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Modified systematic sampling in the presence of linear trend

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Abstract

A new systematic sampling design called "Modified Systematic Sampling (MSS)", proposed by [2] is more general than Linear Systematic Sampling (LSS) and Circular Systematic Sampling (CSS). In the present paper, this scheme is further extended for populations having a linear trend. Expressions for mean and variance of sample mean are obtained for the population having perfect linear trend among population values. Expression for the average variance is also obtained for super population model. Further, efficiency of MSS with respect to CSS is obtained for different sample size.

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1. Introduction

In survey sampling, Linear Systematic Sampling (LSS) is a commonly used design. Generally, it is useful when population size N is a multiple of sample size n, i.e. $N = nk$ (where k is the sampling interval). Thus, we have k samples each of size n. However, LSS is not beneficial when population size N is not a multiple of the sample size n, i.e. $N \neq nk$. Because in this case, LSS cannot provide a constant sample size n, thus, estimate of population mean (total) is biased. Therefore, Circular Systematic Sampling (CSS) was introduced by Lahiri in 1952 (cited in [1, p.139]). Contrary to LSS, CSS is not advantageous when population size N is a multiple of the sample size n, i.e. $N = nk$ as in this case, CSS produces n replicates of k samples. Further, in CSS, the number of samples also rapidly increase to N as compared to k samples of LSS.

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To improve the efficiency of systematic sampling, researchers proposed several modifications in the selection procedure. The considerable work is done by [4], [6] and [7]. In the recent years, [8] proposed Diagonal Systematic Sampling (DSS) under the condition that $n \leq k$ as a competitor of LSS. Later, the condition $n \leq k$ for DSS have relaxed by [9]. The generalization of DSS is suggested by [10]. Some modification in LSS are proposed by [11], in which odd and even sample sizes are dealt separately. Further modification on LSS is also proposed by [12]. Diagonal Circular Systematic Sampling (DCSS) proposed by [5] is an extension of DSS to the circular version of systematic sampling. A note on DCSS has been proposed by [3]. However, some of these schemes are applicable when $N = nk$ while other can be used only when $N \neq nk$.

A new systematic sampling design called "Modified Systematic Sampling (MSS)" proposed by [2], which is applicable in both situations, whether $N = nk$ or $N \neq nk$. According to this design, first compute least common multiple of N and n, i.e. L, then find k_1, m, s and k, where $k_1 = \frac{L}{n}$, $m = \frac{L}{N}$, $s = \frac{N}{k_1}$ and $k = [k_1/m]$ or $k = [N/n]$ is rounded off to integer. Consequently, $ms = n$, which means that there are m sets and each set contains s units in a sample. Thus, in MSS the j^{th} unit of the i^{th} set of a sample of n units can be written as:

$$y_{ij}^{(r)} = r + (i-1)k + (j-1)k_1 - hN \text{ if } hN < r + (i-1)k + (j-1)k_1 \leq (h+1)N \tag{1.1}$$

for $h = 0, 1, 2;\ i = 1, 2, 3, ..., m$ and $j = 1, 2, 3, ..., s$.

This sampling scheme reduces to LSS if $L = N$ or $N = nk$ and CSS if $L = N \times n$, the detail is given below.

If $N = nk$ then $L = N$, $k_1 = k$, $m = 1$ and $s = n$. Thus, Equation (1.1) reduces to

$$y_j^{(r)} = r + (j-1)k, \quad j = 1, 2, 3, ..., n \tag{1.2}$$

which is LSS.

Similarly, if $L = N \times n$, then $k_1 = N$, $m = n$ and $s = 1$. So, Equation (1.1) can be written as

$$y_i^{(r)} = r + (i-1)k - hN \quad \text{if} \quad hN < r + (i-1)k \leq (h+1)N \tag{1.3}$$

for $h = 0, 1, 2;\ i = 1, 2, 3, ..., n$.

Which is CSS.

To study the characteristics of MSS, we use an alternative method by partitioning the total number of samples into different sets of similar samples. To develop an alternative method, let us assume that k_1 can be written as $k_1 = qk + r_m$, where q and r_m are quotient and remainder respectively. Further, we assume that $w = 1$ if $(m - q) \leq 1$ and, $w = (m - q)$ if $(m - q) > 1$. In both cases, there are two types of partitioning, i.e. between samples and within samples(see detail in Subsections 1.1 and 1.2).

1.1. When $w = 1$. In this case partitioning between samples and within samples are given in the Subsections 1.1.1 and 1.1.2.

1.1.1. *Partitioning between samples.* In this case, k_1 possible samples are mainly partitioned into two groups. The first group consists of initial $\{k_1 - (m-1)k\}$ samples and second group contains last $(m-1)k$ samples. However, in the second group, there are $(m-1)$ subgroups, each attains k samples. If a random number r is selected from the first k_1 units of a population, there is a possibility that it is selected from the first group, i.e. $\{k_1 - (m-1)k\}$ or it is selected from the $(m-1)$ subgroups of the second group, i.e. $\{k_1 - (m-u)k\} < r \leq \{k_1 - (m-u-1)k\}$ such that $u = 1, 2, ..., (m-1)$, where integer u is selected corresponding to a random number r.

1.1.2. *Partitioning within samples.* Furthermore, in the first group, all s units of all m sets in each sample are labeled as $r + (i-1)k + (j-1)k_1$, such that $i = 1, 2, ..., m$ and $j = 1, 2, ..., s$; while in the second group, all s units of the first $(m-u)$ sets are labeled as $r + (i-1)k + (j-1)k_1$ such that $i = 1, 2, ..., (m-u)$ and $j = 1, 2, ..., s$ and in each of the last u sets, first $(s-1)$ units are labeled as $r + (i-1)k + (j-1)k_1$ such that $i = (m-u+1), ..., m$; $j = 1, 2, ..., (s-1)$ and last unit is labeled as $r+(i-1)k+(j-1)k_1-N$ such that $i = (m-u+1), ..., m$ and $j = s$.

1.2. When $w = (n-q) > 1$**.** In this case partitioning between samples and within samples are given in the Subsections 1.2.1 and 1.2.2.

1.2.1. *Partitioning between samples.* In this case k_1 samples are mainly partitioned into two groups, the first group consists of the number of samples in which $r \leq \{k_1 - (w-1)k + r_m\}$. The econd group contains the number of samples in which $r > \{k_1 - (w-1)k + r_m\}$. The first group is further partitioned into $\{(m-w) - (w-1) + 2\}$ subgroups in which, there are r_m number of samples in each of the first and the last subgroups, and k samples in each of the middle $\{(m-w) - (w-1)\}$ subgroups. In each subgroup of the first group, corresponding to a random number r, an integer u is picked in such a way that $u = (w-1)$ if $1 \leq r \leq \{k_1 - (m-u-1)k\}$, $u = w, w+1, w+2, ..., (m-w)$ if $\{k_1 - (m-u)k\} < r \leq \{k_1 - (m-u-1)k\}$ and $u = (m-w+1)$ if $\{k_1 - (m-u)k\} < r \leq \{k_1 - (m-u)k + r_m\}$.
The second group consists of last $\{(w-1)k - r_m\} = \{(k-r_m) + (w-2)k\}$ samples, which is the combination of the first $(k-r_m)$ samples and the last $(w-2)$ sets of k samples. These $(w-2)$ sets of k samples further partitioned in such a way that the first r_m of every k samples forms the first subgroup and the last $(k-r_m)$ samples of every k samples together with the first $(k-r_m)$ samples of this group forms the second subgroup. However, when $w = 2$, then we have only $(k-r_m)$ samples in the second group.

1.2.2. *Partitioning within samples.* In each sample of the first group, all s units of the first $(m-u)$ sets are labeled as $r + (i-1)k + (j-1)k_1$ such that $i = 1, 2, ..., (m-u)$ and $j = 1, 2, ..., s$, and in each of the last u sets, the first $(s-1)$ units are labeled as $r + (i-1)k + (j-1)k_1$ such that $i = (m-u+1), ..., m$, $j = 1, 2, ..., (s-1)$ and the last unit is labeled as $r + (i-1)k + (j-1)k_1 - N$ such that $i = (m-u+1), ..., m$ and $j = s$.

In each sample of the first subgroup of the second group, all s units of the first $(w-x)$ sets are labeled as $r + (i-1)k + (j-1)k_1$ such that $i = 1, 2, ..., (w-x)$ and $j = 1, 2, ..., s$; the units of middle $(m-w+1)$ sets are labeled in such a way that, the first $(s-1)$ units of each set are labeled as $r + (i-1)k + (j-1)k_1$ such that $i \in (w-x+1), ..., (m-x+1)$, $j = 1, 2, ..., (s-1)$ and the last unit of each set is labeled as $r + (i-1)k + (j-1)k_1 - N$ such that $i \in (w-x+1), ..., (m-x+1)$ and $j = s$; the units of the last $(x-1)$ sets are labeled in such a way that, the first $(s-2)$ units are labeled as $r+(i-1)k+(j-1)k_1$ such that $i \in (m-x+2), ..., m$, $j = 1, 2, ..., (s-2)$ and the last two units in each set is labeled as $r+(i-1)k+(j-1)k_1-N$ such that $i \in (m-x+2), ..., m$ and $j = (s-1), s$. However, when $s = 1$, the units in these $(x-1)$ sets are labeled as $r + (i-1)k + (j-1)k_1 - 2N$. The possible values of x are $2, 3, ..., (w-1)$.
Note: If $w = 2$, then this set of samples does not exist.
In the second subgroup of the second group, all s units of the the first $(w-x)$ sets are labeled as $r + (i-1)k + (j-1)k_1$ such that $i = 1, 2, ..., (w-x)$ and $j = 1, 2, ..., s$; The units of middle $(m-w)$ sets are labeled in such a way that the first $(s-1)$ units of each set are labeled as $r + (i-1)k + (j-1)k_1$ such that $i \in (w-x+1), ..., (m-x)$ and $j = 1, 2, ..., (s-1)$, the last unit of each set is labeled as $r + (i-1)k + (j-1)k_1 - N$ such that $i \in (w-x+1), ..., (m-x)$ and $j = s$, the units of the last $(x-1)$ sets are labeled

in such a way that, the first $(s-2)$ units are labeled as $r+(i-1)k+(j-1)k_1$ such that $i \in (m-x+2),...,m$, $j=1,2,...,(s-2)$ and the last two units in each set is labeled as $r+(i-1)k+(j-1)k_1-N$ such that $i \in (m-x+2),...,m$ and $j=(s-1),s$. However, when $s=1$, the units in these $(x-1)$ sets are labeled as $r+(i-1)k+(j-1)k_1-2N$, the possible values of x are $1,2...,(w-1)$.

2. Mean and variance of MSS for population having linear trend

The following linear model of hypothetical population is to be considered as

$$Y_t = \alpha + \beta\, t, \qquad t = 1,2,3,...,N \tag{2.1}$$

where α and β are the intercept and slope of the model respectively.

2.1. Mean of MSS. The sample mean for both cases, i.e. $w=1$ and $w>1$ are given below (see detial in Appendix A.1).

Case (i) when $w=1$

$$\bar{y}_{MSS} = \alpha + \beta \begin{cases} \left[r + \frac{1}{2}\{(s-1)k_1 + (m-1)k\}\right], \\ \qquad \text{if} \quad r \le \{k_1 - (m-1)k\} \\ \left[r + \frac{1}{2}\{(s-1)k_1 + (m-1)k\} - u\frac{k_1}{m}\right], \\ \qquad \text{where} \left| \begin{array}{l} u = 1,2,...,(m-1) \text{ if} \\ \{k_1-(m-u)k\} < r \le \{k_1-(m-u-1)k\}. \end{array}\right. \end{cases} \tag{2.2}$$

Case (ii) when $w>2$

$$\bar{y}_{MSS} = \alpha + \beta \begin{cases} \left[r + \frac{1}{2}\{(s-1)k_1 + (m-1)k\} - u\frac{N}{n}\right], \\ \qquad \text{where} \left| \begin{array}{l} u = (w-1) \; if \quad r \le \{k_1-(m-u-1)k \\ u = w, w+1, ..., (m-w) \; if \\ \{N-(m-u)k\} < r \le \{k_1-(m-u-1)k\}, \\ u = (m-w+1) \; if \\ \{k_1-(m-m)k\} < r \le \{k_1-(m-u)k + r_m\} \end{array}\right. \\ \left[r + \frac{1}{2}\{(s-1)k_1 + (m-1)k\} - (m-w-1+2x)\frac{N}{n}\right], \\ \qquad \text{where} \left| \begin{array}{l} x = 2,...,(w-1) \; if \\ \{k_1-(w-x)k\} < r \le \{k_1-(w-x)k + r_m\} \end{array}\right. \\ \left[r + \frac{1}{2}\{(s-1)k_1 + (m-1)k\} - (m-w+2x)\frac{N}{n}\right], \\ \qquad \text{where} \left| \begin{array}{l} x = 1,2,3,...,(w-1) \; if \\ \{k_1-(w-x)k\} + r_m < r \le \{k_1-(w-x)k + k\} \end{array}\right. \end{cases} \tag{2.3}$$

If $w=2$, the Equation (2.3) will reduce to

$$\bar{y}_{MSS} = \alpha + \beta \begin{cases} \left[r + \frac{1}{2}\{(s-1)k_1 + (m-1)k\} - u\frac{N}{n}\right] \\ \qquad \text{where} \left| \begin{array}{l} u = (w-1) \; if \quad r \le \{k_1-(m-u-1)k, \\ u = w, w+1, ..., (m-w) \; if \\ \{N-(m-u)k\} < r \le \{k_1-(m-u-1)k\}, \\ u = (m-w+1) \; if \\ \{k_1-(m-m)k\} < r \le \{k_1-(m-u)k + r_m\} \end{array}\right. \\ \left[r + \frac{1}{2}\{(s-1)k_1 + (m-1)k\} - (m-w+2x)\frac{N}{n}\right], \\ \qquad \text{where} \left| \begin{array}{l} x = 1,2,3,...,(w-1) \; if \\ \{k_1-(w-x)k\} + r_m < r \le \{k_1-(w-x)k + k\} \end{array}\right. \end{cases} \tag{2.4}$$

2.1.1. *Unbiasedness of* $\bar{y}_{MSS}$. (see detial in Appendix A.1.1).
The sample mean ($\bar{y}_{MSS}$) is an unbiased estimator of population mean ($\bar{Y}$)

$$E(\bar{y}_{MSS}) = \alpha + \beta\left\{\frac{N+1}{2}\right\} = \bar{Y}. \tag{2.5}$$

2.1.2. *Variance of* $\bar{y}_{MSS}$. (see detial in Appendix A.2)

(i) when $w = 1$

$$V(\bar{y}_{MSS}) = \frac{1}{12m^2}b^2\left[m^2(k_1^2-1) + m(m^2-1)k(mk-2k_1)\right]. \tag{2.6}$$

Note: In this case, if $N = nk$ then $L = N$, so $m = 1$, thus,

$$V(\bar{y}_{MSS}) = \frac{1}{12}b^2(k^2-1).$$

This is a variance of linear systematic sampling.

(ii) when $w > 1$

$$\begin{aligned}(\bar{y}_{MSS}) = &\tfrac{1}{12m^2}b^2\left[m^2(k_1^2-1) + m(m^2-1)k(mk-2k_1)\right.\\ &\left.+4w(w-1)k_1\left\{3k_1-(3m-2w+1)k\right\}\right].\end{aligned} \tag{2.7}$$

3. Yates corrected estimator

Yates corrected estimator of population mean for MSS is derived below.

3.1. Yates corrected estimator for MSS. The corrected estimator $\bar{y}^c_{MSS}$ of population mean using MSS is given by

$$\bar{y}^c_{MSS} = \tfrac{1}{n}\left[\lambda_{1l}Y_{r1} + \textstyle\sum_{l=2}^{n-1}Y_{rl} + \lambda_{2l}Y_{rn}\right], \tag{3.1}$$

where λ_{1l} and λ_{2l} are selected so that sample mean coincides with the population mean in the presence of linear trend for all choices of $r \in 1, 2, ..., k_1$.
Alternatively Equation (3.1) can be written as

$$\bar{y}^c_{MSS} = \bar{y}_{MSS} + a_l(r)\left(Y_{r1} - Y_{rn}\right), \tag{3.2}$$

where $a_l(r) = \frac{\lambda_{1l}-1}{n} = \frac{1-\lambda_{2l}}{n}$.
Under the model given in (2.1), the population mean is

$$\bar{Y} = \alpha + \beta\frac{N+1}{2}. \tag{3.3}$$

As mentioned earlier, that there are two cases, i.e. (i) $w = 1$ and (ii) $w > 1$.
First, we consider the Case (i).

3.1.1. *Case (i): when* $w = 1$. In this situation, a random start r is selected from k_1 units such that $r \leq \{k_1-(m-1)k\}$ or $r > \{k_1-(m-1)k\}$.
If $r \leq \{k_1-(m-1)k\}$, then $l = 1$ and the last value of each sample is labeled $\{r+(m-1)k+(s-1)k_1\}$. Thus, (3.2) becomes

$$\bar{y}^c_{MSS} = \bar{y}_{MSS} + a_1(r)\left(Y_r - Y_{r+(m-1)k+(s-1)k_1}\right). \tag{3.4}$$

Under the linear model given in (2.1), we have $\bar{y}_{MSS} = \alpha+\beta\left[r+\frac{1}{2}\{(s-1)k_1+(m-1)k\}\right]$, $Y_r = \alpha+\beta\, r$ and $Y_{r+(m-1)k+(s-1)k_1} = \alpha+\beta\{r+(m-1)k+(s-1)k_1\}$.

Putting these values in (3.4), we have

$$(3.5)\qquad \bar{y}^c_{MSS} = \alpha + \beta \left[r + \frac{1}{2}\{(m-1)k + (s-1)k_1\} - a_1(r)\{(m-1)k + (s-1)k_1\} \right].$$

Comparing the coefficients of α and β in (3.3) and (3.5) and solving for $a_1(r)$, we have

$$a_1(r) = \left\{ \frac{2r - 1 + (m-1)k - k_1}{2\{(m-1)k + (s-1)k_1\}} \right\}.$$

Putting $a_1(r)$ in (3.4), we have

$$(3.6)\qquad \bar{y}^c_{MSS} = \bar{y}_{MSS} + \frac{2r - 1 + (m-1)k - k_1}{2\{(m-1)k + (s-1)k_1\}} \left(Y_r - Y_{r+(m-1)k+(s-1)k_1} \right).$$

If $r > \{k_1 - (m-1)k\}$, then $l = 2$ and the last value of each sample is labeled $\{r + (m-1)k + (s-1)k_1 - N\}$. Thus, (3.2) becomes

$$(3.7)\qquad \bar{y}^c_{MSS} = \bar{y}_{MSS} + a_2(r) \left(Y_r - Y_{r+(m-1)k+(s-1)k_1-N} \right).$$

Under the linear model (2.1), we have $\bar{y}_{MSS} = \alpha + \beta \left[r + \frac{1}{2}\{(s-1)k_1 + (m-1)k\} - u\frac{k_1}{m} \right]$, where $u = 1, 2, ..., (m-1)$ if $\{k_1 - (m-u)k\} < r \leq \{k_1 - (m-u-1)k\}$, $Y_r = \alpha + \beta r$ and $Y_{r+(s-1)k_1+(m-1)k-N} = \alpha + \beta\{r + (s-1)k_1 + (m-1)k - N\}$. Putting these values in (3.7), we have

$$(3.8)\qquad \bar{y}^c_{MSS} = \alpha + \beta \left[r + \frac{1}{2}\{(s-1)k_1 + (m-1)k\} - u\frac{k_1}{m} - a_2(r)\{(m-1)k - k_1\} \right].$$

Comparing the coefficients of α and β in (3.3) and (3.8) and solving for $a_2(r)$, we have

$$(3.9)\qquad a_2(r) = \left\{ \frac{2r - (k_1+1) + (m-1)k - 2uk_1/m}{2\{(m-1)k - k_1\}} \right\},$$

where $u = 1, 2, ..., (m-1)$, which are picked corresponding to a random number r such that $\{k_1 - (m-u)k\} < r \leq \{k_1 - (m-u-1)k\}$.
Putting $a_2(r)$ in (3.7), we have

$$(3.10)\qquad \bar{y}^c_{MSS} = \bar{y}_{MSS} + \left\{ \frac{2r - (k_1+1) + (m-1)k - 2uk_1/m}{2\{(m-1)k - k_1\}} \right\} \times \left(Y_r - Y_{r+(m-1)k+(s-1)k_1-N} \right).$$

3.1.2. *Case (ii): when* $w > 1$. As mentioned earlier in Section 1 , when $s = 1$, MSS becomes CSS (see [2]). Therefore, we focus the MSS for $s > 1$. It is also mentioned in Subsection 1.2, all k_1 samples are partitioned into two groups. The first group contains the samples where $r \leq k_1 - (w-1)k + r_m$ and the second group consist of the samples in which $r > k_1 - (w-1)k + r_m$.
The corrected sample mean for each sample in the first group is similar to the corrected sample mean found in Subsection 3.1.1, where $r > k_1 - (m-1)k$, because the pattern of samples in both situations is similar. Further, the weights assigned to the first and the last units of each sample in this group will be similar to the weights given in (3.9), i.e.

$$\bar{y}^c_{MSS} = \bar{y}_{MSS} + a_2(r) \left(Y_r - Y_{r+(m-1)k+(s-1)k_1-N} \right),$$

$$a_2(r) = \left\{ \frac{2r - (k_1+1) + (m-1)k - 2uk_1/n}{2\{(m-1)k - k_1\}} \right\},$$

where $u = (w-1)$ corresponding to a random number r such that $1 \leq r \leq \{k_1 - (m-u-1)k\}$, $u = w, w+1, ..., (m-w)$ if $\{k_1 - (m-u)k\} < r \leq \{k_1 - (m-u-1)k\}$ and

$u = (m-w+1)$ if $\{k_1-(m-u)k\} < r \le \{k_1-(m-u)k+r_m\}$. Thus,

$$(3.11)\qquad y^c_{MSS} = \bar{y}_{MSS} + \left\{\tfrac{2r-(k_1+1)+(m-1)k-2uk_1/n}{2\{(m-1)k-k_1\}}\right\} \times \left(Y_r - Y_{r+(m-1)k+(s-1)k_1-N}\right).$$

The second group having samples in which $r > k_1-(w-1)k+r_m$, and the first subgroup consists of the number of samples in which $\{k_1-(w-x)k\} < r \le \{k_1-(w-x)k+r_m\}$ such that $x = 2, ..., (w-1)$. The Yates corrected estimator with $l = 3$ in (3.2), for the samples of the first subgroup can be written as

$$(3.12)\qquad \bar{y}^c_{MSS} = \bar{y}_{MSS} + a_3(r)\left(Y_r - Y_{r+(m-1)k+(s-1)k_1-N}\right).$$

Under a linear model (2.1) $\bar{y}_{MSS} = \alpha+\beta[r+\frac{1}{2}\{(s-1)k_1+(m-1)k\}-(m-w-1+2x)\frac{k_1}{m}]$, where $x = 2, ..., (w-1)$ if $\{k_1-(w-x)k\} < r \le \{k_1-(w-x)k+r_m\}$, $Y_r = \alpha+\beta\, r$ and $Y_{r+(s-1)k_1+(m-1)k-N} = \alpha+\beta\{r+(s-1)k_1+(m-1)k-N\}$. Putting these values in (3.12), we have

$$(3.13)\qquad \bar{y}^c_{MSS} = \alpha+\beta\Big[r+\tfrac{1}{2}\{(s-1)k_1+(m-1)k\}-(m-w-1+2x)\tfrac{k_1}{m} + a_3(r)\{(m-1)k-k_1\}\Big].$$

Comparing the coefficients of α and β given in (3.3) and (3.13) and solving for $a_3(r)$, we have

$$(3.14)\qquad a_3(r) = \left\{\frac{2r-(k_1+1)+(m-1)k-2k_1(m-w+2x-1)/m}{2\{(m-1)k-k_1\}}\right\}.$$

Putting $a_3(r)$ in the corrected estimator given in (3.12), we have

$$(3.15)\qquad \bar{y}^c_{MSS} = \bar{y}_{MSS} + \left\{\tfrac{2r-(k_1+1)+(m-1)k-2k_1(m-w+2x-1)/m}{2\{(m-1)k-k_1\}}\right\} \times \left(Y_r - Y_{r+(m-1)k+(s-1)k_1-N}\right),$$

where $x = 2, ..., (w-1)$, which are picked corresponding to a random number r such that $\{k_1-(w-x)k\} < r \le \{k_1-(w-x)k+r_m\}$. Similarly, the second subgroup consists of the number of samples in which $\{k_1-(w-x)k+r_m\} < r \le \{k_1-(w-x)k+k\}$ such that $x = 1, 2, ..., (w-1)$. The Yates corrected estimator with $l = 4$ in (3.2), for samples of this subgroup, can be written as

$$(3.16)\qquad \bar{y}^c_{MSS} = \bar{y}_{MSS} + a_4(r)\left(Y_r - Y_{r+(m-1)k+(s-1)k_1-N}\right).$$

Under the linear model (2.1), $\bar{y}_{MSS} = \alpha+\beta[r+\frac{1}{2}\{(s-1)k_1+(m-1)k\}-(m-w+2x)\frac{k_1}{m}]$, where $x = 1, 2, ..., (w-1)$ if $\{k_1-(w-x)k\} < r \le \{k_1-(w-x)k+r_m\}$, $Y_r = \alpha+\beta r$ and $Y_{r+(s-1)k_1+(m-1)k-N} = \alpha+\beta\{r+(s-1)k_1+(m-1)k-N\}$. Putting these values in (3.16), we have

$$(3.17)\qquad \bar{y}^c_{MSS} = \alpha+\beta\Big[r+\tfrac{1}{2}\{(s-1)k_1+(m-1)k\}-(m-w+2x)\tfrac{k_1}{m} + a_4(r)\{(m-1)k-k_1\}\Big].$$

Comparing the coefficients of α and β given in (3.3) and (3.17) and solving for $a_4(r)$, we have

$$(3.18)\qquad a_4(r) = \left\{\frac{2r-(k_1+1)+(m-1)k-2k_1(m-w+2x)/m}{2\{(m-1)k-k_1\}}\right\}.$$

Putting $a_4(r)$ in the corrected estimator given in (3.16), we have

$$y^c_{MSS} = \bar{y}_{MSS} + \left\{\frac{2r-(k_1+1)+(m-1)k-2k_1(m-w+2x)/m}{2\{(m-1)k-k_1\}}\right\} \times \left(Y_r - Y_{r+(m-1)k+(s-1)k_1-N}\right), \tag{3.19}$$

where $x = 1, 2, ..., (w-1)$, which are picked corresponding to a random number r such that $\{k_1 - (w-x)k\} < r \leq \{k_1 - (w-x)k + r_m\}$.

4. Average variance

In real life application, we hardly found such population exhibiting perfect linear trend. Therefore, it is necessary to study the average variance of the corrected estimator under MSS using following super population model.

$$Y_t = \alpha + \beta\, t + e_t, \tag{4.1}$$

where $E(e_t) = 0$, $V(e_t) = E(e_t^2) = \sigma^2 t^g$, $Cov(e_t, e_v) = 0,\ t \neq v = 1, 2, 3, ..., N$ and g is the predetermined constant.
The average variance of $\bar{y}^{(r)}_{MSS}$ under modified systematic sampling for population modeled by $Y_t = \alpha + \beta\, t + e_t$ is given by

Case (i) when $w = 1$ (see detial in Appendix B)

$$\begin{aligned} E\left\{V(\bar{y}^{(r)}_{MSS})\right\} = \frac{\sigma^2}{k_1}\Bigg\{&\sum_{r=1}^{k_1-(m-1)k} \chi_1(u,r) \\ &+ \sum_{u=1}^{m-1} \sum_{r=k_1-(m-u)k+1}^{k_1-(m-u-1)k} \chi_2(u,r) \\ &+ k_1 \sum_{t=1}^{N} t^g \Big/ N^2\Bigg\}, \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} \chi_1(u,r) &= \delta_1^+(r) r^g + \theta \sum_{i=1}^{m} \sum_{j=1}^{s} \{r + (i-1)k + (j-1)k_1\}^g \\ &\quad + \delta_1^-(r)\{r + (m-1)k + (s-1)k_1\}^g, \\ \chi_2(u,r) &= \delta_2^+(r) r^g + \theta \frac{1}{n^2}\Bigg\{\sum_{i=1}^{m-u} \sum_{j=1}^{s} (r + (i-1)k + (j-1)k_1)^g \\ &\quad + \sum_{i=m-u+1}^{m} \Bigg(\sum_{j=1}^{s-1} (r + (i-1)k + (j-1)k_1)^g + \Big(r + (i-1)k \\ &\quad + (s-1)k_1 - N\Big)^g\Bigg)\Bigg\} + \delta_2^-(r)\,(r + (m-1)k + (s-1)k_1 - N)^g, \end{aligned}$$

$\delta_l^+(r) = a_l(r)\{a_l(r) + 2\left(\frac{1}{n} - \frac{1}{N}\right)\}$, $\delta_l^-(r) = a_l(r)\{a_l(r) - 2\left(\frac{1}{n} - \frac{1}{N}\right)\}$ and $\theta = \frac{1}{n}\left(\frac{1}{n} - \frac{2}{N}\right)$, such that $l = 1, 2$.

Case (ii) When $w > 1$ (see detial in Appendix B)

$$\begin{aligned} E\left\{V\left(\bar{y}^{(r)}_{MSS}\right)\right\} = \frac{\sigma^2}{k_1}\Bigg[&\sum_{u=w-1}^{w-1} \sum_{r=1}^{k_1-(m-u-1)k} \chi_2(u,r) \\ &+ \sum_{u=w}^{m-w} \sum_{r=k_1-(m-u)k+1}^{k_1-(m-u-1)k} \chi_2(u,r) \\ &+ \sum_{u=m-w+1}^{m-w+1} \sum_{r=k_1-(m-u)k+1}^{k_1-(m-u-1)k} \chi_2(u,r) \\ &+ \sum_{x=2}^{w-1} \sum_{r=k_1-(w-x)k+1}^{k_1-(w-x)k+r_m} \chi_3(x,r) \\ &+ \sum_{x=1}^{w-1} \sum_{r=k_1-(w-x)k+r_m+1}^{k_1-(w-x)k+k} \chi_4(x,r) \\ &+ k_1 \sum_{t=1}^{N} t^g \Big/ N^2\Bigg], \end{aligned} \tag{4.3}$$

where

$$\chi_2(u,r) = \delta_2^+(r) r^g + \theta \frac{1}{n^2}\Big\{\sum_{i=1}^{m-u}\sum_{j=1}^{s}(r+(i-1)k+(j-1)k_1)^g$$
$$+\sum_{i=m-u+1}^{m}\Big(\sum_{j=1}^{s-1}(r+(i-1)k+(j-1)k_1)^g$$
$$+(r+(i-1)k+(s-1)k_1-N)^g\Big)\Big\} + \delta_2^-(r)\,(r+(m-1)k+(s-1)k_1-N)^g,$$

$$\chi_3(x,r) = \delta_3^+(r) r^g + \theta\Big\{\sum_{i=1}^{w-x}\sum_{j=1}^{s}\{r+(i-1)k+(j-1)k_1\}^g$$
$$+\sum_{i=w-x+1}^{m-x+1}\Big(\sum_{j=1}^{s-1}\{r+(i-1)k+(j-1)k_1\}^g$$
$$+\{r+(i-1)k+(s-1)k_1-N\}^g\Big)$$
$$+\sum_{i=m-x+2}^{m}\Big(\sum_{j=1}^{s-2}\{r+(m-1)k+(j-1)k_1\}^g$$
$$+\sum_{j=s-1}^{s}\{r+(i-1)k+(j-1)k_1-N\}^g\Big)\Big\}$$
$$+\delta_3^-(r)\{r+(m-1)k+(s-1)k_1-N\}^g,$$

$$\chi_4(x,r) = \delta_4^+(r) r^g + \theta\Big\{\sum_{i=1}^{w-x}\sum_{j=1}^{s}\{r+(i-1)k+(j-1)k_1\}^g$$
$$+\sum_{i=w-x+1}^{m-x}\Big(\sum_{j=1}^{s-1}\{r+(i-1)k+(j-1)k_1\}^g$$
$$+\{r+(i-1)k+(s-1)k_1-N\}^g\Big)$$
$$+\sum_{i=m-x+1}^{m}\Big(\sum_{j=1}^{s-2}\{r+(m-1)k+(j-1)k_1\}^g$$
$$+\sum_{j=s-1}^{s}\{r+(i-1)k+(j-1)k_1-N\}^g\Big)\Big\}$$
$$+\delta_4^-(r)\{r+(m-1)k+(s-1)k_1-N\}^g,$$

$\delta_l^+(r) = a_l(r)\{a_l(r) + 2\left(\frac{1}{n} - \frac{1}{N}\right)\}$, $\delta_l^-(r) = a_l(r)\{a_l(r) - 2\left(\frac{1}{n} - \frac{1}{N}\right)\}$ and $\theta = \frac{1}{n}\left(\frac{1}{n} - \frac{2}{N}\right)$, for $l = 2,\ 3,\ 4$

5. Empirical study

Due to the complex nature of the derived expressions, the average variances of MSS and CSS cannot be theoretically compared. Therefore, in this paper, a computer based efficiency comparison of MSS and CSS is made numerically under super population model (4.1). The numerical comparison has been made for $N = 21$, $N = 30$, $N = 50$ and $N = 78$. As mentioned earliar, if $L = N$ then MSS reduces to LSS and if $L = (N \times n)$ then MSS becomes CSS. Therefore, the choice of a sample size considered in this paper is based on the fact that $N < L < (N \times n)$.

The relative efficiency of MSS over CSS is presented in Table 1 under $g = 0, 1, 2, 3$. This table includes 40 different combinations of N and n each at $g = 0, 1,\ 2$ and 3 which are to be considered for efficiency comparison, and it is observed that CSS is not applicable for 4 combinations. Thus, we have $36 \times 4 = 144$ results of efficiency comparison and found that MSS is more efficient than CSS in 135 cases. Further, it is to be noted, whenever $\frac{N}{n} = \left(\frac{n}{2} + \frac{1}{2}\right)$, the efficiency of MSS over CSS is dramatically increased.

5.1. Natural Population. We use the following natural population for efficiency comparison. The results are given in Table 2. Population 1: [Source: [1, page.228]. Table 2 reflacts that MSS is more efficient than CSS.

Table 1. Rercent Relative Efficiency (PRE) of MSS over CSS under linear trend

N	n	$g=0$	$g=1$	$g=2$	$g=3$	N	n	$g=0$	$g=1$	$g=2$	$g=3$
21	6	1235.016	1933.767	2709.724	3246.609	78	4	117.427	136.092	146.405	147.684
	9	117.385	151.130	174.582	186.989		8	129.769	182.162	229.578	259.786
	12	240.000	591.045	1375.565	2521.240		9	192.866	316.722	411.684	454.488
30	4	135.923	162.510	180.204	185.741		10	139.707	208.739	281.749	337.633
	8	246.131	376.018	544.687	700.611		12	6766.915	13989.818	19161.993	21879.417
	9	103.060	138.491	160.844	170.657		14	-	-	-	-
	12	-	-	-	-		15	90.595	152.753	201.481	223.241
	14	103.355	144.513	188.675	222.580		16	191.878	323.211	512.333	718.994
50	4	122.565	143.552	155.795	158.042		18	105.243	205.796	258.754	278.058
	6	101.443	123.039	137.125	142.782		20	257.844	455.482	780.262	1198.619
	8	97.891	125.605	146.267	156.789		21	236.637	633.532	1610.276	3245.550
	12	93.959	130.154	163.181	184.581		22	7.521	14.757	28.913	56.125
	14	43.631	84.386	161.900	305.136		24	114.967	260.473	359.195	399.452
	15	113.396	192.752	237.112	255.203		27	-	-	-	-
	16	93.665	136.682	182.496	217.408		28	129.784	203.929	293.560	381.668
	18	176.253	339.329	667.273	1275.240		30	-	-	-	-
	20	83.942	360.224	621.068	698.071		32	127.980	210.134	322.967	448.077
	22	128.892	193.530	275.040	353.302		33	153.748	309.110	503.525	655.560
	24	101.712	159.836	238.027	313.959		34	129.642	215.246	339.183	482.954
							36	151.165	418.855	673.244	790.552
							38	100.656	170.712	280.950	410.473

The symbol (-) indicates that CSS is not possible

Table 2. Percent Relative Efficiency (PRE) of MSS over CSS for Population 1

$N=80$	Variance		$Eff = \frac{V(CSS)}{V(MSS)} \times 100$
n	MSS	CSS	
6	148053.500	148326.200	100.184
12	37312.340	46858.630	125.585
14	33277.620	37824.400	113.663
15	28206.520	39716.150	140.805
18	29362.610	50787.490	172.967
24	9108.401	37832.020	415.353
25	7983.309	19399.580	243.002
26	7471.210	8915.224	119.328
28	6836.277	12549.070	183.566

Here, $V(MSS)$ = Variance of modified systematic sampling and $V(CSS)$ = Variance of circular systematic sampling.

6. Conclusion

Modified Systematic Sampling (MSS) is a more general scheme than LSS and CSS. Because, when least common multiple of N and n is equal to lower extreme, i.e. $L = N$, MSS coincides with LSS. If it is equal to upper extreme, i.e. $L = (N \times n)$, then MSS coincides with CSS. However, when L lies between these two extreme values, i.e. $N < L < (N \times n)$, MSS is advantageous over CSS. In this case, the number of samples is considerably reduced in MSS as compared to CSS, i.e. minimum reduction is half of the samples.

Contrary to the CSS, the explicit expressions for mean and variance of mean are derived for population having perfect linear trend among the population values. Further, numerical comparison is carried out in this paper clearly favors the use of MSS over CSS for population modeled by a super population model with linear trend as well as for natural population.

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References

[1] Murthy, M.N. *Sampling Theory and Method*, Statistical Publishing Society, Calcutta, 1967.

[2] Khan, Z. Shabbir, J. and Gupta, S. *A new sampling design for systematic sampling*, Communications in Statistics - Theory and Methods,**42**, 2659-2670, 2013.

[3] Khan, Z. Gupta, S and Shabbir, J. *A Note on Diagonal Circular Systematic Sampling*, Journal of Statistical Theory and Practice, **8**, 439-443, 2014.

[4] Madow, W.G. *On the theory of systematic sampling III*, Annals of Mathematics and Statistics, **24**, 101-106, 1953.

[5] Sampath, S and Varalakshmi, V. *Diagonal circular systematic sampling*, Model Assisted Statistics and Applications, **3** (4), 345-352, 2008.

[6] Sethi, V. K. *On optimum pairing of units*, Sankhya **B, 27**, 315-320, 1965.

[7] Singh, D. Jindal, K.K. and Grag, J.N.*On modified systematic sampling*, Biometrika, **55**, 41-546, 1968.

[8] Subramani, J. *Diagonal systematic sampling scheme for finite populations*, Journal of the Indian Society of Agriculture Statistics, **53** (2), 187-195, 2000.

[9] Subramani, J. *Further results on diagonal systematic sampling scheme for finite populations*, Journal of the Indian Society of Agricultural Statistics, **63** (3), 277-282, 2009.

[10] Subramani, J.*Generalization of diagonal systematic sampling scheme for finite populations*, Model Assisted Statistics and Applications, **5**, 117-128, 2010.

[11] Subramani, J. *A Modification on linear systematic sampling*, Model Assisted Statistics and Applications, **8** (3), 215-227, 2013a.

[12] Subramani, J. *A further modification on linear systematic sampling for finite populations*, Journal of Statistical Theory and Practice, **7**, 471-479, 2013b.

Appendix A. Mean and variance of MSS for population having linear trend

The following linear model of hypothetical population is to be considered

$$Y_t = \alpha + \beta\, t, \qquad t = 1, 2, 3, ..., N, \tag{A.1}$$

where α and β are the intercept and slope of the model respectively.

A.1. Mean of MSS. The sample mean for both cases, i.e. $w = 1$ and $w > 1$ are separately discussed below:

Case (i) when $w = 1$
If $r \leq (k_1 - (m-1)k$, the mean, $\bar{y}_{MSS}$ can be written as

$$\bar{y}_{MSS} = \alpha + \beta \frac{1}{ms} \sum_{i=1}^{m-w+1} \sum_{j=1}^{s} \{r + (i-1)k + (j-1)k_1\}$$

After simplification, we have

$$\bar{y}_{MSS} = \alpha + \beta \left[r + \frac{1}{2} \{(s-1)k_1 + (m-1)k\} \right].$$

If $\{k_1 - (m-u)k\} < r \leq \{k_1 - (m-u-1)k\}$ for $u = 1, 2, ..., m-1$, then

$$\begin{aligned} \bar{y}_{MSS} = \alpha + \beta \tfrac{1}{ms} \Big[& \textstyle\sum_{i=1}^{m-u} \sum_{j=1}^{s} \{r + (i-1)k + (j-1)k_1\} \\ & + \textstyle\sum_{i=m-u+1}^{m} \Big\{ \sum_{j=1}^{s-1} \{r + (i-1)k + (j-1)k_1\} \\ & + \{r + (i-1)k + (s-1)k_1 - N\} \Big\} \Big]. \end{aligned}$$

After simplification, we have

$$\bar{y}_{MSS} = \alpha + \beta \left[r + \frac{1}{2} \{(s-1)k_1 + (m-1)k\} - u \frac{k_1}{m} \right].$$

Thus $\bar{y}_{MSS}$ is a piecewise function of r, i.e.

$$\bar{y}_{MSS} = \alpha + \beta \begin{cases} \left[r + \frac{1}{2}\{(s-1)k_1 + (m-1)k\} \right] \\ \qquad \text{if} \quad r \leq \{k_1 - (m-1)k\} \\ \left[r + \frac{1}{2}\{(s-1)k_1 + (m-1)k\} - u\frac{k_1}{m} \right] \\ \qquad \text{where} \left| \begin{array}{l} u = 1, 2, ..., (m-1) \text{ if} \\ \{k_1 - (m-u)k\} < r \leq \{k_1 - (m-u-1)k\}. \end{array} \right. \end{cases} \tag{A.2}$$

Case (ii) when $w > 1$
If $r \leq \{k_1 - (w-1)k + r_m\}$, then r must belongs to any one of the three subgroups which have been discussed in Section 1. Therefore, corresponding to a random number r, an integer u is picked in such a way that $u = (w-1)$ if $1 \leq r \leq \{k_1 - (m-u-1)k\}$; $u = w, w+1, w+2, ..., (m-w)$ if $\{k_1 - (m-u)k\} < r \leq \{k_1 - (m-u-1)k\}$ and $u = (m-w+1)$ if $\{k_1 - (m-u)k\} < r \leq \{k_1 - (m-u)k + r_m\}$.

For each subgroup, $\bar{y}_{MSS}$ can be written as

$$\bar{y}_{MSS} = \alpha + \beta\tfrac{1}{ms}\Big[\textstyle\sum_{i=1}^{m-u}\sum_{j=1}^{s}\{r+(i-1)k+(j-1)k_1\} \\ +\textstyle\sum_{i=m-u+1}^{m}\Big\{\sum_{j=1}^{s-1}\{r+(i-1)k+(j-1)k_1\} \\ +\{r+(i-1)k+(s-1)k_1-N\}\Big].$$

After few steps, we have

$$\bar{y}_{MSS} = \alpha + \beta\left[r + \frac{1}{2}\{(s-1)k_1+(m-1)k\} - u\frac{k_1}{m}\right].$$

If $w > 2$, then it is also possible that $\{k_1-(w-x)k\} < r \leq \{k_1-(w-x)k+r_1\}$, such that $x = 2, 3, ..., w-1$. So,

$$\bar{y}_{MSS} = \alpha + \beta\tfrac{1}{ms}\Big[\textstyle\sum_{i=1}^{w-x}\sum_{j=1}^{s}\{r+(i-1)k+(j-1)k_1\} \\ +\textstyle\sum_{i=w-x+1}^{m-x+1}\Big\{\sum_{j=1}^{s-1}\{r+(i-1)k+(j-1)k_1\} \\ +\{r+(i-1)k+(s-1)k_1-N\} \\ +\textstyle\sum_{i=m-x+2}^{m}\Big\{\sum_{j=1}^{s-2}\{r+(i-1)k+(j-1)k_1\} \\ +\textstyle\sum_{j=s-1}^{s}\{r+(i-1)k+(j-1)k_1-N\}. \tag{A.3}$$

When $s = 2$, then Equation (A.3) can be expressed as

$$\bar{y}_{MSS} = \alpha + \beta\tfrac{1}{ms}\Big[\textstyle\sum_{i=1}^{w-x}\sum_{j=1}^{s}\{r+(i-1)k+(j-1)k_1\} \\ +\textstyle\sum_{i=w-x+1}^{m-x+1}\Big\{\sum_{j=1}^{s-1}\{r+(i-1)k+(j-1)k_1\} \\ +\{r+(i-1)k+(s-1)k_1-N\}\Big\} \\ +\textstyle\sum_{i=m-x+2}^{m}\Big\{\sum_{j=s-1}^{s}\{r+(i-1)k+(j-1)k_1-N\}\Big\}\Big].$$

Also, when $s = 1$, then Equation (A.3) can be expressed as

$$\bar{y}_{MSS} = \alpha + \beta\tfrac{1}{ms}\Big[\textstyle\sum_{i=1}^{w-x}\sum_{j=1}^{s}\{r+(i-1)k+(j-1)k_1\} \\ +\textstyle\sum_{i=w-x+1}^{m-x+1}\{r+(i-1)k+(s-1)k_1-N\} \\ +\textstyle\sum_{i=m-x+2}^{m}\{r+(i-1)k+(s-1)k_1-2N\}\Big].$$

After simplifying of Equation (A.3) for each case, i.e. $s = 1$, $s = 2$ and $s > 2$, we have

$$\bar{y}_{MSS} = \alpha + \beta\left[r + \frac{1}{2}\{(s-1)k_1+(m-1)k\} - (m-w-1+2x)\frac{k_1}{m}\right].$$

If $\{k_1-(w-x)k+r_1\} < r \leq \{k_1-(w-x)k+k\}$, then

$$\bar{y}_{MSS} = \alpha + \beta\tfrac{1}{ms}\Big[\textstyle\sum_{i=1}^{w-x}\sum_{j=1}^{s}\{r+(i-1)k+(j-1)k_1\} \\ +\textstyle\sum_{i=w-x+1}^{m-x}\Big\{\sum_{j=1}^{s-1}\{r+(i-1)k+(j-1)k_1\} \\ +\{r+(i-1)k+(s-1)k_1-N\}\Big\} \\ +\textstyle\sum_{i=m-x+1}^{m}\Big\{\sum_{j=1}^{s-2}\{r+(i-1)k+(j-1)k_1\} \\ +\textstyle\sum_{j=s-1}^{s}\{r+(i-1)k+(j-1)k_1-N\}\Big\}\Big]. \tag{A.4}$$

When $s = 2$, then Equation (A.4) can be expressed as

$$\begin{aligned}\bar{y}_{MSS} = \alpha + \beta\tfrac{1}{ms}\Big[&\textstyle\sum_{i=1}^{w-x}\sum_{j=1}^{s}\{r + (i-1)k + (j-1)k_1\} \\ &+ \textstyle\sum_{i=w-x+1}^{m-x}\Big\{\sum_{j=1}^{s-1}\{r + (i-1)k + (j-1)k_1\} \\ &+ \{r + (i-1)k + (s-1)k_1 - N\}\Big\} \\ &+ \textstyle\sum_{i=m-x+1}^{m}\sum_{j=s-1}^{s}\{r + (i-1)k + (j-1)k_1 - N\}\Big].\end{aligned}$$

When $s = 1$, then Equation (A.4) can be expressed as,

$$\begin{aligned}\bar{y}_{MSS} = \alpha + \beta\tfrac{1}{ms}\Big[&\textstyle\sum_{i=1}^{w-x}\sum_{j=1}^{s}\{r + (i-1)k + (j-1)k_1\} \\ &+ \textstyle\sum_{i=w-x+1}^{m-x}\{r + (i-1)k + (s-1)k_1 - N\} \\ &+ \textstyle\sum_{i=m-x+1}^{m}\{r + (i-1)k + (s-1)k_1 - 2N\}\Big].\end{aligned}$$

After simplification of Equation (A.4) for each case, i.e. $s = 1$, $s = 2$ and $s > 2$, we have

$$\bar{y}_{MSS} = \alpha + \beta\Big[r + \frac{1}{2}\{(s-1)k_1 + (m-1)k\} - (m - w + 2x)\frac{k_1}{m}\Big].$$

Thus, mean of MSS for above model of hypothetical population with random start r is given by:

$$\text{(A.5)} \qquad \bar{y}_{MSS} = \alpha + \beta\begin{cases}\big[r + \frac{1}{2}\{(s-1)k_1 + (m-1)k\} - u\frac{N}{n}\big] \\ \quad\text{where}\ \left|\begin{array}{l} u = (w-1)\ if \quad r \leq \{k_1 - (m-u-1)k, \\ u = w, w+1, ..., (m-w)\ if \\ \{N - (m-u)k\} < r \leq \{k_1 - (m-u-1)k\}, \\ u = (m-w+1)\ if \\ \{k_1 - (m-u)k\} < r \leq \{k_1 - (m-u)k + r_m\}\end{array}\right. \\ \big[r + \frac{1}{2}\{(s-1)k_1 + (m-1)k\} - (m-w-1+2x)\frac{N}{n}\big], \\ \quad\text{where}\ \left|\begin{array}{l} x = 2, ..., (w-1)\ if \\ \{k_1 - (w-x)k\} < r \leq \{k_1 - (w-x)k + r_m\}\end{array}\right. \\ \big[r + \frac{1}{2}\{(s-1)k_1 + (m-1)k\} - (m-w+2x)\frac{N}{n}\big], \\ \quad\text{where}\ \left|\begin{array}{l} x = 1, 2, 3, ..., (w-1)\ if \\ \{k_1 - (w-x)k\} + r_m < r \leq \{k_1 - (w-x)k + k\}\end{array}\right.\end{cases}$$

If $w = 2$, then Equation (A.5) reduces to

$$\bar{y}_{MSS} = \alpha + \beta\begin{cases}\big[r + \frac{1}{2}\{(s-1)k_1 + (m-1)k\} - u\frac{N}{n}\big] \\ \quad\text{where}\ \left|\begin{array}{l} u = (w-1)\ if \quad r \leq \{k_1 - (m-u-1)k, \\ u = w, w+1, ..., (m-w)\ if \\ \{N - (m-u)k\} < r \leq \{k_1 - (m-u-1)k\}, \\ u = (m-w+1)\ if \\ \{k_1 - (m-u)k\} < r \leq \{k_1 - (m-u)k + r_m\}\end{array}\right. \\ \big[r + \frac{1}{2}\{(s-1)k_1 + (m-1)k\} - (m-w+2x)\frac{N}{n}\big], \\ \quad\text{where}\ \left|\begin{array}{l} x = 1, 2, 3, ..., (w-1)\ if \\ \{k_1 - (w-x)k\} + r_m < r \leq \{k_1 - (w-x)k + k\}\end{array}\right.\end{cases}$$

A.1.1. *Unbiasedness of sample mean* $\bar{y}_{MSS}$. We have two cases:

Case (i) when $w = 1$: Taking the expected value of (A.2), we have

$$E(\bar{y}_{MSS}) = \frac{1}{k_1}\Big[\sum_{r=1}^{(k_1-(m-1)k)}\big[r + \tfrac{1}{2}\{(s-1)k_1 + (m-1)k\}\big] + \sum_{u=1}^{m-1}\sum_{r=(k_1-(m-u)k)+1}^{k_1-(m-u-1)k)}\big[r + \tfrac{1}{2}\{(s-1)k_1 + (m-1)k\} - u\tfrac{k_1}{m}\big]\Big].$$

As $sk_1 = N$, then

$$E(\bar{y}_{MSS}) = \frac{1}{k_1}\Big[\sum_{r=1}^{(k_1-(m-1)k)}\big[\alpha + \beta\left\{r + \tfrac{1}{2}\left(N - k_1 + (m-1)k\right)\right\}\big] + \sum_{u=1}^{m-1}\sum_{r=(k_1-(m-u)k)+1}^{k_1-(m-u-1)k)}\Big[\alpha + \beta\Big\{r + \tfrac{1}{2}\left(N - k_1 + (m-1)k\right) - u\tfrac{k_1}{m}\Big\}\Big]\Big].$$

After a little algebra, we have

$$E(\bar{y}_{MSS}) = \alpha + \beta\left\{\frac{N+1}{2}\right\} = \bar{Y},$$

which shows that $\bar{y}_{MSS}$ is an unbiased estimator of $\bar{Y}$.

Case (ii) when $w > 1$:
If $w > 2$, we take the expected value of (A.5), we have

$$\begin{aligned} E(\bar{y}_{MSS}) = \frac{1}{k_1}\Big[&\sum_{u=w-1}^{w-1}\sum_{r=1}^{k_1-(m-u-1)k}\Big[\alpha + \beta\Big\{r + \tfrac{1}{2}\left((s-1)k_1 + (m-1)k - u\tfrac{k_1}{m}\right\}\Big] \\ &+ \sum_{u=w}^{m-w}\sum_{r=k_1-(m-u)k+1}^{k_1-(m-u-1)k}\Big[\alpha + \beta\Big\{r + \tfrac{1}{2}\left((s-1)k_1 + (m-1)k - u\tfrac{k_1}{m}\right\}\Big] \\ &+ \sum_{u=m-w+1}^{m-w+1}\sum_{r=k_1-(m-u)k+1}^{k_1-(m-u)k+r_m}\Big[\alpha + \beta\Big\{r + \tfrac{1}{2}\left((s-1)k_1 + (m-1)k - u\tfrac{k_1}{m}\right\}\Big] \\ &+ \sum_{x=2}^{w-1}\sum_{r=k_1-(w-x)k+1}^{k_1-(w-x)k+r_m}\Big[\alpha + \beta\Big\{r + \tfrac{1}{2}\left((s-1)k_1 + (m-1)k - \tfrac{(m-w-1+2x)k_1}{m}\right\}\Big] \\ &+ \sum_{x=1}^{w-1}\sum_{r=k_1-(w-x)k+r_m+1}^{k_1-(w-x)k+k}\Big[\alpha + \beta\Big\{r + \tfrac{1}{2}\left((s-1)k_1 + (m-1)k - \tfrac{(m-w+2x)k_1}{m}\right\}\Big]\Big]. \end{aligned}$$

But if $w = 2$, we take the expected value of (A.5), given by

$$\begin{aligned} E(\bar{y}_{MSS}) = \frac{1}{k_1}\Big[&\sum_{u=w-1}^{w-1}\sum_{r=1}^{k_1-(m-u-1)k}\Big[\alpha + \beta\Big\{r + \tfrac{1}{2}\left((s-1)k_1 + (m-1)k - u\tfrac{k_1}{m}\right\}\Big] \\ &+ \sum_{u=w}^{m-w}\sum_{r=k_1-(m-u)k+1}^{k_1-(m-u-1)k}\Big[\alpha + \beta\Big\{r + \tfrac{1}{2}\left((s-1)k_1 + (m-1)k - u\tfrac{k_1}{m}\right\}\Big] \\ &+ \sum_{u=m-w+1}^{m-w+1}\sum_{r=k_1-(m-u)k+1}^{k_1-(m-u)k+r_m}\Big[\alpha + \beta\Big\{r + \tfrac{1}{2}\left((s-1)k_1 + (m-1)k - u\tfrac{k_1}{m}\right\}\Big] \\ &+ \sum_{x=1}^{w-1}\sum_{r=k_1-(w-x)k+r_m+1}^{k_1-(w-x)k+k}\Big[\alpha + \beta\Big\{r + \tfrac{1}{2}\left((s-1)k_1 + (m-1)k - \tfrac{(m-w+2x)k_1}{m}\right\}\Big]\Big]. \end{aligned}$$

After few steps, we have

$$E(\bar{y}_{MSS}) = \tfrac{1}{2m}\Big[2\alpha m + \beta\Big\{2k_1 - 2r_1 - 2km + 2wk + 2wr_1$$
$$+2kwm - 2k_1 w - 2kw^2 + Nm + m\Big\}\Big]$$

$$E(\bar{y}_{MSS}) = \Big[\alpha + \beta\tfrac{1}{2m}\Big\{m\,(N+1) - 2k_1(w-1)$$
$$+2r_1(w-1) - 2wk(w-1) + 2km(w-1)\Big\}\Big]$$

$$\text{(A.6)}\quad E(\bar{y}_{MSS}) = \left[\alpha + \beta\left\{\frac{N+1}{2} + \frac{(w-1)}{m}\,(r_1 - k_1 - wk + km)\right\}\right].$$

When $w = (m-q)$ in (A.6), we have

$$E(\bar{y}_{MSS}) = \left[\alpha + \beta\left\{\frac{N+1}{2} + \frac{(m-q-1)}{m}\,(r_1 - k_1 - (m-q)k + km)\right\}\right],$$

$$E(\bar{y}_{MSS}) = \left[\alpha + \beta\left\{\frac{N+1}{2} + \frac{(m-q-1)}{m}\,(qk + r_1 - k_1 - mk + km)\right\}\right],$$

$$E(\bar{y}_{MSS}) = \alpha + \beta\left\{\frac{N+1}{2} + \frac{(m-q-1)}{m}\,(qk + r_1 - k_1)\right\},$$

$$E(\bar{y}_{MSS}) = \alpha + \beta\left\{\frac{N+1}{2}\right\} = \bar{Y}$$

.

the above equation shows that $\bar{y}_{MSS}$ is unbiased estimator of $\bar{Y}$ as $k_1 = qk + r_1$.
Note: Putting $w = 1$ in (A.6), we also have

$$E(\bar{y}_{MSS}) = \alpha + \beta\left\{\frac{N+1}{2}\right\} = \bar{Y}.$$

A.2. The variance of $\bar{y}_{MSS}$.

$$V(\bar{y}_{MSS}) = E(\bar{y}_{MSS} - \bar{Y})^2 = \tfrac{1}{k_1}\textstyle\sum_{r=1}^{k_1}(\bar{y}_{r(MSS)} - \bar{Y})^2.$$

(i) when $w = 1$

$$V(\bar{y}_{MSS}) = \tfrac{1}{k_1}\Big[\textstyle\sum_{r=1}^{k_1-(m-1)k}\big\{\bar{y}_{r(MSS)} - \bar{Y}\big\}^2$$
$$+\textstyle\sum_{u=1}^{m-1}\sum_{r=k_1-(m-u)k+1}^{k_1-(m-u-1)k}\big\{\bar{y}_{r(MSS)} - \bar{Y}\big\}^2\Big].$$

$$V(\bar{y}_{MSS}) = \tfrac{1}{k_1}\Big[\textstyle\sum_{r=1}^{k_1-(m-1)k}\Big[\alpha + \beta\left\{r + \tfrac{1}{2}\left((s-1)k_1 + (m-1)k\right)\right\}$$
$$-\left\{\alpha + \beta\tfrac{N+1}{2}\right\}\Big]^2$$
$$+\textstyle\sum_{u=1}^{m-1}\sum_{r=k_1-(m-u)k+1}^{k_1-(m-u-1)k}\Big[\alpha + \beta\{r + \tfrac{1}{2}\left((s-1)k_1 + (m-1)k\right)$$
$$-u\tfrac{k_1}{m}\Big\} - \left\{\alpha + \beta\tfrac{N+1}{2}\right\}\Big]^2\Big].$$

$$V(\bar{y}_{MSS}) = \tfrac{1}{k_1}\Big[\textstyle\sum_{r=1}^{k_1-(m-1)k}\Big[\beta\left\{r + \tfrac{1}{2}\left((m-1)k - k_1\right)\right\} - \tfrac{\beta}{2}\Big]^2$$
$$+\textstyle\sum_{u=1}^{m-1}\sum_{r=k_1-(m-u)k+1}^{k_1-(m-u-1)k}\Big[\beta\Big\{r + \tfrac{1}{2}\left((m-1)k - k_1\right)$$
$$-u\tfrac{k_1}{m}\Big\} - \tfrac{\beta}{2}\Big]^2\Big].$$

After simplification, we have

$$V(\bar{y}_{MSS}) = \frac{1}{12m^2} b^2 \left[m^2(k_1^2 - 1) + m(m^2 - 1)k(mk - 2k_1) \right], \tag{A.7}$$

Note: If $N = nk$, then $L = N$, so $m = 1$, thus

$$V(\bar{y}_{MSS}) = \frac{1}{12} b^2 (k^2 - 1).$$

This is a variance of linear systematic sampling.

(ii) when $w > 1$
If $w > 2$, then $V(\bar{y}_{MSS})$ will be expressed as:

$$\begin{aligned} V(\bar{y}_{MSS}) = \tfrac{1}{k_1} \Big[& \textstyle\sum_{u=w-1}^{w-1} \sum_{r=1}^{k_1-(m-u-1)k} \left\{ \bar{y}_{r(MSS)} - \bar{Y} \right\}^2 \\ & + \textstyle\sum_{u=w}^{m-w} \sum_{r=k_1-(m-u)k+1}^{k_1-(m-u-1)k} \left\{ \bar{y}_{r(MSS)} - \bar{Y} \right\}^2 \\ & + \textstyle\sum_{u=m-w+1}^{m-w+1} \sum_{r=k_1-(m-u)k+1}^{k_1-(m-u)k+r_m} \left\{ \bar{y}_{r(MSS)} - \bar{Y} \right\}^2 \\ & + \textstyle\sum_{x=2}^{w-1} \sum_{r=(k_1-(w-x)k)+1}^{k_1-(w-x)k+r_m)} \left\{ \bar{y}_{r(MSS)} - \bar{Y} \right\}^2 \\ & + \textstyle\sum_{x=1}^{w-1} \sum_{r=(k_1-(w-x)k)+r_m+1}^{k_1-(w-x)k+k)} \left\{ \bar{y}_{r(MSS)} - \bar{Y} \right\}^2 \Big]. \end{aligned}$$

If $w = 2$, then the term $\sum_{x=2}^{w-1} \sum_{r=(k_1-(w-x)k)+1}^{k_1-(w-x)k+r_m)} \left\{ \bar{y}_{r(MSS)} - \bar{Y} \right\}^2$ will be omitted from $V(\bar{y}_{MSS})$.

$$\begin{aligned} V(\bar{y}_{MSS}) = \tfrac{1}{k_1} \Big[& \textstyle\sum_{u=w-1}^{w-1} \sum_{r=1}^{k_1-(m-u-1)k} \Big[\alpha + \beta \Big\{ r + \frac{1}{2} \Big((s-1)k_1 \\ & +(m-1)k \Big) - u\frac{k_1}{m} \Big\} - \left\{ \alpha + \beta \frac{N+1}{2} \right\} \Big]^2 \\ & + \textstyle\sum_{u=w}^{m-w} \sum_{r=k_1-(m-u)k+1}^{k_1-(m-u-1)k} \Big[\alpha + \beta \Big\{ r + \frac{1}{2} \Big((s-1)k_1 \\ & +(m-1)k \Big) - u\frac{k_1}{m} \Big\} - \left\{ \alpha + \beta \frac{N+1}{2} \right\} \Big]^2 \\ & + \textstyle\sum_{u=m-w+1}^{m-w+1} \sum_{r=k_1-(m-u)k+1}^{k_1-(m-u)k+r_m} \Big[\alpha + \beta \Big\{ r + \frac{1}{2} \Big((s-1)k_1 \\ & +(m-1)k \Big) - u\frac{k_1}{m} \Big\} - \left\{ \alpha + \beta \frac{N+1}{2} \right\} \Big]^2 \\ & + \textstyle\sum_{x=2}^{w-1} \sum_{r=(k_1-(w-x)k)+1}^{k_1-(w-x)k+r_1)} \Big[\alpha + \beta \Big\{ r + \frac{1}{2} \Big\{ (s-1)k_1 \\ & +(m-1)k \Big\} - (m-w-1+2x)\frac{k_1}{m} \Big\} - \left\{ \alpha + \beta \frac{N+1}{2} \right\} \Big]^2 \\ & + \textstyle\sum_{x=1}^{w-1} \sum_{r=(k_1-(w-x)k)+r_1+1}^{k_1-(w-x)k+k)} \Big[\alpha + \beta \Big\{ r + \frac{1}{2} \Big\{ (s-1)k_1 \\ & +(m-1)k \Big\} - (m-w+2x)\frac{k_1}{m} \Big\} - \left\{ \alpha + \beta \frac{N+1}{2} \right\} \Big]^2 \Big], \end{aligned}$$

$$
\begin{aligned}
V(\bar{y}_{MSS}) = \tfrac{1}{k_1}\Big[&\textstyle\sum_{u=w-1}^{w-1}\sum_{r=1}^{k_1-(m-u-1)k}\Big[\beta\Big\{r+\tfrac{1}{2}\left((m-1)k-k_1\right) \\
&-u\tfrac{k_1}{m}\Big\}-\tfrac{\beta}{2}\Big]^2 \\
&+\textstyle\sum_{u=w}^{m-w}\sum_{r=k_1-(m-u)k+1}^{k_1-(m-u-1)k}\Big[\beta\Big\{r+\tfrac{1}{2}\left((m-1)k-k_1\right) \\
&-u\tfrac{k_1}{m}\Big\}-\tfrac{\beta}{2}^2 \\
&+\textstyle\sum_{u=m-w+1}^{m-w+1}\sum_{r=k_1-(m-u)k+1}^{k_1-(m-u)k+r_m}\Big[\beta\Big\{r+\tfrac{1}{2}\left((m-1)k-k_1\right) \\
&-u\tfrac{k_1}{m}\Big\}-\tfrac{\beta}{2}\Big]^2 \\
&+\textstyle\sum_{x=2}^{w-1}\sum_{r=k_1-(w-x)k+1}^{k_1-(w-x)k+r_m}\Big[\alpha+\beta\Big\{r+\tfrac{1}{2}\left((m-1)k-k_1\right) \\
&-(m-w-1+2x)\tfrac{k_1}{m}\Big\}-\tfrac{\beta}{2}\Big]^2 \\
&+\textstyle\sum_{x=1}^{w-1}\sum_{r=k_1-(w-x)k+r_m+1}^{k_1-(w-x)k+k}\Big[\alpha+\beta\Big\{r+\tfrac{1}{2}\left((m-1)k-k_1\right) \\
&-(m-w+2x)\tfrac{k_1}{m}\Big\}-\tfrac{\beta}{2}\Big]^2.
\end{aligned}
\tag{A.8}
$$

If $w = 2$, then Equation (A.8) reduces to

$$
\begin{aligned}
V(\bar{y}_{MSS}) = \tfrac{1}{k_1}\Big[&\textstyle\sum_{u=w-1}^{w-1}\sum_{r=1}^{k_1-(m-u-1)k}\Big[\beta\Big\{r+\tfrac{1}{2}\left((m-1)k-k_1\right) \\
&-u\tfrac{k_1}{m}\Big\}-\tfrac{\beta}{2}\Big]^2 \\
&+\textstyle\sum_{u=w}^{m-w}\sum_{r=k_1-(m-u)k+1}^{k_1-(m-u-1)k}\Big[\beta\Big\{r+\tfrac{1}{2}\left((m-1)k-k_1\right) \\
&-u\tfrac{k_1}{m}\Big\}-\tfrac{\beta}{2}\Big]^2 \\
&+\textstyle\sum_{u=m-w+1}^{m-w+1}\sum_{r=k_1-(m-u)k+1}^{k_1-(m-u)k+r_m}\Big[\beta\Big\{r+\tfrac{1}{2}\left((m-1)k-k_1\right) \\
&-u\tfrac{k_1}{m}\Big\}-\tfrac{\beta}{2}\Big]^2 \\
&+\textstyle\sum_{x=1}^{w-1}\sum_{r=k_1-(w-x)k+r_m+1}^{k_1-(w-x)k+k}\Big[\alpha+\beta\Big\{r+\tfrac{1}{2}\left((m-1)k-k_1\right) \\
&-(m-w+2x)\tfrac{k_1}{m}\Big\}-\tfrac{\beta}{2}\Big]^2\Big].
\end{aligned}
$$

After simplification, we have

$$
\begin{aligned}
V(\bar{y}_{MSS}) = \tfrac{1}{12m^2}b^2\Big[&m^2(k_1^2-1)+m(m^2-1)k(mk-2k_1) \\
&+4(w-1)\Big\{3k(m-q-w)\left\{m(k_1-qk)+(k_1-mk)\right\} \\
&+k_1w\ \left\{3k_1-(3m-2w+1)k\right\}\Big\}\Big].
\end{aligned}
$$

The term $(w-1)\left[3k(m-q-w)\left\{m(k_1-qk)+(k_1-mk)\right\}\right]$ will be vanished in both situations, when $w = 1$ or $w = (m-q)$. So, we are left with

$$
\begin{aligned}
V(\bar{y}_{MSS}) = \tfrac{1}{12m^2}b^2\big[&m^2(k_1^2-1)+m(m^2-1)k(mk-2k_1) \\
&+4w(w-1)k_1\left\{3k_1-(3m-2w+1)k\right\}\big].
\end{aligned}
\tag{A.9}
$$

Appendix B. Average variance

In real life application, we hardly found such population exhibiting perfect linear trend, therefore, it is necessary to study the average variance of the corrected estimator under MSS using following super population model.

$$Y_t = \alpha + \beta t + e_t, \tag{B.1}$$

where $E(e_t) = 0$, $V(e_t) = E(e_t^2) = \sigma^2 t^g$, $Cov(e_t, e_v) = 0$, $t \neq v = 1, 2, 3, ..., N$and g is a predetermine constant.
Under the above super population model (B.1), the average variance expression of MSS is given below:

Case (i) when $w = 1$
Consider that l^{th} sum of squares (SSl) are given by

$$SSl = \left[\bar{y}_{MSS}^{(r)} - \bar{Y}\right]^2 = \left[\{\bar{y}_{MSS} + a_l(r)\left(Y_{r1} - Y_{rn}\right)\} - \bar{Y}\right]^2, \tag{B.2}$$

where $l = 1$ if $r \leq k_1 - (m-1)k$ and $l = 2$ if $r > k_1 - (m-1)k$.
When $r \leq k_1 - (m-1)k$, the expressions of $\bar{y}_{MSS}$, $\bar{Y}$, $Y_{r1}$and Y_{rn} under the model $Y_t = \alpha + \beta\, t + e_t$, can be expressed as:
$\bar{y}_{MSS} = \alpha + \beta\left[r + \frac{1}{2}\left\{(s-1)k_1 + (m-1)k\right\}\right] + \frac{1}{ms}\sum_{i=1}^{m}\sum_{j=1}^{s} e_{r+(i-1)k+(j-1)k_1}$,
$\bar{Y} = \alpha + \beta\frac{N+1}{2} + \frac{1}{N}\sum_{t=1}^{N} e_t$, $Y_{r1} = \alpha + \beta r + e_r$ and $Y_{rn} = \alpha + \beta\left\{r + (m-1)k + (s-1)k_1\right\} + e_{r+(m-1)k+(s-1)k_1}$.
Substituting these expressions in (B.2), we have

$$SS1 = \left[\bar{y}_{MSS}^{(r)} - \bar{Y}\right]^2 = \Bigg[\frac{1}{n}\Bigg\{\sum_{i=1}^{m}\sum_{j=1}^{s} e_{r+(i-1)k+(j-1)k_1} + na_1(r)\left(e_r - e_{r+(m-1)k+(s-1)k_1}\right)\Bigg\} - \frac{1}{N}\sum_{t=1}^{N} e_t\Bigg]^2.$$

Similarly, if $r > k_1 - (m-1)k$ the expressions of $\bar{y}_{MSS}$, $\bar{Y}$, Y_{r1} and Y_{rn} under the super population model $Y_t = \alpha + \beta\, t + e_t$, can be written as

$$\bar{y}_{MSS} = \alpha + \beta\left[r + \frac{1}{2}\left\{(s-1)k_1 + (m-1)k\right\} - u\frac{k_1}{m}\right] + \frac{1}{ms}\Bigg[\sum_{i=1}^{m-u}\sum_{j=1}^{s} e_{r+(i-1)k+(j-1)k_1} + \sum_{i=m-u+1}^{m}\Bigg\{\sum_{j=1}^{s-1} e_{r+(i-1)k+(j-1)k_1} + e_{r+(i-1)k+(s-1)k_1-N}\Bigg\}\Bigg],$$

$\bar{Y} = \alpha + \beta\frac{N+1}{2} + \frac{1}{N}\sum_{t=1}^{N} e_t$, $Y_{r1} = \alpha + \beta r + e_r$ and $Y_{rn} = \alpha + \beta\left\{r + (m-1)k + (s-1)k_1 - N\right\} + e_{r+(m-1)k+(s-1)k_1-N}$.
Thus,

$$SS2 = \left[\bar{y}_{MSS}^{(r)} - \bar{Y}\right]^2 = \Bigg[\frac{1}{n}\Bigg\{\sum_{i=1}^{m-u}\sum_{j=1}^{s} e_{r+(i-1)k+(j-1)k_1} + \sum_{i=m-u+1}^{m}\Bigg(\sum_{j=1}^{s-1} e_{r+(i-1)k+(j-1)k1} + e_{r+(i-1)k+(s-1)k_1-N}\Bigg) + na_2(r)(e_r - e_{r+(m-1)k+(s-1)k_1-N})\Bigg\} - \frac{1}{N}\sum_{t=1}^{N} e_t\Bigg]^2.$$

The average variance of the corrected sample mean can be written as:

$$E\left\{V(\bar{y}_{MSS}^{(r)})\right\} = \frac{1}{k_1}\Bigg\{\sum_{r=1}^{k_1-(m-1)k} E\left(SS1\right) + \sum_{u=1}^{m-1}\sum_{r=k_1-(m-u)k+1}^{k_1-(m-u-1)k} E\left(SS2\right)\Bigg\} \tag{B.3}$$

under the assumptions of super population model.

$$
\begin{aligned}
E\,(SS1) = \Bigg[&\frac{1}{n^2}\Bigg\{\sum_{i=1}^{m}\sum_{j=1}^{s} E(e^2_{r+(i-1)k+(j-1)k_1}) + n^2 a_1^2(r)\Bigg\{E(e_r^2)\\
&+E(e^2_{r+(m-1)k+(s-1)k_1})\Bigg\} + 2na_1(r)\Bigg\{E(e_r^2)\\
&-E(e^2_{r+(m-1)k+(s-1)k_1})\Bigg\}\Bigg\}\\
&+\frac{1}{N^2}\sum_{t=1}^{N} E(e_t^2) - \frac{2}{nN}\Bigg\{\sum_{i=1}^{m}\sum_{j=1}^{s} E(e^2_{r+(i-1)k+(j-1)k_1})\\
&+na_1(r)\Bigg\{E(e_r^2) - E(e^2_{r+(m-1)k+(s-1)k_1})\Bigg\}\Bigg\}\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
E\,(SS1) = \sigma^2\Bigg[&\frac{1}{n^2}\Bigg\{\sum_{i=1}^{m}\sum_{j=1}^{s}(r+(i-1)k+(j-1)k_1)^g\\
&+n^2 a_1^2(r)\left\{r^g + (r+(m-1)k+(s-1)k_1)^g\right\}\\
&+2na_1(r)\left\{r^g - (r+(m-1)k+(s-1)k_1)^g\right\}\Bigg\}\\
&-\frac{2}{nN}\Bigg\{\sum_{i=1}^{m}\sum_{j=1}^{s}(r+(i-1)k+(j-1)k_1)^g\\
&+na_1(r)\left\{r^g - (r+(m-1)k+(s-1)k_1)^g\right\}\Bigg\}\\
&+\frac{1}{N^2}\sum_{t=1}^{N} t^g\Bigg].
\end{aligned}
$$

$$
\begin{aligned}
E\,(SS1) = \sigma^2\Bigg\{&a_1(r)\left(a_1(r) + 2\left(\frac{1}{n} - \frac{1}{N}\right)\right) r^g\\
&+\frac{1}{n}\left(\frac{1}{n} - \frac{2}{N}\right)\sum_{i=1}^{m}\sum_{j=1}^{s}\{r+(i-1)k+(j-1)k_1\}^g\\
&+a_1(r)\left(a_1(r) - 2\left(\frac{1}{n} - \frac{1}{N}\right)\right)\{r+(m-1)k+(s-1)k_1\}^g\\
&+\frac{1}{N^2}\sum_{t=1}^{N} t^g\Bigg\}.
\end{aligned}
\tag{B.4}
$$

Similarly,

$$
\begin{aligned}
E\,(SS2) = \Bigg[&\frac{1}{n^2}\Bigg\{\sum_{i=1}^{m-u}\sum_{j=1}^{s} E(e^2_{r+(i-1)k+(j-1)k_1})\\
&+\sum_{i=m-u+1}^{m}\Bigg(\sum_{j=1}^{s} E(e^2_{r+(i-1)k+(j-1)k_1})\\
&+E(e^2_{r+(i-1)k+(s-1)k_1-N})\Bigg) + n^2 a_2^2(r)\Bigg\{E(e_r^2)\\
&+E(e^2_{r+(m-1)k+(s-1)k_1-N})\Bigg\} + 2na_2(r)\Bigg\{(E(e_r^2)\\
&-E(e^2_{r+(m-1)k+(s-1)k_1-N})\Bigg\}\Bigg\}\\
&-\frac{2}{nN}\Bigg\{\sum_{i=1}^{m-u}\sum_{j=1}^{s} E(e^2_{r+(i-1)k+(j-1)k_1})\\
&+\sum_{i=m-u+1}^{m}\Bigg(\sum_{j=1}^{s} E(e^2_{r+(i-1)k+(j-1)k_1})\\
&+E(e^2_{r+(i-1)k+(s-1)k_1-N})\Bigg)\\
&+ na_2(r)\Bigg\{(E(e_r^2) - E(e^2_{r+(m-1)k+(s-1)k_1-N})\Bigg\}\Bigg\}\\
&+\frac{1}{N^2}\sum_{t=1}^{N} E(e_t^2)\Bigg],
\end{aligned}
$$

$$E\left(SS2\right)=\Big[\tfrac{1}{n^2}\Big\{\sum_{i=1}^{m-u}\sum_{j=1}^{s}\left(r+(i-1)k+(j-1)k_1\right)^g$$
$$+\sum_{i=m-u+1}^{m}\Big(\sum_{j=1}^{s-1}\left(r+(i-1)k+(j-1)k_1\right)^g$$
$$+\left(r+(i-1)k+(s-1)k_1-N\right)^g\Big)+n^2a_2^2(r)\Big\{r^g$$
$$+\left(r+(m-1)k+(s-1)k_1-N\right)^g\Big\}$$
$$+2na_2(r)\left\{r^g-\left(r+(m-1)k+(s-1)k_1-N\right)^g\right\}\Big\}$$
$$-\tfrac{2}{nN}\Big\{\sum_{i=1}^{m-u}\sum_{j=1}^{s}\left(r+(i-1)k+(j-1)k_1\right)^g.$$
$$+\sum_{i=m-u+1}^{m}\Big(\sum_{j=1}^{s-1}\left(r+(i-1)k+(j-1)k_1\right)^g$$
$$+\left(r+(i-1)k+(s-1)k_1-N\right)^g\Big)$$
$$+\,na_2(r)\left\{r^g-\left(r+(m-1)k+(s-1)k_1-N\right)^g\right\}\Big\}$$
$$+\tfrac{1}{N^2}\sum_{t=1}^{N}t^g\Big],$$

(B.5)
$$E\left(SS2\right)=\Big[a_2(r)\left(a_2(r)+2\left(\tfrac{1}{n}-\tfrac{1}{N}\right)\right)r^g$$
$$+\tfrac{1}{n}\left(\tfrac{1}{n}-\tfrac{2}{N}\right)\tfrac{1}{n^2}\Big\{\sum_{i=1}^{m-u}\sum_{j=1}^{s}\left(r+(i-1)k+(j-1)k_1\right)^g$$
$$+\sum_{i=m-u+1}^{m}\Big(\sum_{j=1}^{s-1}\left(r+(i-1)k+(j-1)k_1\right)^g$$
$$+\left(r+(i-1)k+(s-1)k_1-N\right)^g\Big)\Big\}$$
$$+a_2(r)\left(a_2(r)-2\left(\tfrac{1}{n}-\tfrac{1}{N}\right)\right)\Big(r+(m-1)k+(s-1)k_1$$
$$-N\Big)^g+\tfrac{1}{N^2}\sum_{t=1}^{N}t^g\Big].$$

Equations (B.4) and (B.5) can be written as:

$$E\left(SS1\right)=\sigma^2\Big\{\delta_1^+(r)r^g+\theta\sum_{i=1}^{m}\sum_{j=1}^{s}\{r+(i-1)k+(j-1)k_1\}^g$$
$$+\delta_1^-(r)\{r+(m-1)k+(s-1)k_1\}^g+\tfrac{1}{N^2}\sum_{t=1}^{N}t^g\Big\}$$

and

$$E\left(SS2\right)=\sigma^2\Big[\delta_2^+(r)r^g+\theta\tfrac{1}{n^2}\Big\{\sum_{i=1}^{m-u}\sum_{j=1}^{s}\left(r+(i-1)k+(j-1)k_1\right)^g$$
$$+\sum_{i=m-u+1}^{m}\Big(\sum_{j=1}^{s-1}\left(r+(i-1)k+(j-1)k_1\right)^g.$$
$$+\left(r+(i-1)k+(s-1)k_1-N\right)^g\Big\}$$
$$+\delta_2^-(r)\left(r+(m-1)k+(s-1)k_1-N\right)^g+\tfrac{1}{N^2}\sum_{t=1}^{N}t^g\Big],$$

where $\delta_l^+(r)=a_l(r)\{a_l(r)+2\left(\frac{1}{n}-\frac{1}{N}\right)\}$, $\delta_l^-(r)=a_l(r)\{a_l(r)-2\left(\frac{1}{n}-\frac{1}{N}\right)\}$ and $\theta=\frac{1}{n}\left(\frac{1}{n}-\frac{2}{N}\right)$, such that $l=1,2$. Also

(B.6)
$$E\left(SS1\right)=\sigma^2\left\{\chi_1\left(u,r\right)+\tfrac{1}{N^2}\sum_{t=1}^{N}t^g\right\}$$

and

(B.7)
$$E\left(SS2\right)=\sigma^2\left\{\chi_2\left(u,r\right)+\tfrac{1}{N^2}\sum_{t=1}^{N}t^g\right\},$$

where

$$\chi_1(u,r) = \delta_1^+(r)r^g + \theta \sum_{i=1}^{m} \sum_{j=1}^{s} \{r+(i-1)k+(j-1)k_1\}^g \\ +\delta_1^-(r)\{r+(m-1)k+(s-1)k_1\}^g$$

,

and

$$\chi_2(u,r) = \delta_2^+(r)r^g + \theta \frac{1}{n^2}\Big\{ \sum_{i=1}^{m-u} \sum_{j=1}^{s} (r+(i-1)k+(j-1)k_1)^g \\ + \sum_{i=m-u+1}^{m} \Big(\sum_{j=1}^{s-1} (r+(i-1)k+(j-1)k_1)^g \\ + (r+(i-1)k+(s-1)k_1-N)^g \Big) \Big\} \\ +\delta_2^-(r)\,(r+(m-1)k+(s-1)k_1-N)^g .$$

Substituting the values of $E(SS1)$ and $E(SS2)$ in (B.3), we have

(B.8)
$$E\left\{V(\bar{y}_{MSS}^{(r)})\right\} = \frac{\sigma^2}{k_1}\Big\{ \sum_{r=1}^{k_1-(m-1)k} \chi_1(u,r) \\ + \sum_{u=1}^{m-1} \sum_{r=k_1-(m-u)k+1}^{k_1-(m-u-1)k} \chi_2(u,r) \\ + k_1 \sum_{t=1}^{N} t^g \Big/ N^2 \Big\}.$$

Case (ii) when $w > 1$

We can write

(B.9)
$$SSl = \left[\bar{y}_{MSS}^{(r)} - \bar{Y}\right]^2 = \left[\{\bar{y}_{MSS} + a_l(r)\,(Y_{r1} - Y_{rn})\} - \bar{Y}\right]^2,$$

where $l = 2$ if $r \leq k_1 - (w-1)k + r_m$, $l = 3$ if $k_1 - (w-x)k < r \leq k_1 - (w-x)k + r_m$ such that $x = 2, ..., (m-1)$
and $l = 4$ if $k_1 - (w-x)k + r_m < r \leq k_1 - (w-x)k + k$ such that $x = 1, 2, ..., m-1$. Furthermore, when $r \leq k_1 - (w-1)k + r_m$, we realize whether $1 \leq r \leq k_1 - (m-u-1)k$ such that $u = w-1$, $k_1 - (m-u)k < r \leq k_1 - (m-u-1)k$ such that $u = w, w+1, ..., (m-w)$ or $k_1 - (m-u)k < r \leq k_1 - (m-u)k + r_m$ such that $u = (m-w+1)$. However, for each of these subgroups $E(SS2)$ will be used. Thus, the average variance of the corrected sample mean can be expressed as

(B.10)
$$E\left[V\left(\bar{y}_{MSS}^{(r)}\right)\right] = \frac{1}{N}\Big[\sum_{u=w-1}^{w-1} \sum_{r=1}^{k_1-(m-u-1)k} E[SS2] \\ + \sum_{u=w}^{n-w} \sum_{r=k_1-(m-u)k+1}^{k_1-(m-u-1)k} E[SS2] \\ + \sum_{u=m-w+1}^{m-w+1} \sum_{r=k_1-(m-u)k+1}^{k_1-(m-u)k+r_m} E[SS2] \\ + \sum_{x=2}^{w-1} \sum_{r=k_1-(w-x)k+1}^{k_1-(w-x)k+r_m} E[SS3] \\ + \sum_{x=1}^{w-1} \sum_{r=k_1-(w-x)k+r_m+1}^{k_1-(w-x)k+k} E[SS4] \Big].$$

The $E(SS2)$ is already obtained in case of $w = 1$, i.e.

(B.11)
$$E(SS2) = \chi_2(u,r) + \frac{1}{N^2} \sum_{t=1}^{N} t^g.$$

Now consider

(B.12)
$$E(SS3) = E\left[\{\bar{y}_{MSS} + a_3(r)\,(Y_{r1} - Y_{rn})\} - \overline{Y}\right]^2.$$

Under the super population model, we have

$$\begin{aligned}\bar{y}_{MSS} &= \alpha + \beta\Big\{r + \tfrac{1}{2}\{(s-1)k_1 + (m-1)k\} - (m-w-1+2x)\tfrac{k_1}{m}\Big\} \\ &+ \tfrac{1}{n}\Big[\textstyle\sum_{i=1}^{w-x}\sum_{j=1}^{s} e_{r+(i-1)k+(j-1)k_1} \\ &+ \textstyle\sum_{i=w-x+1}^{m-x+1}\Big\{\sum_{j=1}^{s-1} e_{r+(i-1)k+(j-1)k_1} + e_{r+(i-1)k+(s-1)k_1-N}\Big\} \\ &+ \textstyle\sum_{i=m-x+2}^{m}\Big\{\sum_{j=1}^{s-2} e_{r+(i-1)k+(j-1)k_1} \\ &+ \textstyle\sum_{j=s-1}^{s} e_{r+(i-1)k+(s-1)k_1-N}\Big\}\Big],\end{aligned}$$

$\bar{Y} = \alpha + \beta\frac{N+1}{2} + \frac{1}{N}\sum_{t=1}^{N} e_t$, $Y_{r1} = \alpha + \beta r + e_r$ and $Y_{rn} = \alpha + \beta\{r + (m-1)k + (s-1)k_1\} + e_{r+(m-1)k+(s-1)k_1}$.
Substituting these expressions in (B.11), we have

$$\begin{aligned}E\,(SS3) &= E\Big[\tfrac{1}{n}\Big\{\textstyle\sum_{i=1}^{w-x}\sum_{j=1}^{s} e_{r+(i-1)k+(j-1)k_1} \\ &+ \textstyle\sum_{i=w-x+1}^{m-x+1}\Big\{\sum_{j=1}^{s-1} e_{r+(i-1)k+(j-1)k_1} + e_{r+(i-1)k+(s-1)k_1-N}\Big\} \\ &+ \textstyle\sum_{i=m-x+2}^{m}\Big\{\sum_{j=1}^{s-2} e_{r+(i-1)k+(j-1)k_1} + \sum_{j=s-1}^{s} e_{r+(i-1)k+(j-1)k_1-N}\Big\} \\ &+ na_2(r)(e_r - e_{r+(m-1)k+(s-1)k_1-N})\Big\} - \tfrac{1}{N}\textstyle\sum_{t=1}^{N} e_t\Big]^2.\end{aligned}$$

Applying the assumption of super population model, we have

$$\begin{aligned}E\,(SS3) =&\Big[\tfrac{1}{n^2}\Big\{\textstyle\sum_{i=1}^{w-x}\sum_{j=1}^{s} E(e^2_{r+(i-1)k+(j-1)k_1}) \\ &+ \textstyle\sum_{i=w-x+1}^{m-x+1}\Big\{\sum_{j=1}^{s-1} E(e^2_{r+(i-1)k+(j-1)k_1}) \\ &+ E(e^2_{r+(i-1)k+(s-1)k_1-N})\Big\} \\ &+ \textstyle\sum_{i=m-x+2}^{m}\Big\{\sum_{j=1}^{s-2} E(e^2_{r+(i-1)k+(j-1)k_1}) \\ &+ \textstyle\sum_{j=s-1}^{s} E(e^2_{r+(i-1)k+(j-1)k_1-N})\Big\} \\ &+ n^2a_2^2(r)\Big\{E(e_r^2) + E(e^2_{r+(m-1)k+(s-1)k_1-N})\Big\} \\ &+ 2na_2(r)\Big\{E(e_r^2) - E(e^2_{r+(m-1)k+(s-1)k_1-N})\Big\}\Big\} \\ &- 2\tfrac{1}{nN}\Big\{\textstyle\sum_{i=1}^{w-x}\sum_{j=1}^{s} E(e^2_{r+(i-1)k+(j-1)k_1}) \\ &+ \textstyle\sum_{i=w-x+1}^{m-x+1}\Big\{\sum_{j=1}^{s-1} E(e^2_{r+(i-1)k+(j-1)k_1}) \\ &+ E(e^2_{r+(i-1)k+(s-1)k_1-N}) \\ &+ \textstyle\sum_{i=m-x+2}^{m}\Big\{\sum_{j=1}^{s-2} E(e^2_{r+(i-1)k+(j-1)k_1}) \\ &+ \textstyle\sum_{j=s-1}^{s} E(e^2_{r+(i-1)k+(j-1)k_1-N})\Big\} \\ &+ na_2(r)\Big\{E(e_r^2) - E(e^2_{r+(m-1)k+(s-1)k_1-N})\Big\}\Big\} \\ &+ \tfrac{1}{N^2}\textstyle\sum_{t=1}^{N} E(e_t^2)\Big].\end{aligned}$$

$$
\begin{aligned}
E\left(SS3\right) = & \Big[\tfrac{1}{n^2}\Big\{\textstyle\sum_{i=1}^{w-x}\sum_{j=1}^{s}\left(r+(i-1)k+(j-1)k_1\right)^g \\
& +\textstyle\sum_{i=w-x+1}^{m-x+1}\Big\{\sum_{j=1}^{s-1}\left(r+(i-1)k+(j-1)k_1\right)^g \\
& +\left(r+(i-1)k+(j-1)k_1-N\right)^g\Big\} \\
& +\textstyle\sum_{i=m-x+2}^{m}\Big\{\sum_{j=1}^{s-2}\left(r+(i-1)k+(j-1)k_1\right)^g \\
& +\textstyle\sum_{j=s-1}^{s}\left(r+(i-1)k+(j-1)k_1-N\right)^g\Big\} \\
& +n^2a_2^2(r)\left\{r^g+\left(r+(i-1)k+(j-1)k_1-N\right)^g\right\} \\
& +2na_2(r)\left\{r^g-\left(r+(i-1)k+(j-1)k_1-N\right)^g\right\}\Big\} \\
& -2\tfrac{1}{nN}\Big\{\textstyle\sum_{i=1}^{w-x}\sum_{j=1}^{s}\left(r+(i-1)k+(j-1)k_1\right)^g \\
& +\textstyle\sum_{i=w-x+1}^{m-x+1}\Big\{\sum_{j=1}^{s-1}\left(r+(i-1)k+(j-1)k_1\right)^g \\
& +\left(r+(i-1)k+(j-1)k_1-N\right)^g\Big\} \\
& +\textstyle\sum_{i=m-x+2}^{m}\Big\{\sum_{j=1}^{s-2}\left(r+(i-1)k+(j-1)k_1\right)^g \\
& +\textstyle\sum_{j=s-1}^{s}\left(r+(i-1)k+(j-1)k_1-N\right)^g\Big\} \\
& +na_2(r)\left\{r^g-\left(r+(i-1)k+(j-1)k_1-N\right)^g\right\}\Big\} \\
& +\tfrac{1}{N^2}\textstyle\sum_{t=1}^{N}t^g\Big],
\end{aligned}
$$

$$
\begin{aligned}
E\left(SS3\right) = & a_3(r)\left(a_3(r)+2\left(\tfrac{1}{n}-\tfrac{1}{N}\right)\right)r^g \\
& +\textstyle\sum_{x=2}^{w-1}\sum_{r=k_1-(w-x)k+1}^{k_1-(w-x)k+r_1}\Big\{\tfrac{1}{n}\left(\tfrac{1}{n}-\tfrac{2}{N}\right)\Big\{\sum_{i=1}^{w-x}\sum_{j=1}^{s}\{r \\
& +(i-1)k+(j-1)k_1\}^g \\
& +\textstyle\sum_{i=w-x+1}^{m-x+1}\Big(\sum_{j=1}^{s-1}\{r+(i-1)k+(j-1)k_1\}^g \\
& +\{r+(i-1)k+(s-1)k_1-N\}^g\Big) \\
& +\textstyle\sum_{i=m-x+2}^{m}\Big(\sum_{j=1}^{s-2}\{r+(m-1)k+(j-1)k_1\}^g \\
& +\textstyle\sum_{j=s-1}^{s}\{r+(i-1)k+(j-1)k_1-N\}^g\Big)\Big\} \\
& +a_3(r)\left(a_3(r)-2\left(\tfrac{1}{n}-\tfrac{1}{N}\right)\right)\{r+(m-1)k+(s-1)k_1-N\}^g \\
& +\tfrac{1}{N^2}\textstyle\sum_{t=1}^{N}t^g,
\end{aligned}
$$

$$
\begin{aligned}
E\left(SS3\right) = & \delta_3^+(r)r^g+\theta\Big\{\textstyle\sum_{i=1}^{w-x}\sum_{j=1}^{s}\{r+(i-1)k+(j-1)k_1\}^g \\
& +\textstyle\sum_{i=w-x+1}^{m-x+1}\Big(\sum_{j=1}^{s-1}\{r+(i-1)k+(j-1)k_1\}^g \\
& +\{r+(i-1)k+(s-1)k_1-N\}^g\Big) \\
& +\textstyle\sum_{i=m-x+2}^{m}\Big(\sum_{j=1}^{s-2}\{r+(m-1)k+(j-1)k_1\}^g \\
& +\textstyle\sum_{j=s-1}^{s}\{r+(i-1)k+(j-1)k_1-N\}^g\Big)\Big\} \\
& +\delta_3^-(r)\{r+(m-1)k+(s-1)k_1-N\}^g+\tfrac{1}{N^2}\textstyle\sum_{t=1}^{N}t^g,
\end{aligned}
$$

where

$$\delta_3^+(r)=a_3(r)\{a_3(r)+2\left(\tfrac{1}{n}-\tfrac{1}{N}\right)\} \text{ and } \delta_3^-(r)=a_3(r)\{a_3(r)-2\left(\tfrac{1}{n}-\tfrac{1}{N}\right)\}.$$

Also

(B.13) $E\left(SS3\right)=\chi_3\left(x,r\right)+\frac{1}{N^2}\sum_{t=1}^{N}t^g,$

where

$$\begin{aligned}\chi_3\left(x,r\right)=&\,\delta_3^{+}(r)r^g+\theta\Big\{\sum_{i=1}^{w-x}\sum_{j=1}^{s}\{r+(i-1)k+(j-1)k_1\}^g\\&+\sum_{i=w-x+1}^{m-x+1}\Big(\sum_{j=1}^{s-1}\{r+(i-1)k+(j-1)k_1\}^g\\&+\{r+(i-1)k+(s-1)k_1-N\}^g\Big)\\&+\sum_{i=m-x+2}^{m}\Big(\sum_{j=1}^{s-2}\{r+(m-1)k+(j-1)k_1\}^g\\&+\sum_{j=s-1}^{s}\{r+(i-1)k+(j-1)k_1-N\}^g\Big)\Big\}\\&+\delta_3^{-}(r)\{r+(m-1)k+(s-1)k_1-N\}^g.\end{aligned}$$

Similarly,

(B.14) $E\left(SS4\right)=\chi_4\left(x,r\right)+\frac{1}{N^2}\sum_{t=1}^{N}t^g,$

where

$$\begin{aligned}\chi_4\left(x,r\right)=&\,\delta_4^{+}(r)r^g+\theta\Big\{\sum_{i=1}^{w-x}\sum_{j=1}^{s}\{r+(i-1)k+(j-1)k_1\}^g\\&+\sum_{i=w-x+1}^{m-x}\Big(\sum_{j=1}^{s-1}\{r+(i-1)k+(j-1)k_1\}^g\\&+\{r+(i-1)k+(s-1)k_1-N\}^g\Big)\\&+\sum_{i=m-x+1}^{m}\Big(\sum_{j=1}^{s-2}\{r+(m-1)k+(j-1)k_1\}^g\\&+\sum_{j=s-1}^{s}\{r+(i-1)k+(j-1)k_1-N\}^g\Big)\Big\}\\&+\delta_4^{-}(r)\{r+(m-1)k+(s-1)k_1-N\}^g.\end{aligned}$$

Putting $E\left(SSl\right)$ for $l=2,\,3,\,4$ in (B.10), we have

$$\begin{aligned}E\left\{V\left(\bar{y}_{MSS}^{(r)}\right)\right\}=&\,\frac{\sigma^2}{k_1}\Big[\sum_{u=w-1}^{w-1}\sum_{r=1}^{k_1-(m-u-1)k}\chi_2\left(u,r\right)\\&+\sum_{u=w}^{m-w}\sum_{r=k_1-(m-u)k+1}^{k_1-(m-u-1)k}\chi_2\left(u,r\right)\\&+\sum_{u=m-w+1}^{m-w+1}\sum_{r=k_1-(m-u)k+1}^{k_1-(m-u-1)k}\chi_2\left(u,r\right)\\&+\sum_{x=2}^{w-1}\sum_{r=k_1-(w-x)k+1}^{k_1-(w-x)k+r_m}\chi_3\left(x,r\right)\\&+\sum_{x=1}^{w-1}\sum_{r=k_1-(w-x)k+r_m+1}^{k_1-(w-x)k+k}\chi_4\left(x,r\right)\\&+k_1\sum_{t=1}^{N}t^g\Big/N^2\Big].\end{aligned}$$