $(D_X, \mathcal{D}_X, \delta_X)$.

Proof. The reader may easily verify that the molecules of $\mathbb{D}_X$ are of two kinds, namely

$$(\{x\}, X \setminus \{x\}),\ x \in X, \text{ and } (\emptyset, X \setminus \{x\}),\ x \in X.$$

Clearly the first kind of molecule corresponds to an intuitinistic point in the sense of Çoker [12] (see also Çoker and Demirci [15]), while the second kind corresponds to a vanishing intuitonistic point. On the other hand, for $(A, B) \in \mathbb{D}_X$ we have

$$\widehat{(A, B)} = \{(\{x\}, X \setminus \{x\}) \mid x \in A\} \cup \{(\emptyset, X \setminus \{x\}) \mid x \in X \setminus B\}.$$

It is now clear that the mapping

$$(\{x\}, X \setminus \{x\}) \mapsto (x, 1),\ x \in X,$$
$$(\emptyset, X \setminus \{x\}) \mapsto (x, \tfrac{1}{2}),\ x \in X,$$

sets up a complemented textural isomorphism between the complemented texture associated with $\mathbb{D}_X$ as in [7], and the complemented texture $(D_X, \mathcal{D}_X, \delta_X)$. $\square$

It is clear that under this isomorphism $\widehat{(A,B)}$ corresponds to $\beta_X(A,B)$, so the two representations of $\mathbb{D}_X$ are the same up to isomorphism. We refer to such textures as *double-set textures*.

For the remainder of this section we consider double-set textures in their own right, only returning to their relation with double subsets of X in the next section. It will therefore be convenient to rewrite $\mathcal{D}_X$ and δ_X in the equivalent forms

$$\mathcal{D}_X = \{(A \times \{1\}) \cup (C \times \{\tfrac{1}{2}\}) \mid A \subseteq C \subseteq X\}, \text{ and}$$
$$\delta_X((A \times \{1\}) \cup (C \times \{\tfrac{1}{2}\})) = ((X \setminus C) \times \{1\}) \cup ((X \setminus A) \times \{\tfrac{1}{2}\}).$$

The following results are now immediate from the definitions.

2.2. Lemma. *The p-sets and q-sets for $(D_X, \mathcal{D}_X, \delta_X)$ are as follows:*

$$P_{(x,\frac{1}{2})} = \{x\} \times \{\tfrac{1}{2}\},$$
$$P_{(x,1)} = \{x\} \times \{\tfrac{1}{2}, 1\},$$
$$Q_{(x,\frac{1}{2})} = (X \setminus \{x\}) \times \{\tfrac{1}{2}, 1\},$$
$$Q_{(x,1)} = (X \times \{\tfrac{1}{2}\}) \cup ((X \setminus \{x\}) \times \{1\}).$$

Also,

$$\delta_X(P_{(x,\frac{1}{2})}) = Q_{(x,1)} \text{ and } \delta_X(P_{(x,1)}) = Q_{(x,\frac{1}{2})},$$
$$\delta_X(Q_{(x,\frac{1}{2})}) = P_{(x,1)} \text{ and } \delta_X(Q_{(x,1)}) = P_{(x,\frac{1}{2})}.$$

In particular we note for future reference that for $x, x' \in X$, $k, k' \in \{\frac{1}{2}, 1\}$ we have $P_{(x,k)} \not\subseteq Q_{(x',k')} \iff x = x'$ and $k' \le k$.

We will be particularly interested in difunctions between double-set textures. It is known that a difunction (f, F) between simple textures $(S, \mathcal{S})$ and $(T, \mathcal{T})$ may be represented by a unique point function $\varphi : S \to T$ in the sense that $(f, F) = (f_\varphi, F_\varphi)$ where

$$f_\varphi = \bigvee\{\overline{P}_{(s,t)} \mid \exists v \in S \text{ satisfying } P_s \not\subseteq Q_v \text{ and } P_{\varphi(v)} \not\subseteq Q_t\},$$
$$F_\varphi = \bigcap\{\overline{Q}_{(s,t)} \mid \exists v \in S \text{ satisfying } P_v \not\subseteq Q_s \text{ and } P_t \not\subseteq Q_{\varphi(v)}\}$$

and φ satisfies the conditions

(a) $s, s' \in S$, $P_s \not\subseteq Q_{s'} \implies P_{\varphi(s)} \not\subseteq Q_{\varphi(s')}$.
(b) $P_{\varphi(s)} \not\subseteq B$, $B \in \mathcal{T} \implies \exists\, s' \in S$ with $P_s \not\subseteq Q_{s'}$ for which $P_{\varphi(s')} \not\subseteq B$.

([9], Proposition 3.6). If, in addition, the textures $(S, \mathcal{S})$ and $(T, \mathcal{T})$ are plain then (b) is automatically satisfied and we have

$$f_\varphi = \bigcup\{\overline{P}_{(s,\varphi(s))} \mid s \in S\}, \ F_\varphi = \bigcap\{\overline{Q}_{(s,\varphi(s))} \mid s \in S\}.$$

Applying the above to a difunction (f, F) from the plain simple texture $(D_X, \mathcal{D}_X)$ to the plain simple texture $(D_Y, \mathcal{D}_Y)$ we have a point function $\varphi : D_X \to D_Y$ satisfying (a) for which $(f, F) = (f_\varphi, F_\varphi)$.

2.3. Lemma. *The point function $\varphi : D_X \to D_Y$ satisfies (a) if and only if*

(i) *$\varphi = \langle \varphi_1, \varphi_2 \rangle$, where $\varphi_1 : X \to Y$ and $\varphi_2 : D_X \to \{\frac{1}{2}, 1\}$.*
(ii) *For $x \in X$, $k, k' \in \{\frac{1}{2}, 1\}$ we have $k \le k' \implies \varphi_2(x, k) \le \varphi_2(x, k')$.*

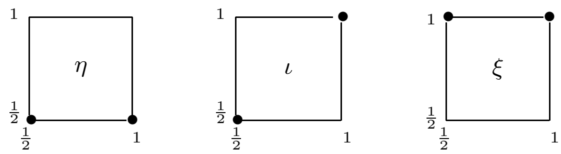
Proof. First suppose that (a) is satisfied. For $x \in X$ we have $P_{(x,1)} \not\subseteq Q_{(x,\frac{1}{2})}$, and so $P_{(\varphi_1(x,1),\varphi_2(x,1))} \not\subseteq Q_{(\varphi_1(x,\frac{1}{2}),\varphi_2(x,\frac{1}{2}))}$. This gives us $\varphi_1(x,1) = \varphi_1(x,\frac{1}{2})$ and $\varphi_2(x,\frac{1}{2}) \leq \varphi_2(x,1)$. From the above equality we see that the value of φ_1 depends only on x, and so this may be regarded as a point function from X to Y. This proves (i), and (ii) is an immediate consequence of the above inequality.

Conversely, if φ satisfies (i) and (ii) it trivially satisfies (a), as the reader may easily verify. □

The formulae for $f = f_\varphi$ and $F = F_\varphi$ specialise to

$$f_\varphi = \bigcup\{\overline{P}_{((x,k),(\varphi_1(x),\varphi_2(x,k)))} \mid x \in X,\ k = \tfrac{1}{2}, 1\},$$
$$F_\varphi = \bigcap\{\overline{Q}_{((x,k),(\varphi_1(x),\varphi_2(x,k)))} \mid x \in X,\ k = \tfrac{1}{2}, 1\}.$$

We note that by Lemma 2.3 (ii), for fixed $x \in X$ the function $\varphi_2(x,k)$ of k can have any of the following three forms:



This shows that even if X only has one point, for each function φ_1 from X to Y there is more than one difunction from $(D_X, \mathcal{D}_X)$ to $(D_Y, \mathcal{D}_Y)$.

Now let us consider the complement of a difunction (f,F) from $(D_X, \mathcal{D}_X, \delta_X)$ to $(D_Y, \mathcal{D}_Y, \delta_Y)$.

2.4. Proposition. *Let (f,F) be a difunction from $(D_X, \mathcal{D}_X, \delta_X)$ to $(D_Y, \mathcal{D}_Y, \delta_Y)$ that corresponds to the point function φ, and let the complement $(f,F)' = (F', f')$ correspond to the point function φ'. Then $\varphi'_1(x) = \varphi_1(x)$ for all $x \in X$ and for all fixed $x \in X$ the functions $\varphi_2(x,k)$, $\varphi'_2(x,k)$ of k are related by*

$$\varphi'_2(x,k) = \begin{cases} \xi(k), & \text{if } \varphi_2(x,k) = \eta(k), \\ \iota(k), & \text{if } \varphi_2(x,k) = \iota(k), \\ \eta(k), & \text{if } \varphi_2(x,k) = \xi(k). \end{cases}$$

Proof. It will be sufficient to calculate $(F_\varphi)'$ and compare it with $f_{\varphi'}$. Now, by ([9], Definition 2.18),

$$F' = \bigcup\{\overline{P}_{((x,k),(y,l))} \mid \exists u \in X, v \in Y, m,n \in \{\tfrac{1}{2},1\} \text{ with } \overline{P}_{((u,m),(v,n))} \not\subseteq F,$$
$$P_{(u,m)} \not\subseteq \delta_X(P_{(x,k)}) \text{ and } \delta_Y(Q_{(y,l)}) \not\subseteq Q_{(v,n)}\},$$

since the textures here are plain. Now using Lemma 2.2 and the above formula for $F = F_\varphi$ we obtain

$$(F_\varphi)' = \bigcup\{\overline{P}_{((x,k),(\varphi_1(x),l))} \mid \varphi_2(x,k') \leq l'\},$$

where for $k, l \in \{\frac{1}{2}, 1\}$ we have $k', l' \in \{\frac{1}{2}, 1\}$, $k' \neq k$, $l' \neq l$. On the other hand

$$f_{\varphi'} = \bigcup\{\overline{P}_{((x,k),(\varphi'_1(x),\varphi'_2(x,k)))} \mid x \in X, k \in \{\tfrac{1}{2},1\}\},$$

and we deduce that $\varphi'_1(x) = \varphi_1(x)\ \forall\, x \in X$, and that for each fixed $x \in X$, $\varphi'_2(x,k)$ is the largest value of l satisfying $\varphi_2(x,k') \leq l'$. Let us consider the three possible cases for the function $\varphi_2(x,k)$ of k.

Case 1. Let $\varphi_2(x,k)=\eta(k)$, $k=\frac{1}{2},1$. We may take $l=1$ since then $l'=\frac{1}{2}$ and $\varphi_2(x,k')\le\frac{1}{2}$ for each k. Hence $\varphi_2'(x,k)=1$, that is $\varphi_2'(x,k)=\xi(k)$, $k=\frac{1}{2},1$.

Case 2. Let $\varphi_2(x,k)=\iota(k)$, $k=\frac{1}{2},1$. For $k=1$ we have $k'=\frac{1}{2}$ and $\varphi_2(x,\frac{1}{2})=\frac{1}{2}$, so as above we may take $l=1$. Thus $\varphi_2'(x,1)=1$. On the other hand for $k=\frac{1}{2}$ we have $k'=1$ and $\varphi_2(x,1)=1$, so we must take $l=\frac{1}{2}$. Thus $\varphi_2'(x,\frac{1}{2})=\frac{1}{2}$, and we have shown that $\varphi_2'(x,k)=\iota(k)$, $k=\frac{1}{2},1$.

Case 3. Let $\varphi_2(x,k)=\xi(k)$, $k=\frac{1}{2},1$. The proof is similar to that for the previous cases, and is omitted. □

2.5. Corollary. *The difunction* $(f,F)=(f_\varphi,F_\varphi)$ *is complemented if and only if* $\varphi=\langle\varphi_1,\iota\rangle$, *that is* $\varphi(x,k)=(\varphi_1(x),k)$ *for all* $x\in X$, $k=\frac{1}{2},1$.

Proof. By Proposition 2.4 the only way in which the functions φ, φ' can be equal is for $\varphi_2(x,k)$ to have the value $\iota(k)$ for all fixed $x\in X$. Hence $(f,F)'=(f,F)\iff\varphi'=\varphi\iff\varphi=\langle\varphi_1,\iota\rangle$. □

It follows that to each point function $\varphi_1:X\to Y$ corresponds just one complemented difunction from $(D_X,\mathcal{D}_X,\delta_X)$ to $(D_Y,\mathcal{D}_Y,\delta_Y)$. This contrasts strongly with the situation for complemented difunctions between Hutton textures of $\mathbb{I}$-valued sets [9, Examples 3.11 (4)].

Now we consider (inverse) images and (inverse) co-images under a difunction from $(D_X,\mathcal{D}_X)$ to $(D_Y,\mathcal{D}_Y)$.

2.6. Proposition. *Let* $(f,F)=(f_\varphi,F_\varphi):(D_X,\mathcal{D}_X)\to(D_Y,\mathcal{D}_Y)$ *and take* $(A\times\{1\})\cup(C\times\{\frac{1}{2}\})\in\mathcal{D}_Y$. *Then*

$$f^{\leftarrow}(A\times\{1\})\cup(C\times\{\tfrac{1}{2}\})=F^{\leftarrow}(A\times\{1\})\cup(C\times\{\tfrac{1}{2}\})=(A'\times\{1\})\cup(C'\times\{\tfrac{1}{2}\}),$$

where

$$A'=\{x\in\varphi_1^{-1}[A]\mid\varphi_2(x,1)=1\}\cup\{x\in\varphi_1^{-1}[C]\mid\varphi_2(x,1)=\tfrac{1}{2}\},$$
$$C'=\{x\in\varphi_1^{-1}[A]\mid\varphi_2(x,\tfrac{1}{2})=1\}\cup\{x\in\varphi_1^{-1}[C]\mid\varphi_2(x,\tfrac{1}{2})=\tfrac{1}{2}\}.$$

Proof. From [9, Lemma 3.4 and Lemma 3.9] we have

$$f_\varphi^{\leftarrow}B=F_\varphi^{\leftarrow}B=\varphi^{-1}[B]$$

for all $B=(A\times\{1\})\cup(C\times\{\frac{1}{2}\})\in\mathcal{D}_Y$. But $(x,k)\in\varphi^{-1}[B]\iff(\varphi_1(x),\varphi_2(x,k))\in B\iff\varphi_1(x)\in A,\ \varphi_2(x,k)=1$ or $\varphi_1(x)\in C,\ \varphi_2(x,k)=\frac{1}{2}$. The result now follows by setting $k=1$ for A' and $k=\frac{1}{2}$ for C'. □

2.7. Corollary. *For the special cases* $\varphi=\langle g,\eta\rangle$, $\varphi=\langle g,\iota\rangle$ *and* $\varphi=\langle g,\xi\rangle$, *where* $g:X\to Y$ *and* η, ι *and* ξ *are defined as above, the inverse image and inverse co-image of* $(A\times\{1\})\cup(C\times\{\frac{1}{2}\})\in\mathcal{D}_Y$ *under* (f_φ,F_φ) *are given respectively by* $g^{-1}[C]\times\{\frac{1}{2},1\}$, $(g^{-1}[A]\times\{1\})\cup(g^{-1}[C]\times\{\frac{1}{2}\})$ *and* $g^{-1}[A]\times\{\frac{1}{2},1\}$.

Proof. Clear from Proposition 2.6 and the fact that $g^{-1}[A]\subseteq g^{-1}[C]$ since $A\subseteq C$. □

2.8. Proposition. *For* $(f_\varphi,F_\varphi):(D_X,\mathcal{D}_X)\to(D_Y,\mathcal{D}_Y)$ *and* $(A\times\{1\})\cup(C\times\{\frac{1}{2}\})\in\mathcal{D}_X$ *we have*

$$f_\varphi^{\rightarrow}(A\times\{1\})\cup(C\times\{\tfrac{1}{2}\})=(A'\times\{1\})\cup(\varphi_1[C]\times\{\tfrac{1}{2}\}),$$

where

$$A'=\{\varphi_1(x)\mid x\in A,\ \varphi_2(x,1)=1\ \textit{or}\ x\in C,\ \varphi_2(x,\tfrac{1}{2})=1\}.$$

Proof. By ([9], Definition 2.5 (1)), we have for $B = (A \times \{1\}) \cup (C \times \{\frac{1}{2}\}) \in \mathcal{D}_X$,

$$\begin{aligned} f_\varphi^{\to} B &= \bigcap \{Q_{(y,l)} \mid \forall\,(x,k) \in X \times \{\tfrac{1}{2}, 1\},\ f_\varphi \not\subseteq \overline{Q}_{((x,k),(y,l))} \implies B \subseteq Q_{(x,k)}\} \\ &= \bigcap \{Q_{(y,l)} \mid y = \varphi_1(x),\ l \leq \varphi_2(x,k) \implies B \subseteq Q_{(x,k)}\ \forall\,(x,k)\}. \end{aligned}$$

Now, since we are dealing with plain textures,

$$\begin{aligned} (w,s) \in f_\varphi^{\to} B &\iff f_\varphi^{\to} B \not\subseteq Q_{(w,s)} \\ &\iff \exists\,(x,k) \text{ with } w = \varphi_1(x),\ s \leq \varphi_2(x,k) \text{ and } B \not\subseteq Q_{(x,k)}. \end{aligned}$$

There are two cases to be considered.

Case 1. $s = \frac{1}{2}$. In this case $s \leq \varphi_2(x,k)$ is automatically satisfied for both values of k, and $B \not\subseteq Q_{(x,k)}$ gives $x \in A$ $(k = 1)$ or $x \in C$ $(k = \frac{1}{2})$, so in both cases $w \in \varphi_1[C]$.

Case 2. $s = 1$. In this case we have the condition $\varphi_2(x,k) = 1$ for $x \in A$, $k = 1$, and for $x \in C$, $k = \frac{1}{2}$. This is clearly equivalent to $w \in A'$. □

2.9. Corollary. *For the special cases $\varphi = \langle g, \eta \rangle$, $\varphi = \langle g, \iota \rangle$ and $\varphi = \langle g, \xi \rangle$, where $g : X \to Y$ and η, ι and ξ are defined as above, the image of $(A \times \{1\}) \cup (C \times \{\frac{1}{2}\}) \in \mathcal{D}_X$ under (f_φ, F_φ) is given respectively by $g[C] \times \{\frac{1}{2}\}$, $(g[A] \times \{1\}) \cup (g[C] \times \{\frac{1}{2}\})$ and $g[C] \times \{\frac{1}{2}, 1\}$.*

Proof. Clear from Proposition 2.8. □

2.10. Proposition. *For $(f_\varphi, F_\varphi) : (D_X, \mathcal{D}_X) \to (D_Y, \mathcal{D}_Y)$ and $(A \times \{1\}) \cup (C \times \{\frac{1}{2}\}) \in \mathcal{D}_X$ we have*

$$F_\varphi^{\to} (A \times \{1\}) \cup (C \times \{\tfrac{1}{2}\}) = (\varphi_1[A] \times \{1\}) \cup (C' \times \{\tfrac{1}{2}\}),$$

where

$$\begin{aligned} C' = \{\varphi_1(x) \mid x \in A,\ \varphi_2(x,k) = \tfrac{1}{2},\ k = \tfrac{1}{2}, 1 \text{ or } x \in C,\ \varphi_2(x, \tfrac{1}{2}) = \tfrac{1}{2}, \\ \varphi_2(x,1) = 1 \text{ or } x \in X,\ \varphi(x,k) = 1, k = \tfrac{1}{2}, 1\}. \end{aligned}$$

Proof. For $B = (A \times \{1\}) \cup (C \times \{\frac{1}{2}\}) \in \mathcal{D}_X$ we have

$$\begin{aligned} F_\varphi^{\to} B &= \bigcup \{P_{(y,l)} \mid \forall\,(x,k) \in X \times \{\tfrac{1}{2}, 1\},\ \overline{P}_{((x,k),(y,l))} \not\subseteq F_\varphi \implies P_{(x,k)} \subseteq B\} \\ &= \bigcup \{P_{(y,l)} \mid y = \varphi_1(x),\ \varphi_2(x,k) \leq l \implies (x,k) \in B,\ \forall\, k = \tfrac{1}{2}, 1\} \end{aligned}$$

by [9, Definition 2.5 (2)] and the fact that we are dealing with plain textures. For $l = 1$ the given implication can hold for both values of k only if $x \in A$, which gives $\varphi_1(x) \in \varphi_1[A]$. Likewise, the set C' is easily obtained by considering the given implication when $l = \frac{1}{2}$. □

2.11. Corollary. *For the special cases $\varphi = \langle g, \eta \rangle$, $\varphi = \langle g, \iota \rangle$ and $\varphi = \langle g, \xi \rangle$, where $g : X \to Y$ and η, ι and ξ are defined as above, the co-image of $(A \times \{1\}) \cup (C \times \{\frac{1}{2}\}) \in \mathcal{D}_X$ under (f_φ, F_φ) is given respectively by $g[A] \times \{\frac{1}{2}, 1\}$, $(g[A] \times \{1\}) \cup (g[C] \times \{\frac{1}{2}\})$ and $(g[A] \times \{1\}) \cup (g[X] \times \{\frac{1}{2}\})$.*

Proof. Clear from Proposition 2.10. □

We end this section by considering the surjectivity and injectivity of difunctions [9, Definition 2.30] between double-set textures. First we give the following.

2.12. Lemma. *Let $(S, \mathcal{S})$, $(T, \mathcal{T})$ be plain textures and $\varphi : S \to T$ satisfy (a). Then (f_φ, F_φ) is surjective if and only if φ is onto.*

Proof. First let (f_φ, F_φ) be surjective and take $t \in T$. Since $(T, \mathcal{T})$ is plain we have $P_t \not\subseteq Q_t$ and so by surjectivity we have $s \in S$ with $f_\varphi \not\subseteq \overline{Q}_{(s,t)}$ and $\overline{P}_{(s,t)} \not\subseteq F_\varphi$. This now gives $P_{\varphi(s)} \not\subseteq Q_t$ and $P_t \not\subseteq Q_{\varphi(s)}$, whence $P_t = P_{\varphi(s)}$ and so $t = \varphi(s)$. Thus φ is onto.

The converse follows from [4, Lemma 2.7] since the condition (b) is automatically satisfied for plain textures. □

2.13. Proposition. *The following are equivalent for* $(f_\varphi, F_\varphi) : (D_X, \mathcal{D}_X) \to (D_Y, \mathcal{D}_Y)$.

(1) (f_φ, F_φ) *is surjective.*
(2) $\varphi = \langle \varphi_1, \varphi_2 \rangle : D_X \to D_Y$ *is onto.*
(3) *Given* $y \in Y$ *there exists* $x \in X$ *with* $y = \varphi_1(x)$, $\varphi_2(x,k) = \iota(k)$, $k = \frac{1}{2}, 1$, *or there exist* $x_1, x_2 \in X$ *with* $y = \varphi_1(x_1) = \varphi_1(x_2)$, $\varphi_2(x_1,k) = \eta(k)$, $\varphi_2(x_2,k) = \xi(k)$, $k = \frac{1}{2}, 1$.

Proof. Since double-set textures are plain the equivalence of (1) and (2) follows at once from Lemma 2.12. The equivalence of (2) and (3) is clear. □

2.14. Corollary. *If* (f_φ, F_φ) *is surjective then* φ_1 *is onto. The converse is true if* $\varphi = \langle \varphi_1, \iota \rangle$.

Proof. Immediate from Proposition 2.13. □

Now we turn our attention to injectivity.

2.15. Lemma. *Let* $(S, \mathcal{S})$, $(T, \mathcal{T})$ *be plain textures, and let* $\varphi : S \to T$ *satisfy (a). Then the injectivity of* (f_φ, F_φ) *implies that* φ *is one-to-one.*

Proof. Take $s, s' \in X$ with $\varphi(s) = \varphi(s')$, and denote this element of T by t. Then since $(T, \mathcal{T})$ is plain, $P_{\varphi(s)} \not\subseteq Q_t$ and $P_t \not\subseteq Q_{\varphi(s')}$, so $f_\varphi \not\subseteq \overline{Q}_{(s,t)}$ and $\overline{P}_{(s',t)} \not\subseteq F_\varphi$. The injectivity of (f_φ, F_φ) now gives $P_s \not\subseteq Q_{s'}$, whence $P_{s'} \subseteq P_s$. Reversing the roles of s, s' gives $P_s \subseteq P_{s'}$, so $P_s = P_{s'}$ and so $s = s'$ since the texturing $\mathcal{S}$ separates the points of S. This shows that φ is one-to-one. □

2.16. Proposition. *The following are equivalent for* $(f_\varphi, F_\varphi) : (D_X, \mathcal{D}_X) \to (D_Y, \mathcal{D}_Y)$.

(1) (f_φ, F_φ) *is injective.*
(2) $\varphi = \langle \varphi_1, \varphi_2 \rangle : D_X \to D_Y$ *is one-to-one.*
(3) $\varphi = \langle \varphi_1, \iota \rangle$ *and* $\varphi_1 : X \to Y$ *is one-to-one.*

Proof. (1) $\Longrightarrow$ (2) This follows from Lemma 2.15 since double-set textures are plain.

(2) $\Longrightarrow$ (3) For $x \in X$ we have $\varphi(x, \frac{1}{2}) \neq \varphi(x, 1)$ since φ is one-to-one. Hence $\varphi_2(x, \frac{1}{2}) \neq \varphi_2(x, 1)$ and by Lemma 2.3 (ii) we deduce that $\varphi_2(x, \frac{1}{2}) < \varphi_2(x, 1)$. It now follows that $\varphi_2(x, \frac{1}{2}) = \frac{1}{2}$, $\varphi_2(x, 1) = 1$ so $\varphi = \langle \varphi_1, \iota \rangle$. Now let $\varphi_1(x) = \varphi_1(x')$ for $x, x' \in X$. Then $\varphi(x, 1) = (\varphi_1(x), 1) = (\varphi_1(x'), 1) = \varphi(x', 1)$, whence $x = x'$ since φ is one-to-one. Hence φ_1 is also one-to-one.

(3) $\Longrightarrow$ (1) Suppose that $f_\varphi \not\subseteq \overline{Q}_{((x,k),(y,l))}$ and $\overline{P}_{((x',k'),(y,l))} \not\subseteq F_\varphi$. Then using the formulae for f_φ, F_φ and the fact that $\varphi_2 = \iota$ we obtain $\overline{P}_{((x,k),(\varphi_1(x),k))} \not\subseteq \overline{Q}_{((x,k),(y,l))}$ and $\overline{P}_{((x',k'),(y,l))} \not\subseteq \overline{Q}_{((x',k'),(\varphi_1(x'),k'))}$. Hence $P_{(\varphi_1(x),k)} \not\subseteq Q_{(y,l)}$ and $P_{(y,l)} \not\subseteq Q_{(\varphi_1(x'),k')}$, so $P_{(\varphi_1(x),k)} \not\subseteq Q_{(\varphi_1(x'),k')}$. By Lemma 2.2 we deduce $\varphi_1(x) = \varphi_1(x')$ and $k' \leq k$. By hypothesis φ_1 is one-to-one, so $x = x'$, and again by Lemma 2.2 we deduce that $P_{(x,k)} \not\subseteq Q_{(x',k')}$. This shows that (f_φ, F_φ) is injective. □

2.17. Proposition. *The following are equivalent for* $(f_\varphi, F_\varphi) : (D_X, \mathcal{D}_X) \to (D_Y, \mathcal{D}_Y)$.

(1) (f_φ, F_φ) *is bijective.*

(2) *$\varphi = \langle \varphi_1, \varphi_2 \rangle : D_X \to D_Y$ is one-to-one and onto.*
(3) *$\varphi = \langle \varphi_1, \iota \rangle$ and $\varphi_1 : X \to Y$ is one-to-one and onto.*
(4) *φ is onto and φ_1 is one-to-one.*

Proof. (1) $\implies$ (2) Clear from Proposition 2.13 and Proposition 2.16.

(2) $\implies$ (3) Clear from Corollary 2.14 and Proposition 2.16.

(3) $\implies$ (4) Since $\varphi = \langle \varphi_1, \iota \rangle$ and φ_1 is onto we see (f_φ, F_φ) is surjective by Corollary 2.14, so φ is onto by Proposition 2.13.

(4) $\implies$ (1) Since φ is onto it satisfies condition (3) of Proposition 2.13. Take $u \in X$ and let $y = \varphi_1(u) \in Y$. Since φ_1 is assumed to be one-to-one the existence of $x_1, x_2 \in X$ with $y = \varphi_1(x_1) = \varphi_1(x_2)$ and $\varphi_2(x_1, k) = \eta(k)$, $\varphi_2(x_2, k) = \xi(k)$ would lead to $x_1 = x_2$ and hence the contradiction $\eta(k) = \xi(k)$, $k = \frac{1}{2}, 1$. Hence there exists $x \in X$ with $y = \varphi_1(x)$ and $\varphi_2(x, k) = \iota(k)$, $k = \frac{1}{2}, 1$. But in particular $\varphi_1(u) = \varphi_1(x)$, so $x = u$ as φ_1 is one-to-one, and we have established that $\varphi = \langle \varphi_1, \iota \rangle$. Moreover φ_1 is onto, so (f_φ, F_φ) is surjective by Corollary 2.14, and φ_1 is one-to-one so (f_φ, F_φ) is injective by Proposition 2.16. This shows that (f_φ, F_φ) is bijective. □

In view of Corollary 2.5, the above results lead immediately to the following.

2.18. Proposition. *Let $(f_\varphi, F_\varphi) : (D_X, \mathcal{D}_X) \to (D_Y, \mathcal{D}_Y)$. Then:*

(1) *If (f_φ, F_φ) is injective it is complemented.*
(2) *If (f_φ, F_φ) is complemented then it is surjective (injective, bijective) if and only if φ_1 is onto (respectively, one-to-one, one-to-one and onto).*

The reader might find it instructive to compare the above results on the injectivity and surjective of difunctions between double-set textures with the corresponding results for difunctions between Hutton textures of classical fuzzy sets as presented in [21].

3. Categories of Double-Set Textures

In this section we consider categories of double-set textures and study their relation with certain known categories.

It was announced in [5], and proved in [7], that the double-subsets of X correspond to the $\mathbb{L}$-fuzzy subsets of X for $\mathbb{L} = \mathbf{3} = \{0, \frac{1}{2}, 1\}$. The proof given in [7] was indirect, and we now obtain explicit mappings between $\mathbb{D}_X$ and $\mathbf{3}^X$ that demonstrate the isomorphism of these Hutton algebras. We again work via the texture $(D_X, \mathcal{D}_X, \delta_X)$, thereby confirming that up to isomorphism this is the Hutton texture of both $\mathbb{D}_X$ and $\mathbf{3}^X$.

First let us note that the set of molecules of $\mathbf{3}$ is clearly $\{\frac{1}{2}, 1\}$, so the set of molecules of $\mathbf{3}^X$ is $X \times \{\frac{1}{2}, 1\} = D_X$. The texturing on this set corresponding to $\mathbf{3}^X$ as in [7] consists of the sets $\widehat{\mu} = \{(x, \alpha) \in D_X \mid \alpha \le \mu(x)\}$. If we set

$$A_\mu = \{x \in X \mid \mu(x) = 1\} \text{ and } B_\mu = \{x \in X \mid \mu(x) = 0\}$$

it is easy to see that $\widehat{\mu} = (A_\mu \times \{1\}) \cup ((X \setminus B_\mu) \times \{\frac{1}{2}\})$, and clearly $A_\mu \cap B_\mu = \emptyset$. Hence $(A_\mu, B_\mu) \in \mathbb{D}_X$ and $\widehat{\mu} = \beta_X(A_\mu, B_\mu)$, so $\{\widehat{\mu} \mid \mu \in \mathbf{3}^X\} \subseteq \mathcal{D}_X$.

Conversely, take $(A, B) \in \mathbb{D}_X$. Then for $\mu \in \mathbf{3}^X$ we clearly have $A = A_\mu$, $B = B_\mu$ if and only if $\mu = \mu_{(A,B)}$, where

$$\mu_{(A,B)}(x) = \begin{cases} 1 & \text{if } x \in A, \\ \frac{1}{2} & \text{if } x \in X \setminus (A \cup B), \\ 0 & \text{if } x \in B. \end{cases}$$

This confirms that $\{\widehat{\mu} \mid \mu \in \mathbf{3}^X\} = \mathcal{D}_X$. On the other hand if μ' is the complement of μ in $\mathbf{3}^X$ then $\mu'(x) = 1 - \mu(x)$, $x \in X$, and so $A_{\mu'} = B_\mu$, $B_{\mu'} = A_\mu$. It follows that $\widehat{\mu'} = \delta_X(\widehat{\mu})$, and we have proved that the Hutton texture of $\mathbf{3}^X$ is precisely $(D_X, \mathcal{D}_X, \delta_X)$. The above argument clearly shows that the mapping

$$\mu \mapsto (A_\mu, B_\mu),$$

which we will denote by ζ_X, is an isomorphism $\zeta_X : \mathbf{3}^X \to \mathbb{D}_X$, while

$$(A, B) \mapsto \mu_{(A,B)}$$

is the inverse isomorphism $\zeta_X^{-1} : \mathbb{D}_X \to \mathbf{3}^X$.

In view of the above, the functor $\mathfrak{F} = \mathfrak{F}_{(\mathbf{3},\prime)} : \mathbf{Set} \to \mathbf{ctmSTex}^{\mathrm{op}}$ given in [10] may be defined by $\mathfrak{F}(X) = (D_X, \mathcal{D}_X, \delta_X)$, $\mathfrak{F}(\psi) = \widehat{\psi^{-1}}$, and now maps into $\mathbf{ctmPSTex}^{\mathrm{op}}$ since $(D_X, \mathcal{D}_X, \delta_X)$ is plain.

$$\begin{array}{ccccc} X & & \mathbf{3}^X \equiv \mathbb{D}_X & & \mathcal{D}_X \\ \psi \big\downarrow & & \big\uparrow \psi^{-1} & & \big\uparrow \widehat{\psi^{-1}} \\ Y & & \mathbf{3}^Y \equiv \mathbb{D}_Y & & \mathcal{D}_Y \end{array}$$

Here $\psi^{-1}(\mu)(x) = \mu(\psi(x))$, that is $\psi^{-1}(\mu) = \mu \circ \psi$. In terms of double-sets, if $(A, B) \in \mathbb{D}_Y$ then the result of applying ψ^{-1} will be $(A_{\mu_{(A,B)} \circ \psi},\ B_{\mu_{(A,B)} \circ \psi})$. Since

$$\begin{aligned} A_{\mu_{(A,B)} \circ \psi} &= \{x \in X \mid \mu_{(A,B)}(\psi(x)) = 1\} \\ &= \{x \in X \mid \psi(x) \in A\} \\ &= \psi^{-1}[A], \end{aligned}$$

and likewise $B_{\mu_{(A,B)} \circ \psi} = \psi^{-1}[B]$, we see that $\psi^{-1}(A, B) = (\psi^{-1}[A], \psi^{-1}[B])$ which is precisely the inverse image as defined in [13]. It follows that the mapping $\widehat{\psi^{-1}}$ from $\mathcal{D}_Y$ to $\mathcal{D}_X$, which we will denote by θ_ψ for short, is given explicitly by

$$(A \times \{1\}) \cup ((Y \setminus B) \times \{\tfrac{1}{2}\}) \mapsto (\psi^{-1}[A] \times \{1\}) \cup ((X \setminus \psi^{-1}[B]) \times \{\tfrac{1}{2}\}).$$

Now let us recall from [10] that the category $\mathbf{ctmPSTex}^{\mathrm{op}}$ is isomorphic to the category $\mathbf{cdfPSTex}$ of plain simple textures and complemented difunctions, which in turn is isomorphic to the construct $\mathbf{cfPSTex}$ of plain simple textures and point functions φ between the base sets satisfying the condition

(a) $P_{s'} \not\subseteq Q_s \implies P_{\varphi(s')} \not\subseteq Q_{\varphi(s)}$,

and which are complemented in the sense that the corresponding difunction is complemented. Note that the condition (b) mentioned in [9] is automatically satisfied for plain textures. If we restrict our attention to the subcategories $\mathbf{ctmPSTex}^{\mathrm{op}}_{\mathbf{dbl}}$, $\mathbf{cdfPSTex}_{\mathbf{dbl}}$ and $\mathbf{cfPSTex}_{\mathbf{dbl}}$, respectively, whose objects have the form $(D_X, \mathcal{D}_X, \delta_X)$, $X \in \mathrm{Ob}\mathbf{Set}$, the same isomorphisms hold. Finally, these isomorphisms hold also in the more general case where the morphisms in these categories are not assumed to preserve complementation, so we will be interested in the following functors:

$$\mathbf{Set} \xrightarrow{\ \mathfrak{F}\ } \mathbf{ctmPSTex}^{\mathbf{op}}_{\mathbf{dbl}} \xrightarrow{\ \mathfrak{N}_c\ } \mathbf{cdfPSTex}_{\mathbf{dbl}} \xrightarrow{\ \mathfrak{V}_c\ } \mathbf{cfPSTex}_{\mathbf{dbl}}$$

and

$$\mathbf{Set} \xrightarrow{\ \mathfrak{F}\ } \mathbf{tmPSTex}^{\mathbf{op}}_{\mathbf{dbl}} \xrightarrow{\ \mathfrak{N}\ } \mathbf{dfPSTex}_{\mathbf{dbl}} \xrightarrow{\ \mathfrak{V}\ } \mathbf{fPSTex}_{\mathbf{dbl}}$$

where $\mathfrak{N}_c$, $\mathfrak{N}$ and $\mathfrak{V}_c$, $\mathfrak{V}$ are the isomorphisms mentioned above. Since $\mathfrak{N}_c$, $\mathfrak{N}$ are identities on objects by [10, Theorem 4.2], we need only determine the image $(f^{\theta_\psi}, F^{\theta_\psi})$ of θ_ψ under these isomorphisms.

3.1. Lemma. *For $\psi : X \to Y$, the image of θ_ψ under $\mathfrak{N}_c$ or $\mathfrak{N}$ is the difunction $(f^{\theta_\psi}, F^{\theta_\psi})$ given by*

$$f^{\theta_\psi} = \bigcup\{\overline{P}_{((x,k),(\psi(x),k)} \mid x \in X,\ k = \tfrac{1}{2}, 1\},$$
$$F^{\theta_\psi} = \bigcap\{\overline{Q}_{((x,k),(\psi(x),k))} \mid x \in X,\ k = \tfrac{1}{2}, 1\}.$$

Proof. We prove the formula for f^{θ_ψ}, leaving the dual proof of the formula for F^{θ_ψ} to the interested reader.

Since we are dealing with plain textures it follows from [10, Proposition 4.1] that

$$f^{\theta_\psi} = \bigcup\{\overline{P}_{((x,k),(y,l))} \mid P_{(x,k)} \subseteq \theta_g(C) \implies P_{(y,l)} \subseteq C\ \forall C \in \mathcal{D}_Y\}.$$

Hence, to obtain the required equality it will be sufficient to show that

$$(y = \psi(x),\ l = k) \implies ((x,k) \in \theta_\psi(C) \implies (y,l) \in C\,\forall C \in \mathcal{D}_Y) \implies (y = \psi(x),\ l \le k).$$

Suppose first that $y = \psi(x)$, $l = k$ and take any $C = (A \times \{1\}) \cup ((Y \setminus B) \times \{\frac{1}{2}\}) \in \mathcal{D}_Y$. Then $(x,k) \in \theta_\psi(C) = (\psi^{-1}[A] \times \{1\}) \cup ((X \setminus \psi^{-1}[B]) \times \{\frac{1}{2}\})$ so $y = \psi(x) \in A$, $l = k = 1$ or $y = \psi(x) \in Y \setminus B$, $l = k = \frac{1}{2}$, which gives $(y,l) \in C$ as required.

Conversely, suppose that $(x,k) \in \theta_\psi(C) \implies (y,l) \in C\ \forall C \in \mathcal{D}_Y$. There are two cases to be considered.

Case 1. Suppose $k = 1$ and let $A = Y \setminus B = \{\psi(x)\}$. Then $C = A \times \{\frac{1}{2}, 1\} \in \mathcal{D}_Y$ and $(x,1) \in \theta_\psi(C) = \psi^{-1}[A] \times \{\frac{1}{2}, 1\}$ so by hypothesis $(y,l) \in C = A \times \{\frac{1}{2}, 1\}$, which gives $y = \psi(x)$, and trivially we have $l \le k$.

Case 2. Suppose $k = \frac{1}{2}$ and let $A = \emptyset$, $B = Y \setminus \{\psi(x)\}$, whence $C = ((Y \setminus B) \times \{\frac{1}{2}\} = \{\psi(x)\} \times \{\frac{1}{2}\} \in \mathcal{D}_Y$. Then $(x,k) \in \theta_\psi(C) = \psi^{-1}[\{\psi(x)\}]$, so by hypothesis $(y,l) \in C$ and we obtain $y = \psi(x)$ and $l = \frac{1}{2} = k$.

This completes the proof of the required equality. □

Comparing the above expressions for f^{θ_ψ}, F^{θ_ψ} with the formulae for f_φ, F_φ, where $\varphi = \langle \varphi_1, \varphi_2 \rangle : D_X \to D_Y$, given following Lemma 2.3, we see that $(f^{\theta_\psi}, F^{\theta_\psi}) = (f_\varphi, F_\varphi)$ if and only if $\varphi = \langle \psi, \iota \rangle$. Hence we have proved:

3.2. Corollary. *For $\psi : X \to Y$ in* **Set** *we have*

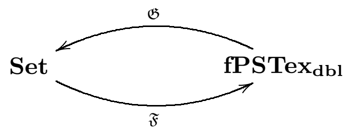
$$\mathfrak{V} \circ \mathfrak{N} \circ \mathfrak{F}(\psi) = \langle \psi, \iota \rangle = \mathfrak{V}_c \circ \mathfrak{N}_c \circ \mathfrak{F}(\psi).$$

We recall from Corollary 2.5 that all the complemented difunctions (f_φ, F_φ) between the double-set textures $(D_x, \mathcal{D}_X, \delta_X)$ and $(D_Y, \mathcal{D}_Y, \delta_Y)$ have the form $\langle \psi, \iota \rangle$ for $\psi : X \to Y$, whence the functor $\mathfrak{V}_c \circ \mathfrak{N}_c \circ \mathfrak{F}$ is a bijection on morphisms as well as objects and hence an isomorphism between **Set** and $\mathbf{cfPSTex_{dbl}}$. For simplicity we rename this functor $\mathfrak{F}$, that is we redefine $\mathfrak{F}$ by setting $\mathfrak{F}(X) = (D_X, \mathcal{D}_X, \delta_X)$ and $\mathfrak{F}(\psi) = \langle \psi, \iota \rangle$. On the other hand $\mathbf{cfPSTex_{dbl}}$ is a full subconstruct of **cfPSTex**, so we have established the following:

3.3. Theorem. *The functor $\mathfrak{F}$ is a full embedding of* **Set** *in* **cfPSTex**.

On the other hand we know from Lemma 2.3 that there exist (non-complemented) difunctions between double-set textures which are not of the form $\langle \psi, \iota \rangle$. Hence $\mathfrak{F}$, regarded as a functor from **Set** to $\mathbf{fPSTex_{dbl}}$, is not a full embedding. In this case, on analogy with the functor $\mathfrak{G}$ considered in [10], we may define a functor $\mathfrak{G} : \mathbf{fPSTex_{dbl}} \to \mathbf{Set}$

by setting $\mathfrak{G}(D_X, \mathcal{D}_X, \delta_X) = X$ and $\mathcal{G}(\varphi) = \varphi_1$, where φ_1 is as in Lemma 2.3. It is straightforward to verify that $\mathfrak{G}$ is indeed a functor, and we omit the details. Hence we have:



It is easy to show that $\mathfrak{G}$ is not an adjoint of $\mathfrak{F}$, precisely because there are morphisms in $\mathbf{fPSTex_{dbl}}$ that are not of the form $\langle \psi, \iota \rangle$, and likewise $\mathfrak{G}$ is not a co-adjoint of $\mathfrak{F}$.

4. Constructs of Ditopological Double-Set Textures

In this final section we briefly consider ditopologies on double-set textures and their relation to topologies on double sets and to topologies in the classical sense.

We call T a *double-topology* on X if $T \subseteq \mathbb{D}_X$ is an intuitionistic topology on X in the sense of [14], that is if it contains $(X, \emptyset)$, $(\emptyset, X)$ and is closed under finite intersections and arbitrary unions. We will denote by **Dbl-Top** the construct whose objects are pairs (X, T) where T is a double topology on X, and whose morphisms are continuous functions between these spaces.

In view of the isomorphism between $\mathbb{D}_X$ and $\mathbf{3}^X$ described above, it follows that double topologies on X are essentially the same thing as $\mathbf{3}$-topologies on X. Hence to show that **Dbl-Top** is isomorphic to the construct **3 - Top**, it will suffice to show that $\psi : X \to Y$ is continuous as a mapping from (X, T) to (Y, V), where T, V are double-topologies, if and only if $\{g \circ \psi \mid g \in \zeta_Y^{-1}[V]\} \subseteq \zeta_X^{-1}[T]$ (cf. [10, Example 2.8]). But if ψ is T–V continuous and $g \in \zeta_Y^{-1}[V]$ then $\zeta_Y(g) = (A_g, B_g) \in V$ so $\psi^{-1}(A_g, B_g) = (\psi^{-1}[A_g], \psi^{-1}[B_g]) \in T$ by the continuity of ψ. On the other hand $\psi^{-1}(\zeta_Y(g)) = \zeta_X(g \circ \psi)$, and so $g \circ \psi \in \zeta_X^{-1}[T]$ as required. The proof of the converse is similar, and is omitted. Hence we have:

4.1. Theorem. *The functor* $\mathfrak{Z}$: **Dbl-Top** $\to$ **3-Top** *defined by* $\mathfrak{Z}(D_X) = \mathbf{3}^X$, $\mathfrak{Z}(\psi) = \psi$, *is an isomorphism.*

From the general discussion in [10] it is known that the $\mathbf{3}$-topologies on X correspond in a one-to-one way with complemented ditopologies on the Hutton texture of $\mathbf{3}^X$, and that this texture is the product of the discrete texture $(X, \mathcal{P}(X), \pi_X)$ and the Hutton texture $(N, \mathcal{N}, \nu)$ of $\mathbf{3}$ given by $N = \{\frac{1}{2}, 1\}$, $\mathcal{N} = \{\emptyset, \{\frac{1}{2}\}, N\}$ and $\mu(\emptyset) = N$, $\nu(\{\frac{1}{2}\} = \{\frac{1}{2}\}$, $\nu(N) = \emptyset$. In view of the isomorphism between $\mathbb{D}$ and $\mathbf{3}^X$ this product texture must be isomorphic with $(D_X, \mathcal{D}_X, \delta_X)$, and it is easy to check that in fact they are equal.

If on analogy with the construct $\mathbb{I}$-**Top** given in [10, Example 2.8] we define the construct **3-Ditop** to have as objects the textures $(D_X, \mathcal{D}_X, \delta_X)$ equipped with a complemented ditopology (τ_X, κ_X), and as morphisms bicontinuous mappings of the form $\langle \psi, \iota \rangle$, we see that **3-Top** is isomorphic to **3-Ditop** under the functor $\mathfrak{T}$ taking the $\mathbf{3}$-topological space (X, T) to $(D_X, \mathcal{D}_X, \delta_X, \tau_T, \kappa_T)$, and the morphism $\psi : X \to Y$ to $\langle \psi, \iota \rangle$. Moreover, in view of Corollary 2.5, we clearly have **3-Ditop** = $\mathbf{cfPSDitop_{dbl}}$, while **3-Ditop** is a non-full subconstruct of $\mathbf{fPSDitop_{dbl}}$.

In the context of double-set textures we will find it convenient to refer to **3-Ditop** as **Dbl-CDitop**, while **Dbl-Ditop** will denote the construct of double-set textures $(D_X, \mathcal{D}_X, \delta_X)$ equipped with an arbitrary ditopology (τ_X, κ_X), and morphisms which are bicontinuous mappings of the form $\langle \psi, \iota \rangle$. Hence **Dbl-CDitop** is a full subconstruct of **Dbl-Ditop**, which in turn is a non-full subconstruct of $\mathbf{fPSDitop_{dbl}}$.

We now return to the functors $\mathfrak{F}$ and $\mathfrak{G}$ considered earlier.

Let (τ_0, κ_0) be a fixed ditopology on $(N, \mathcal{N}, \nu)$. Then for any bitopology [19] (u, v) on X the pair (u, v^c) is a ditopology on $(X, \mathcal{P}(X), \pi_X)$ and so the product $(u \otimes \tau_0, v^c \otimes \kappa_0)$ is a ditopology on $(X, \mathcal{P}(X), \pi_X) \otimes (N, \mathcal{N}, \nu) = (D_X, \mathcal{D}_X, \delta_X)$. We denote this ditopology by $(\tau_{(u,v)}, \kappa_{(u,v)})$. If $\psi : X_1 \to X_2$ is (u_1, v_1)–(u_2, v_2) bicontinuous (also called pairwise continuous), that is u_1–u_2 continuous and v_1–v_2 continuous, then $\langle \psi, \iota \rangle$ is $(\tau_{(u_1,v_1)}, \kappa_{(u_1,v_1)})$–$(\tau_{(u_2,v_2)}, \kappa_{(u_2,v_2)})$ bicontinuous. Indeed, a subbase for $u_2 \otimes \tau_0$ consists of the sets $X_2 \times G$, $G \in \tau_0$ and $U \times N$, $U \in u_2$, while $\langle \psi, \iota \rangle^{-1}[X_2 \times G] = X_1 \times G$, $\langle \psi, \iota \rangle^{-1}[U \times N] = \psi^{-1}[U] \times N \in u_1 \otimes \tau_0$, whence $\langle \psi, \iota \rangle$ is continuous. Likewise, a similar argument shows that it is cocontinuous. Hence,

4.2. Theorem. *Let (τ_0, κ_0) be a fixed ditopology on $(N, \mathcal{N}, \nu)$. Then we may specialize $\mathfrak{F}$ to a functor $\mathfrak{F}_{(\tau_0,\kappa_0)}$:* **Bitop** $\to$ **Dbl-Ditop** *by setting $\mathfrak{F}_{(\tau_0,\kappa_0)}(X, u, v) = (D_X, \mathcal{D}_X, \delta_X, \tau_{(u,v)}, \kappa_{(u,v)})$ and $\mathfrak{F}_{(\tau_0,\kappa_0)}(\psi) = \langle \psi, \iota \rangle$ for any bitopology (u, v) on X and $\psi \in$ Mor***Bitop**.

In the special case where (τ_0, κ_0) is complemented and the bitopology (u, v) has the form $(\mathcal{T}, \mathcal{T})$ for some topology $\mathcal{T}$ on X it is clear that the corresponding ditopology $(\tau_{(u,v)}, \kappa_{(u,v)})$, which we will denote by $(\tau_{\mathcal{T}}, \kappa_{\mathcal{T}})$, is a complemented ditopology on $(D_X, \mathcal{D}_X, \delta_X)$. Hence,

4.3. Corollary. *Let (τ_0, κ_0) be a fixed complemented ditopology on $(N, \mathcal{N}, \nu)$. Then we may specialize $\mathfrak{F}$ to a functor $\mathfrak{F}_{(\tau_0,\kappa_0)}$:* **Top** $\to$ **Dbl-CDitop** *by setting $\mathfrak{F}_{(\tau_0,\kappa_0)}(X, \mathcal{T}) = (D_X, \mathcal{D}_X, \delta_X, \tau_{\mathcal{T}}, \kappa_{\mathcal{T}})$ for any topology $\mathcal{T}$ on X and $\psi \in$ Mor***Bitop**.

We note in passing that there are just two complemented ditopologies on $(N, \mathcal{N}, \nu)$, namely the discrete, codiscrete ditopology $(\mathcal{N}, \mathcal{N})$ and the indiscrete, co-indiscrete ditopology $(\{\emptyset, N\}, \{\emptyset, N\})$. Hence Corollary 4.3 gives two functors from **Top** to **Dbl-CDitop**. There are also two non-complemented ditopologies on $(N, \mathcal{N}, \nu)$, namely the one for which the set $\{\frac{1}{2}\}$ is open but not closed, and the one for which this set is closed but not open. Hence Theorem 4.2 gives a total of four functors from **Bitop** to **Dbl-Ditop**.

In order to specialize the functor $\mathfrak{G}$ in a similar way let (τ, κ) be a ditopology on $(D_X, \mathcal{D}_X, \delta_X) = (X \times N, \mathcal{P}(X) \otimes \mathcal{N}, \pi_X \otimes \nu)$ and define

$$\tau^1 = \{G \in \mathcal{P}(X) \mid G \times N \in \tau\}, \qquad \tau^2 = \{G \in \mathcal{N} \mid X \times G \in \tau\},$$
$$\kappa^1 = \{K \in \mathcal{P}(X) \mid K \times N \in \kappa\}, \qquad \kappa^2 = \{K \in \mathcal{N} \mid X \times K \in \kappa\}.$$

Clearly (τ^1, κ^1) is a ditopology on $(X, \mathcal{P}(X), \pi_X)$ and (τ^2, κ^2) a ditopology on $(N, \mathcal{N}, \nu)$ whose product is clearly coarser than (τ, κ). The following gives necessary and sufficient conditions for equality.

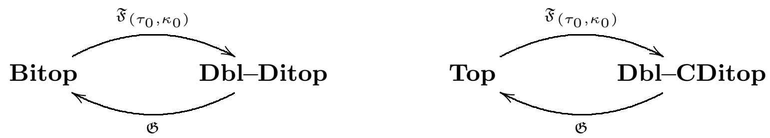
4.4. Lemma. *The following are equivalent:*

(1) *The product of (τ^1, κ^1) and (τ^2, κ^2) coincides with (τ, κ).*
(2) *The following conditions hold:*
  a) *Given $H \in \tau$, $(x, k) \in H$, there exist $G_1 \in \tau^1$, $G_2 \in \tau^2$ so that $(x, k) \in (G_1 \times N) \cap (X \times G_2) \subseteq H$.*
  b) *Given $K \in \kappa$, $(x, k) \notin K$, there exist $K_1 \in \kappa^1$, $K_2 \in \kappa^2$ so that $K \subseteq (K_1 \times N) \cup (X \times K_2)$ and $(x, k) \notin (K_1 \times N) \cup (X \times K_2)$.*

Proof. Immediate by the definition of product ditopology and the fact that we are dealing with a plain texture. □

If we note that $(\tau^1, (\kappa^1)^c)$ is a bitopology on X and that the bicontinuity of a $\mathbf{fPSTex_{dbl}}$ morphism φ trivially implies the bicontinuity of φ_1 we may clearly specialize

$\mathfrak{G}$ to a functor $\mathfrak{G} : \mathbf{fPSDitop_{dbl}} \to \mathbf{Bitop}$. However, since there will again be no chance of an adjoint situation with any of the functors $\mathfrak{F}_{(\tau_0,\kappa_0)}$ we will regard $\mathfrak{G}$ as a functor from **Dbl-Ditop** to **Bitop**. In case (τ, κ) is a complemented ditopology then (τ^1, κ^1), (τ^2, κ^2) are also complemented, and in particular we may regard (τ^1, κ^1) as representing the topology τ^1 with family of closed sets κ^1. Hence, we may also regard $\mathfrak{G}$ as a functor from **Dbl-CDitop** to **Top**.



For the left-hand diagram (τ_0, κ_0) is arbitrary and for the right-hand diagram it is a complemented ditopology.

Now we may state:

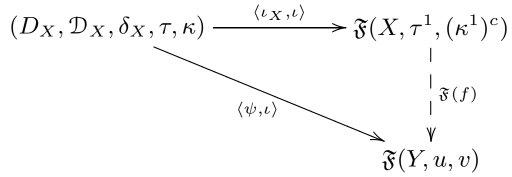
4.5. Theorem. *If* (τ_0, κ_0) *is taken to be the indiscrete, co-indiscrete ditopology on* $(N, \mathcal{N}, \nu)$ *then* $\mathfrak{F} = \mathfrak{F}_{(\tau_0,\kappa_0)}$ *is an adjoint of* $\mathfrak{G}$.

Proof. We give the proof for $\mathfrak{F}$, $\mathfrak{G}$ as in the left-hand diagram, the complemented case being essentially a special case of this.

Take $(D_X, \mathcal{D}_X, \delta_X, \tau, \kappa) \in$ **Dbl-Ditop** and consider $(\langle \iota_X, \iota\rangle, (X, \tau^1, (\kappa^1)^c))$, where ι_X denotes the identity function on X. Since for $\tau_0 = \{\emptyset, N\}$ we clearly have $\tau^1 \otimes \tau_0 \subseteq \tau$ we see that

$$(D_X, \mathcal{D}_X, \delta_X, \tau, \kappa) \xrightarrow{\langle \iota_X, \iota\rangle} (D_X, \mathcal{D}_X, \delta_X, \tau^1 \otimes \tau_0, \kappa^1 \otimes \kappa_0) = \mathfrak{F}(X, \tau^1, (\kappa^1)^c)$$

is continuous, and likewise it is cocontinuous. Hence $(\langle \iota_X, \iota\rangle, (X, \tau^1, (\kappa^1)^c))$ is a structured $\mathfrak{F}$ arrow with domain $(D_X, \mathcal{D}_X, \delta_X, \tau, \kappa)$. Since $(X, \tau^1, (\kappa^1)^c) = \mathfrak{G}(D_X, \mathcal{D}_X, \delta_X, \tau, \kappa)$ it remains to show that the structured arrow above has the universal property [1]. To this end let $(\langle \psi, \iota\rangle, (Y, u, v))$ be any structured $\mathfrak{F}$ arrow with domain $(D_X, \mathcal{D}_X, \delta_X, \tau, \kappa)$. We must prove the existence of a unique **Bitop** morphism $f : (X, \tau^1, (\kappa^1)^c) \to (Y, u, v)$ making the following diagram commutative.



Since $\mathfrak{F}(f) = \langle f, \iota\rangle$ we see that the only possible choice for f is ψ, so it remains only to prove that $\psi : (X, \tau^1, (\kappa^1)^c) \to (Y, u, v)$ is bicontinuous. Take $U \in u$. Then $U \times N \in u \otimes \tau_0$, so $\langle \psi, \iota\rangle^{-1}[U \times N] \in \tau$ by the continuity of $\langle \psi, \iota\rangle$. But $\langle \psi, \iota\rangle^{-1}[U \times N] = \psi^{-1}[U] \times N$ so $\psi^{-1}[U] \in \tau^1$ and we see that ψ is τ^1–u continuous. A similar argument shows it is $(\kappa^1)^c$–v continuous. □

The following will be useful when working directly in terms of **3-Top**.

4.6. Lemma. *Let* T *be a* **3***–topology on* X*, and* (τ_T, κ_T) *the corresponding complemented ditopology on* $(D_X, \mathcal{D}_X, \delta_X)$*. Then:*

(1) $\tau_T^1 = \{G \subseteq X \mid \chi_G \in T\}$, $\tau_T^2 = \begin{cases} \mathcal{N} & \text{if } \frac{1}{2} \in T, \\ \{\emptyset, N\} & \text{if } \frac{1}{2} \notin T. \end{cases}$

(2) *The following are equivalent:*

(a) $\tau_T = \tau_T^1 \otimes \tau_T^2$.
(b) *For* $g \in T$ *and* $x \in X$ *with* $g(x) > 0$ *there exists* $Y \subseteq X$ *with* $x \in Y$ *and* $\chi_Y \in T$ *so that* $\chi_Y \leq g$ *if* $g(x) = 1$*, otherwise* $\frac{\mathbf{1}}{\mathbf{2}} \in T$ *and* $\chi_Y \wedge \frac{\mathbf{1}}{\mathbf{2}} \leq g$.
(c) *There exists a subbase* B *of* T *so that the above condition holds for all* $g \in B$ *and* $x \in X$ *with* $g(x) > 0$.
(d) $(X, T) = \mathfrak{F}_{(\tau_T^2, \kappa_T^2)}(\mathfrak{G}(X, T))$.

Proof. (1) The first equality follows from $\widehat{\chi_G} = G \times N$, and the second is clear.

(2) (a) $\implies$ (b). Suppose (a) holds and take $g \in T$, $x \in X$, with $g(x) > 0$. Then $(x, g(x)) \in \widehat{g} \in \tau_T$, so by Lemma 4.4 there exist $Y \in \tau_T^1$ and $Z \in \tau_T^2$ with $(x, g(x)) \in Y \times Z \subseteq \widehat{g}$. Hence $x \in Y$ and $\chi_Y \in T$ by (1). If $g(x) = 1$ then $1 \in Z$ so $Z = N$ and $Y \times N \subseteq \widehat{g}$ gives $\chi_Y \leq g$. Otherwise, $g(x) = \frac{1}{2}$ and we deduce from $Y \times Z \subseteq \widehat{g}$ that $1 \notin Z$. Since $g(x) \in Z$ we have $Z \neq \emptyset$ so $Z = \{\frac{1}{2}\}$. Hence $\frac{\mathbf{1}}{\mathbf{2}} \in T$, and $Y \times Z \subseteq \widehat{g}$ gives $\chi_Y \wedge \frac{\mathbf{1}}{\mathbf{2}} \leq g$.

(b) $\implies$ (c). Immediate.

(c) $\implies$ (a). Assume (c) holds and take $(x, k) \in H \in \tau_T$. Then we have $h \in T$ with $H = \widehat{h}$, and so $(x, k) \in \widehat{h}$. If B is the subbase of T mentioned in (c) we may choose $g_1, g_2, \dots, g_n \in B$ with $\bigwedge_{i=}^n g_i \leq h$ and $k \leq \bigwedge_{i=1}^n g_i(x)$ since **3** is finite. Now $k \in N$ implies $g_i(x) > 0$ for $i = 1, 2, \dots, n$, so there exist $x \in Y_i \subseteq X$ with $\chi_{Y_i} \in T$ satisfying the conditions given in (b). If we let $G_1 = \bigcap_{i=1}^n Y_i$, $G_2 = N$ if $g_i(x) = 1$ for all $i = 1, 2, \dots, n$ and $G_2 = \{\frac{1}{2}\}$ otherwise, it is straightforward to verify the condition $(x, k) \in G_1 \times G_2 \subseteq H$ of Lemma 4.4 (2a).

(a) $\iff$ (d). Immediate from the definitions. □

We will refer to a **3**–topology satisfying the equivalent conditions of Lemma 4.6 (2) as *productive.* Note that equivalent dual conditions may be written down for the cotopologies, and we omit the details.

In view of the isomorphism $\mathfrak{Z}$: **Dbl-Top** $\to$ **3-Top** given in Theorem 4.1, we may express Lemma 4.6 in terms of a double-set topology T on X as below. The proof is straightforward, and is omitted.

4.7. Corollary. *Let* T *be a double-set topology on* X*, and* (τ_T, κ_T) *the corresponding complemented ditopology on* $(D_X, \mathcal{D}_X, \delta_X)$*. Then:*

(1) $\tau_T^1 = \{G \subseteq X \mid (G, X \setminus G) \in T\}$, $\tau_T^2 = \begin{cases} \mathcal{N} & \text{if } (\emptyset, \emptyset) \in T, \\ \{\emptyset, N\} & \text{if } (\emptyset, \emptyset) \notin T. \end{cases}$

(2) *The following are equivalent:*
(a) $\tau_T = \tau_T^1 \otimes \tau_T^2$.
(b) *For* $(A, B) \in T$ *and* $x \in X$ *with* $x \notin B$ *there exists* $Y \subseteq X$ *with* $x \in Y$*,* $B \cap Y = \emptyset$ *and* $(Y, X \setminus Y) \in T$ *so that* $Y \subseteq A$ *if* $x \in A$ *and* $(\emptyset, \emptyset) \in T$ *if* $x \notin A$.
(c) *There exists a subbase* B *of* T *so that the above condition holds for all* $(A, B) \in B$ *and* $x \in X$ *with* $x \notin B$.
(d) $(X, T) = \mathfrak{F}_{(\tau_T^2, \kappa_T^2)}(\mathfrak{G}(X, T))$.

Again, a double-set topology satisfying the equivalent conditions of Corollary 4.7 (2) will be called *productive.*

On analogy with the classical case, the mapping which takes a topology $\mathcal{T}$ on X to the **3**-topology on X with subbase $\{g \in \mathbf{3}^X \mid \{x \mid k < g(x)\} \in \mathcal{T}\ \forall k \in \mathbf{3}\}$ is called the Lowen mapping ω [20], while the Lowen mapping ι takes a **3**-topology T to the topology on X with subbase $\{\{x \mid k < g(x)\} \mid k \in \mathbf{3},\ g \in T\}$. For simplicity of notation we will also denote by ω, ι the corresponding functors $\omega : \mathbf{Top} \to \mathbf{3\text{-}Top}$ and $\iota : \mathbf{3\text{-}Top} \to \mathbf{Top}$. Naturally, in view of the isomorphism between **3-Top** and **Dbl-Top**, these functors also give rise to functors between **Top** and **Dbl-Top**.

4.8. Theorem. (1) *For any topology $\mathcal{T}$ on X, the $\mathbf{3}$-topology $\omega(\mathcal{T})$ is productive and contains all the constant $\mathbf{3}$-sets.*

(2) *$\mathfrak{T} \circ \omega$ coincides with the functor $\mathfrak{F}_{(\mathcal{N},\mathcal{N})} : \mathbf{Top} \to \mathbf{3\text{-}Ditop}$ corresponding to the discrete, codiscrete ditopology on $(N, \mathcal{N}, \nu)$.*

Proof. (1). It is clear from the definition that $\omega(\mathcal{T})$ contains all the constant $\mathbf{3}$-sets. A direct proof of productivity may be obtained by verifying Lemma 4.6 (2c) for the given subbase B of $\omega(\mathcal{T})$. The details are left to the interested reader. Alternatively, since $\mathbf{3}$ is a chain, the productivity is also a trivial consequence of [20, Theorem 3.3] which says that $\{\chi_G \mid G \in \mathcal{T}\} \cup \{\mathbf{k} \mid k \in \mathbf{3}\}$ is also a subbase of $\omega(\mathcal{T})$.

(2). Immediate from (1). □

4.9. Theorem. *The functor $\mathfrak{G} \circ \mathfrak{T} : \mathbf{3\text{-}Top} \to \mathbf{Top}$ assigns to a $\mathbf{3}$-topology T on X a (possibly strictly) weaker topology than does the Lowen functor ι. In case T is topological these topologies coincide.*

Proof. The proof of $\mathfrak{G}(\mathfrak{T}(T)) \subseteq \iota(T)$ is essentially the same as for the classical case [10, Theorem 5.15], and we omit the details. Likewise, taking $X = \{a, b\}$ and letting $T = \{\mathbf{0}, \chi_{\{a\}} \wedge \frac{\mathbf{1}}{\mathbf{2}}, \mathbf{1}\}$ as before leads to $\tau_T = \{\emptyset, \{a\} \times \{\frac{1}{2}\}, D_X\}$, whence $\mathfrak{G}(\mathfrak{T}(T)) = \tau_T^1 = \{\emptyset, X\}$, whereas $\iota(T) = \{\emptyset, \{a\}, X\}$. Finally, the proof of equality when T is topological is formally the same as in [10, Theorem 5.15] and we again omit the details. □

It is well known [20] that ι is an adjoint of ω, but just as in the classical case the example given in the proof of Theorem 4.9 may be used to show that $\mathfrak{G}$ is not an adjoint of $\mathfrak{F}_{(\mathcal{N},\mathcal{N})}$.

Finally we note that, exactly as for the classical case:

4.10. Theorem. *Let T be a $\mathbf{3}$-topology on X.*

(1) *If T is productive it is topological, but not conversely.*
(2) *T is topological if and only if $G_k(T)$ is productive.*

Proof. (1). The proof that productive implies topological follows the same lines as the proof of [10, Theorem 5.16], but using Lemma 4.6 (2c) in place of [10, Lemma 5.13 (2ii)]. Likewise, the $\mathbf{3}$-topology $T = \{\mathbf{0}, \chi_{\{a\}} \wedge \frac{\mathbf{1}}{\mathbf{2}}, \chi_{\{a\}}, \mathbf{1}\}$ is easily seen to be topological but not productive.

(2). If $G_k(T)$ is productive it is topological by (1), hence T is topological also. Conversely, if T is topological then $G_k(T)$ is topological so from the equality $\omega(\iota(G_k(T))) = G_k(T)$ and the fact that $\mathfrak{T}$ is an isomorphism we find $\mathfrak{F}_{(\mathcal{N},\mathcal{N})}(\mathfrak{G}(X, G_k(T))) = (X, G_k(T))$. However $G_k(T)$ contains all the constant $\mathbf{3}$-sets so $(\tau^2_{G_k(T)}, \kappa^2_{G_k(T)}) = (\mathcal{N}, \mathcal{N})$ and we deduce from Lemma 4.6 (2d) that $G_k(T)$ is productive. □

If we call a double-set topology T *topological* if the corresponding $\mathbf{3}$-topology is topological, and denote by $G_k(T)$ the double set topology generated by T and $(\emptyset, \emptyset)$, then we may state:

4.11. Corollary. *Let T be a double-set topology on X.*

(1) *If T is productive it is topological, but not conversely.*
(2) *T is topological if and only if $G_k(T)$ is productive.*

Clearly, the family of functors $\mathfrak{F}$ continues to play as important a role in the study of $\mathbf{3}$-topologies and double-set topologies as it does for classical $\mathbb{I}$-topologies.

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