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A new generalized intuitionistic fuzzy set

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Abstract

A generalized intuitionistic fuzzy set ($GIFS_B$) is proposed. It is shown that Atanassov's intuitionistic fuzzy set, intuitionistic fuzzy sets of root type and intuitionistic fuzzy sets of second type are special cases of this new one. Some important notions, basic algebraic properties of $GIFS_B$, three operators and their relationship are discussed. The algebraic properties include being closed under union, being closed under intersection, being closed under a necessity measure, being closed under a possibility measure and de Morgan type identities.

Keywords: Generalized intuitionistic fuzzy set, Intuitionistic fuzzy set, Intuitionistic fuzzy sets of root type, Intuitionistic fuzzy sets of second type.

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1. Introduction

The concept of fuzzy sets was introduced by Zadeh [22] whose basic component is only a degree of membership. Atanassov [2] generalized this idea to intuitionistic fuzzy sets (IFS) using a degree of membership and a degree of non-membership, under the constraint that the sum of the two degrees does not exceed one. A fuzzy set can be considered as IFS, since the sum of these grades is one. However, there are different situations when the sum of two degrees is smaller than one, which means that there is a certain ambiguity in the decision of membership or non-membership. For such cases the IFS is an appropriate tool.

A generalized intuitionistic fuzzy set (GIFS) were proposed by Mondal and Samanta [14] under the constraint that the minimum of the two degrees does not exceed half. Following the definition of IFS, Atanassov [3] [4] and Atanassov and Gargov [5] introduced

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interval valued IFSs, IFSs of second type, and temporal IFS. Srinivasan and Palaniappan [19] introduced IFSs of root type.

Some other extensions of the IFSs have also been introduced: IF soft sets due to Maji et al. [12]; IF rough sets due to Samanta and Mondal [18]; rough IFSs due to Rizvi et al. [17].

Some recent applications of IFSs have been: sustainable energy planning in Malaysia (Abdullah and Najib [1]); image fusion (Balasubramaniam and Ananthi [6]); agricultural production planning from a small farm holder perspective (Bharati and Singh [7]); medical diagnosis (Bora et al. [8]); pattern recognition (Chu et al. [9]); reservoir flood control operation (Hashemi et al. [10]); reliability optimization of complex system (Mahapatra and Roy [11]); fault diagnosis using dissolved gas analysis for power transformer (Mani and Jerome [13]); prioritizing the components of SWOT matrix in the Iranian insurance industry (Nikjoo and Saeedpoor [15]); prediction of the best quality of two-wheelers (Pathinathan et al. [16]); study of the decision framework of wind farm project plan selection (Wu et al. [21]).

The aim of this paper is to introduce new generalized IFSs and to derive their properties. The derived properties include: i) if A and B are generalized IFSs then their union and intersection are also generalized IFSs; ii) if A, B and C are generalized IFSs, A is a subset of B and B is a subset of C then A is a subset of C; iii) if A is a generalized IFS then its necessity and possibility measures are also generalized IFSs; iv) if the degree of non-determinacy of an element of a generalized IFS is zero then that for the nth power of the set is also zero; v) if A is a generalized IFS then the necessity measure of the nth power of A is the same as the nth power of the necessity measure of A; vi) if A is a generalized IFS then the possibility measure of the nth power of A is the same as the nth power of the necessity measure of A; vii) if A is a generalized IFS and $m \geq n$ then the mth power of A is a subset of the nth power of A; viii) if A is a generalized IFS and $m \geq n$ then nA is a subset of mA; ix) if A and B are generalized IFSs and A is a subset of B then nA is a subset of nB; x) if A and B are generalized IFSs and A is a subset of B then the nth power of A is a subset of nth power of B; xi) if A and B are generalized IFSs then the nth power of the union of A and B is the same as the union of the nth powers of A and B; xii) if A and B are generalized IFSs then the nth power of the intersection of A and B is the same as the intersection of the nth powers of A and B; xiii) if A and B are generalized IFSs then n times the union of A and B is the same as the union of nA and nB; xiv) if A and B are generalized IFSs then n times the intersection of A and B is the same as the intersection of nA and nB.

2. Preliminaries

In this section, we give some definitions of various types of IFS. We also define triangular norms and triangular conorms. Let X denote a non-empty set.

1. Definition. (Atanassov [2]). An IFS A in X is defined as an object of the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$, where the functions $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ denote, respectively, the degree of membership and degree of non-membership functions of A, and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$.

2. Definition. (Atanassov [3]). An intuitionistic fuzzy set of second type (IFSST) A in X is defined as an object of the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$, where the functions $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ denote, respectively, the degree of membership and degree of non-membership functions of A, and $0 \leq [\mu_A(x)]^2 + [\nu_A(x)]^2 \leq 1$ for each $x \in X$.

3. Definition. (Srinivasan and Palaniappan [20]). An intuitionistic fuzzy set of root type (IFSRT) A in X is defined as an object of the form

$$A = \{\langle x, \mu_A(x), \nu_A(x) : x \in X\rangle\},$$

where the functions $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ denote, respectively, the degree of membership and degree of non-membership functions of A, and $0 \leq \frac{1}{2}\sqrt{\mu_A(x)} + \frac{1}{2}\sqrt{\nu_A(x)} \leq 1$ for each $x \in X$.

4. Definition. (Atanassov [4]). A temporal IFS A in X is defined as an object of the form $A(T) = \{(x,t), \mu_A(x,t), \nu_A(x,t) : (x,t) \in E \times T\}$, where the functions $\mu_A(x,t)$ and $\nu_A(x,t)$ denote, respectively, the degree of membership and degree of non-membership functions of A of the element $x \in X$ at the time-moment $t \in T$, $A \subset E$ is a fixed set and $0 \leq \mu_A(x,t) + \nu_A(x,t) \leq 1$ for each $(x,t) \in E \times T$.

5. Definition. A triangular norm is a binary operation on $[0,1]$, i.e., an operator $T : [0,1]^2 \to [0,1]$ such that for all $x, y, z \in [0,1]$ the following conditions are satisfied:

i) Communicativity: $T(x,y) = T(y,x)$,
ii) Associativity: $T(x, T(y,z)) = T(T(x,y), z)$,
iii) Monotonicity: $T(x,y) \leq T(x,z)$ whenever $y \leq z$,
iv) Boundary condition: $T(x,1) = x$.

6. Definition. A triangular conorm is a binary operation on $[0,1]$, i.e., an operator $S : [0,1]^2 \to [0,1]$ such that for all $x, y, z \in [0,1]$ the following conditions are satisfied:

i) Communicativity: $S(x,y) = S(y,x)$,
ii) Associativity: $S(x, T(y,z)) = S(T(x,y), z)$,
iii) Monotonicity: $S(x,y) \leq S(x,z)$ whenever $y \leq z$,
iv) Boundary condition: $T(x,0) = x$.

Generalized fuzzy intuitionistic metric spaces can be defined based on triangular norms and triangular conorms.

3. New generalized intuitionistic fuzzy sets

7. Definition. Let X denote a non-empty set. Our generalized IFS A in X is defined as an object of the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$, where the functions $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ denote, respectively, the degree of membership and degree of non-membership functions of A, and $0 \leq \mu_A(x)^\delta + \nu_A(x)^\delta \leq 1$ for each $x \in X$ and $\delta = n$ or $\frac{1}{n}, n = 1, 2, \ldots, N$. The collection of all of our generalized IFSs is denoted by $GIFS_B(\delta, X)$.

One of the geometrical interpretations of the $GIFS_B(\delta, X)$ is shown in Figures 1 and 2. Let X denote a universal set and F a subset in the Euclidean plane with cartesian coordinates. For a $GIFS_B A$, a function f_A from X to F can be constructed such that if $x \in X$ then $p = (\nu_A(x), \mu_A(x)) = f_A(x) \in F$, $0 \leq \mu_A(x)$, $\nu_A(x) \leq 1$.

Let X be a set of ages of men over $[0, 75]$. Let A be a set of young men whose ages are between 20 and 30. Define the membership and non membership functions of A as

$$\mu_A(x) = \begin{cases} \left(\dfrac{x-10}{10}\right)^{1/2}, & \text{if } 10 \leq x \leq 20, \\ 1, & \text{if } 20 \leq x \leq 30, \\ \left(\dfrac{40-x}{10}\right)^{1/2}, & \text{if } 30 \leq x \leq 40, \\ 0, & \text{otherwise,} \end{cases}$$

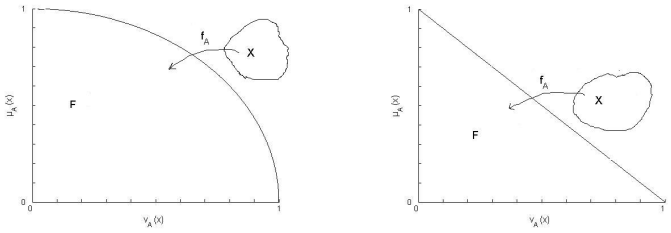

Figure 1. A geometrical interpretation of $GIFS_B$ with $\delta = 1$ and 2.

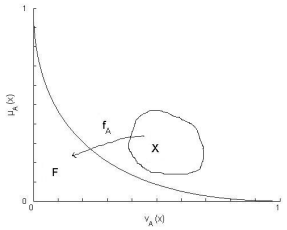

Figure 2. A geometrical interpretation of $GIFS_B$ with $\delta = 0.5$.

and

$$\nu_A(x) = \begin{cases} \left(\dfrac{20-x}{15}\right)^{1/2}, & \text{if } 5 \le x \le 20, \\ 0, & \text{if } 20 \le x \le 30, \\ \left(\dfrac{x-30}{15}\right)^{1/2}, & \text{if } 30 \le x \le 45, \\ 1, & \text{otherwise.} \end{cases}$$

Since $0 \le \mu_A(x)^2 + \nu_A(x)^2 \le 1$, $\forall x \in X$, $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ is a $GIFS_B(2)$.

Also, we can define the membership and non membership functions of A as

$$\mu_A(x) = \begin{cases} \left(\dfrac{x-10}{10}\right)^2, & \text{if } 10 \le x \le 20, \\ 1, & \text{if } 20 \le x \le 30, \\ \left(\dfrac{40-x}{10}\right)^2, & \text{if } 30 \le x \le 40, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\nu_A(x) = \begin{cases} \left(\dfrac{20-x}{15}\right)^2, & \text{if } 5 \le x \le 20, \\ 0, & \text{if } 20 \le x \le 30, \\ \left(\dfrac{x-30}{15}\right)^2, & \text{if } 30 \le x \le 45, \\ 1, & \text{otherwise.} \end{cases}$$

Since $0 \le \mu_A(x)^{0.5} + \nu_A(x)^{0.5} \le 1$, $\forall x \in X$, $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ is a $GIFS_B(0.5)$.

3.1. Remark. It is obvious that for all real numbers $\alpha, \beta \in [0,1]$,

(i) if $0 \le \alpha+\beta \le 1$ and $\delta \ge 1$ then we have $0 \le \alpha^\delta+\beta^\delta \le 1$. With this consideration if $A \in IFS$ then $A \in GIFS_B$.
(ii) if $0 \le \alpha^\delta + \beta^\delta \le 1$ and $\delta \le 1$ then $0 \le \alpha + \beta \le 1$. With this consideration if $A \in GIFS_B$ then $A \in IFS$.
(iii) if $\delta_1 \le \delta_2$ then $\alpha^{\delta_2} \le \alpha^{\delta_1}$ and $\beta^{\delta_2} \le \beta^{\delta_1}$. It follows that $GIFS_B(\delta_1) \subset GIFS_B(\delta_2)$.

3.2. Remark. $GIFS_B(1) = IFS$, $GIFS_B(2) = GIFSST$, and $GIFS_B\left(\frac{1}{2}\right) = GIFSRT$.

8. Definition. Let X denote a non-empty set. Let A and B denote two $GIFS_B$s such that $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ and $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle : x \in X\}$. Define the following relations and operations on A and B:

i. $A \subset B$ if and only if $\mu_A(x) \le \mu_B(x)$ and $\nu_A(x) \ge \nu_A(x)$, $\forall x \in X$,
ii. $A = B$ if and only if $\mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x)$, $\forall x \in X$,
iii. $A \cup B = \{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))\rangle : x \in X\}$,
iv. $A \cap B = \{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))\rangle : x \in X\}$,
v. $A + B = \left\{\left\langle x, \mu_A(x)^\delta + \mu_B(x)^\delta - \mu_A(x)^\delta\mu_B(x)^\delta, \nu_A(x)^\delta\nu_B(x)^\delta\right\rangle : x \in X\right\}$, so

$$2A = \left\{\left\langle x, 1 - \left(1 - \mu_A(x)^\delta\right)^2, \nu_A(x)^{2\delta}\right\rangle : x \in X\right\}$$

and

$$nA = \left\{\left\langle x, 1 - \left(1 - \mu_A(x)^\delta\right)^n, \nu_A(x)^{n\delta}\right\rangle : x \in X\right\},$$

vi. $A.B = \left\{\left\langle x, \mu_A(x)^\delta.\mu_B(x)^\delta, \nu_A(x)^\delta + \nu_B(x)^\delta - \nu_A(x)^\delta\nu_B(x)^\delta\right\rangle : x \in X\right\}$, so

$$A^2 = \left\{\left\langle x, \mu_A(x)^{2\delta}, 1 - \left(1 - \nu_A(x)^\delta\right)^2\right\rangle : x \in X\right\}$$

and

$$A^n = \left\{\left\langle x, \mu_A(x)^{n\delta}, 1 - \left(1 - \nu_A(x)^\delta\right)^n\right\rangle : x \in X\right\},$$

vii. $\overline{A} = \{\langle x, \nu_A(x), \mu_A(x)\rangle : x \in X\}$.

Proposition 3.1 For $A, B, C \in GIFS_B$, we have

i. $\overline{\overline{A}} = A$,
ii. $A \subset B$, $B \subset C \Rightarrow A \subset C$.

Proof. The proof is obvious. □

Proposition 3.2 For $A, B \in GIFS_B$, we have

i. $A \cup B \in GIFS_B$,
ii. $A \cap B \in GIFS_B$,

iii. $\delta \geq 1 \Rightarrow A+B \in GIFS_B$, $\delta < 1 \Rightarrow A+B \in IFS$,
iv. $\delta \geq 1 \Rightarrow A.B \in GIFS_B$, $\delta < 1 \Rightarrow A.B \in IFS$.

Proof. (i) Suppose $\max(\mu_A(x), \mu_B(x)) = \mu_A(x)$. Since $\min(\nu_A(x), \nu_B(x)) \leq \nu_A(x)$, we have

$$\begin{aligned} 0 &\leq \mu_{A\cup B}(x)^\delta + \nu_{A\cup B}(x)^\delta \\ &= (\max(\mu_A(x), \mu_B(x)))^\delta + (\min(\nu_A(x), \nu_B(x)))^\delta \\ &= \mu_A(x)^\delta + (\min(\nu_A(x), \nu_B(x)))^\delta \\ &\leq \mu_A(x)^\delta + \nu_A(x)^\delta \leq 1. \end{aligned}$$

Suppose now $\max(\mu_A(x), \mu_B(x)) = \mu_B(x)$. Since $\min(\nu_A(x), \nu_B(x)) \leq \nu_B(x)$, we have

$$\begin{aligned} 0 &\leq (\max(\mu_A(x), \mu_B(x)))^\delta + (\min(\nu_A(x), \nu_B(x)))^\delta \\ &= \mu_B(x)^\delta + (\min(\nu_A(x), \nu_B(x)))^\delta \\ &\leq \mu_B(x)^\delta + \nu_B(x)^\delta \leq 1. \end{aligned}$$

The proof of (i) is complete.
(ii) Proof of (i) is similar.

(iii) Since

$$A+B = \left\{ \left\langle x, \mu_A(x)^\delta + \mu_B(x)^\delta - \mu_A(x)^\delta \mu_B(x)^\delta, \nu_A(x)^\delta \nu_B(x)^\delta \right\rangle : x \in X \right\},$$

we have

$$\begin{aligned} & \mu_{A+B}(x)^\delta + \nu_{A+B}(x)^\delta \\ = & \left(\mu_A(x)^\delta + \mu_B(x)^\delta - \mu_A(x)^\delta \mu_B(x)^\delta\right)^\delta + \left(\nu_A(x)^\delta \nu_B(x)^\delta\right)^\delta \\ = & \left(\mu_A(x)^\delta \left(1 - \mu_B(x)^\delta\right) + \mu_B(x)^\delta\right)^\delta + \left(\nu_A(x)^\delta \nu_B(x)^\delta\right)^\delta \geq 0 \end{aligned}$$

and

$$\begin{aligned} & \mu_{A+B}(x)^\delta + \nu_{A+B}(x)^\delta \\ = & \left(\mu_A(x)^\delta + \mu_B(x)^\delta - \mu_A(x)^\delta \mu_B(x)^\delta\right)^\delta + \left(\nu_A(x)^\delta \nu_B(x)^\delta\right)^\delta \\ \leq & \left(\left(1 - \nu_A(x)^\delta\right) + \left(1 - \nu_B(x)^\delta\right) - \left(1 - \nu_A(x)^\delta\right)\left(1 - \nu_B(x)^\delta\right)\right)^\delta \\ & + \left(\nu_A(x)^\delta \nu_B(x)^\delta\right)^\delta \\ = & \left(1 - \nu_A(x)^\delta \nu_B(x)^\delta\right)^\delta + \left(\nu_A(x)^\delta \nu_B(x)^\delta\right)^\delta \\ = & (1-u)^\delta + u^\delta, \end{aligned}$$

where $u = \nu_A(x)^\delta \nu_B(x)^\delta$. If $\delta \geq 1$ then $(1-u)^\delta + u^\delta \leq 1$, hence $A+B \in GIFS_B$. If $\delta < 1$ then $(1-u)^\delta + u^\delta \leq 1$, if and only if $\nu_A(x) = 0$ or $\nu_B(x) = 0$. But for any δ, we have

$$\begin{aligned} & \mu_{A+B}(x) + \nu_{A+B}(x) \\ = & \mu_A(x)^\delta + \mu_B(x)^\delta - \mu_A(x)^\delta \mu_B(x)^\delta + \nu_A(x)^\delta \nu_B(x)^\delta \\ \leq & \mu_A(x)^\delta + \mu_B(x)^\delta - \mu_A(x)^\delta \mu_B(x)^\delta + \left(1 - \mu_A(x)^\delta\right)\left(1 - \mu_B(x)^\delta\right) \\ = & 1, \end{aligned}$$

hence $A+B \in IFS$. The proof of (iii) is complete.
(iv). The proof of (iii) is similar. □

9. Definition. The degree of non-determinacy (uncertainty) of an element $x \in X$ to the $GIFS_B A$ is defined by

$$\pi_A(x) = \left(1 - \mu_A(x)^\delta - \nu_A(x)^\delta\right)^{\frac{1}{\delta}}.$$

3.3. Remark. It can be easily shown that $\pi_A(x)^\delta + \mu_A(x)^\delta + \nu_A(x)^\delta = 1$.

10. Definition. For every $GIFS_B A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$, we define the modal logic operators, the necessity measure on A and the possibility measure on A, as

$$\Box A = \left\{\left\langle x, \mu_A(x), \left(1 - \mu_A(x)^\delta\right)^{\frac{1}{\delta}}\right\rangle : x \in X\right\}$$

and

$$\Diamond A = \left\{\left\langle x, \left(1 - \nu_A(x)^\delta\right)^{\frac{1}{\delta}}, \nu_A(x)\right\rangle : x \in X\right\},$$

respectively.

11. Definition. Let X denote a non-empty finite set. For every $GIFS_B$ as

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\},$$

two analogues of the topological operators, closure (C) and intersection (I), can be defined on $GIFS_B$s as

$$C(A) = \{\langle x, K, L\rangle : x \in X\}, \quad K = \max_{y \in X} \mu_A(y), \quad L = \min_{y \in X} \nu_A(y)$$

and

$$I(A) = \{\langle x, k, l\rangle : x \in X\}, \quad k = \min_{y \in X} \mu_A(y), \quad l = \max_{y \in X} \nu_A(y).$$

It is obvious that both $C(A)$ and $I(A)$ are $GIFS_B$. These two operators transform a given $GIFS_B$ to a new $GIFS_B$.

12. Definition. Let X denote a non-empty finite set and let A denote a finite $GIFS_B$. The normalization of A denoted by $NORM(A)$ is defined by

$$NORM(A) = \left\{\left\langle x, \frac{\mu_A(x)^\delta}{\sup \mu_A(x)^\delta}, \frac{\nu_A(x)^\delta - \inf \nu_A(x)^\delta}{1 - \inf \nu_A(x)^\delta}\right\rangle : x \in X\right\}.$$

Proposition 3.3 Let $A, B \in GIFS_B$. We have

i. $\Box A \in GIFS_B$,
ii. $\Diamond A \in GIFS_B$,
iii. $\pi_A(x) = 0 \Rightarrow \pi_{A^n}(x) = 0$.

Proof. (i) Follows by noting that

$$\mu_{\Box A}(x)^\delta + \nu_{\Box A}(x)^\delta = \mu_A(x)^\delta + \left(\left(1 - \mu_A(x)^\delta\right)^{\frac{1}{\delta}}\right)^\delta = 1.$$

Proof of (ii) is similar to that of (i). (iii) Since

$$\pi_A(x) = \left(1 - \mu_A(x)^\delta - \nu_A(x)^\delta\right)^{\frac{1}{\delta}},$$

we have

$$\begin{aligned} & \pi_A(x) = 0 \\ \Rightarrow \quad & \left(1 - \mu_A(x)^\delta - \nu_A(x)^\delta\right)^{\frac{1}{\delta}} = 0 \\ \Rightarrow \quad & \mu_A(x)^\delta + \nu_A(x)^\delta = 1 \\ \Rightarrow \quad & \mu_A(x)^\delta = 1 - \nu_A(x)^\delta. \end{aligned}$$

By using this result, we have

$$\begin{aligned} A^n &= \left\{\left\langle x, \mu_A(x)^{n\delta}, 1-\left(1-\nu_A(x)^{\delta}\right)^n\right\rangle : x \in X\right\} \\ &= \left\{\left\langle x, \mu_A(x)^{n\delta}, 1-\mu_A(x)^{n\delta}\right\rangle : x \in X\right\}. \end{aligned}$$

It is now obvious that $\pi_{A^n}(x) = 0$. □

3.4. Proposition. Let A denote a $GIFS_B$ and n any positive real number. Then, the following relations are true at the extreme values of $\mu_A(x)$ and $\nu_A(x)$:

i. $\Box A^n = (\Box A)^n$,
ii. $\Diamond A^n = (\Diamond A)^n$.

Proof. (i) Since

$$A^n = \left\{\left\langle x, \mu_A(x)^{n\delta}, 1-\left(1-\nu_A(x)^{\delta}\right)^n\right\rangle : x \in X\right\},$$

we have

$$\Box A^n = \left\{\left\langle x, \mu_A(x)^{n\delta}, \left(1-\mu_A(x)^{n\delta^2}\right)^{\frac{1}{\delta}}\right\rangle : x \in X\right\}.$$

Also since

$$\Box A = \left\{\left\langle x, \mu_A(x), \left(1-\mu_A(x)^{\delta}\right)^{\frac{1}{\delta}}\right\rangle : x \in X\right\},$$

we have

$$\begin{aligned} (\Box A)^n &= \left\{\left\langle x, \mu_A(x)^{n\delta}, 1-\left(1-\left(1-\mu_A(x)^{\delta}\right)\right)^n\right\rangle : x \in X\right\} \\ &= \left\{\left\langle x, \mu_A(x)^{n\delta}, 1-\mu_A(x)^{n\delta}\right\rangle : x \in X\right\}. \end{aligned}$$

Assume $\Box A^n = (\Box A)^n$. Consequently, we must have

$$\begin{aligned} &\left(1-\mu_A(x)^{n\delta^2}\right)^{\frac{1}{\delta}} = 1-\mu_A(x)^{n\delta}, \\ &\left(1-\mu_A(x)^{n\delta^2}\right) = \left(1-\mu_A(x)^{n\delta}\right)^{\delta}, \\ &1-u^{\delta} = (1-u)^{\delta}, \qquad u = \left(1-\mu_A(x)^{n\delta}\right). \end{aligned}$$

Hence, (i) is true if and only if $\mu_A(x) = 0$ or 1, $\forall x \in X$.
(ii) We know that

$$A^n = \left\{\left\langle x, \mu_A(x)^{n\delta}, 1-\left(1-\nu_A(x)^{\delta}\right)^n\right\rangle : x \in X\right\}$$

and

$$\Diamond A^n = \left\{\left\langle x, \left(1-\left(1-\left(1-\nu_A(x)^{\delta}\right)^n\right)^{\delta}\right)^{\frac{1}{\delta}}, 1-\left(1-\nu_A(x)^{\delta}\right)^n\right\rangle : x \in X\right\}.$$

Also

$$\Diamond A = \left\{\left\langle x, \left(1-\nu_A(x)^{\delta}\right)^{\frac{1}{\delta}}, \nu_A(x)\right\rangle : x \in X\right\},$$

so

$$\begin{aligned} (\Diamond A)^n &= \left\{\left\langle x, \left(1-\nu_A(x)^{\delta}\right)^{\frac{n\delta}{\delta}}, 1-\left(1-\nu_A(x)^{\delta}\right)^n\right\rangle : x \in X\right\} \\ &= \left\{\left\langle x, \left(1-\nu_A(x)^{\delta}\right)^{n}, 1-\left(1-\nu_A(x)^{\delta}\right)^n\right\rangle : x \in X\right\}. \end{aligned}$$

Assume $\diamond A^n = (\diamond A)^n$. Consequently, we must have

$$\left(1-\left(1-\left(1-\nu_A(x)^\delta\right)^n\right)^\delta\right)^{\frac{1}{\delta}} = \left(1-\nu_A(x)^\delta\right)^n,$$
$$\left(1-\left(1-\nu_A(x)^\delta\right)^n\right)^\delta = 1-\left(1-\nu_A(x)^\delta\right)^{n\delta},$$
$$(1-u)^\delta = 1-u^\delta, \qquad u = \left(1-\nu_A(x)^\delta\right)^n.$$

Hence, (ii) is true if and only if $\nu_A(x) = 0$ or 1, $\forall x \in X$. □

3.5. Proposition. For every $GIFS_B A$, we have

i. $m \geq n \Rightarrow A^m \subset A^n$,
ii. $m \geq n \Rightarrow nA \subset mA$,
iii. $A^n = \overline{n\overline{A}}$,

where m and n are both positive numbers.

Proof. (i) Since

$$A^n = \left\{\left\langle x, \mu_A(x)^{n\delta}, 1-\left(1-\nu_A(x)^\delta\right)^n\right\rangle : x \in X\right\},$$

we have

$$A^m = \left\{\left\langle x, \mu_A(x)^{m\delta}, 1-\left(1-\nu_A(x)^\delta\right)^m\right\rangle : x \in X\right\}.$$

Since $m \geq n$, we have $\mu_A(x)^n \geq \mu_A(x)^m$, so $\mu_A(x)^{n\delta} \geq \mu_A(x)^{m\delta}$ and $\mu_{A^n}(x) \geq \mu_{A^m}(x)$. Also since $\nu_A(x) \leq 1$, we have $\left(1-\nu_A(x)^\delta\right)^m \leq \left(1-\nu_A(x)^\delta\right)^n$, so

$$1-\left(1-\nu_A(x)^\delta\right)^n \leq 1-\left(1-\nu_A(x)^\delta\right)^m \Rightarrow \nu_{A^n}(x) \leq \nu_{A^m}(x),$$

completing the proof. The proof of (ii) is similar to that of (i). The proof of (iii) is immediate. □

3.6. Proposition. Let $A, B \in GIFS_B$. We have

i. $A \subset B \Rightarrow nA \subset nB$,
ii. $A \subset B \Rightarrow A^n \subset B^n$,
iii. $(A \cup B)^n = A^n \cup B^n$,
iv. $(A \cap B)^n = A^n \cap B^n$,
v. $n(A \cup B) = nA \cup nB$,
vi. $n(A \cap B) = nA \cap nB$.

Proof. (i) Since $A \subset B$, we have $\mu_A(x) \leq \mu_B(x)$ and

$$\mu_A(x)^\delta \leq \mu_B(x)^\delta \Rightarrow 1-\mu_B(x)^\delta \leq 1-\mu_A(x)^\delta \Rightarrow \left(1-\mu_B(x)^\delta\right)^n \leq \left(1-\mu_A(x)^\delta\right)^n,$$

so

$$1-\left(1-\mu_A(x)^\delta\right)^n \leq 1-\left(1-\mu_B(x)^\delta\right)^n \Rightarrow \mu_{nA}(x) \leq \mu_{mB}(x).$$

Also since $A \subset B$, we have $\nu_B(x) \leq \nu_A(x)$ and

$$\nu_B(x)^{n\delta} \leq \nu_A(x)^{n\delta} \Rightarrow \nu_{nB}(x) \leq \nu_{nA}(x),$$

completing the proof.
(ii) follows since

$$A \subset B \Rightarrow \overline{B} \subset \overline{A} \Rightarrow n\overline{B} \subset n\overline{A} \Rightarrow \overline{n\overline{A}} \subset \overline{n\overline{B}} \Rightarrow A^n \subset B^n.$$

(iii) follows since

$$A \cup B = \{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))\rangle : x \in X\}$$

and

$$\begin{aligned}
& (A \cup B)^n \\
= & \left\{\left\langle x, \left(\max\left(\mu_A(x), \mu_B(x)\right)\right)^{n\delta}, 1-\left(1-\min\left(\nu_A(x), \nu_B(x)\right)^{\delta}\right)^n\right\rangle : x \in X\right\} \\
= & \left\{\left\langle x, \max\left(\mu_A(x)^{n\delta}, \mu_B(x)^{n\delta}\right), 1-\left(1-\min\left(\nu_A(x)^{\delta}, \nu_B(x)^{\delta}\right)\right)^n\right\rangle : x \in X\right\} \\
= & \left\{\left\langle x, \max\left(\mu_A(x)^{n\delta}, \mu_B(x)^{n\delta}\right), 1-\left(\max\left(1-\nu_A(x)^{\delta}, 1-\nu_B(x)^{\delta}\right)\right)^n\right\rangle : x \in X\right\} \\
= & \left\{\left\langle x, \max\left(\mu_A(x)^{n\delta}, \mu_B(x)^{n\delta}\right), 1-\max\left(\left(1-\nu_A(x)^{\delta}\right)^n, \left(1-\nu_B(x)^{\delta}\right)^n\right)\right\rangle : x \in X\right\} \\
= & \left\{\left\langle x, \max\left(\mu_A(x)^{n\delta}, \mu_B(x)^{n\delta}\right), \min\left(1-\left(1-\nu_A(x)^{\delta}\right)^n, 1-\left(1-\nu_B(x)^{\delta}\right)^n\right)\right\rangle : x \in X\right\} \\
= & A^n \cup B^n.
\end{aligned}$$

(iv) follows since

$$A \cap B = \{\langle x, \min\left(\mu_A(x), \mu_B(x)\right), \max\left(\nu_A(x), \nu_B(x)\right)\rangle : x \in X\}$$

and

$$\begin{aligned}
& (A \cap B)^n \\
= & \left\{\left\langle x, \left(\min\left(\mu_A(x), \mu_B(x)\right)\right)^{n\delta}, 1-\left(1-\max\left(\nu_A(x), \nu_B(x)\right)^{\delta}\right)^n\right\rangle : x \in X\right\} \\
= & \left\{\left\langle x, \min\left(\mu_A(x)^{n\delta}, \mu_B(x)^{n\delta}\right), 1-\left(1-\max\left(\nu_A(x)^{\delta}, \nu_B(x)^{\delta}\right)\right)^n\right\rangle : x \in X\right\} \\
= & \left\{\left\langle x, \min\left(\mu_A(x)^{n\delta}, \mu_B(x)^{n\delta}\right), 1-\left(\min\left(1-\nu_A(x)^{\delta}, 1-\nu_B(x)^{\delta}\right)\right)^n\right\rangle : x \in X\right\} \\
= & \left\{\left\langle x, \min\left(\mu_A(x)^{n\delta}, \mu_B(x)^{n\delta}\right), 1-\min\left(\left(1-\nu_A(x)^{\delta}\right)^n, \left(1-\nu_B(x)^{\delta}\right)^n\right)\right\rangle : x \in X\right\} \\
= & \left\{\left\langle x, \min\left(\mu_A(x)^{n\delta}, \mu_B(x)^{n\delta}\right), \max\left(1-\left(1-\nu_A(x)^{\delta}\right)^n, 1-\left(1-\nu_B(x)^{\delta}\right)^n\right)\right\rangle : x \in X\right\} \\
= & A^n \cap B^n.
\end{aligned}$$

(v) follows since

$$\begin{aligned}
& n\,(A \cup B) \\
= & \left\{\left\langle x, 1-\left(1-\max\left(\mu_A(x), \mu_B(x)\right)^{\delta}\right)^n, \min\left(\nu_A(x), \nu_B(x)\right)^{n\delta}\right\rangle : x \in X\right\} \\
= & \left\{\left\langle x, 1-\left(1-\max\left(\mu_A(x)^{\delta}, \mu_B(x)^{\delta}\right)\right)^n, \min\left(\nu_A(x)^{n\delta}, \nu_B(x)^{n\delta}\right)\right\rangle : x \in X\right\} \\
= & \left\{\left\langle x, 1-\left(\min\left(1-\mu_A(x)^{\delta}, 1-\mu_B(x)^{\delta}\right)\right)^n, \min\left(\nu_A(x)^{n\delta}, \nu_B(x)^{n\delta}\right)\right\rangle : x \in X\right\} \\
= & \left\{\left\langle x, 1-\min\left(\left(1-\mu_A(x)^{\delta}\right)^n, \left(1-\mu_B(x)^{\delta}\right)^n\right), \min\left(\nu_A(x)^{n\delta}, \nu_B(x)^{n\delta}\right)\right\rangle : x \in X\right\} \\
= & \left\{\left\langle x, \max\left(1-\left(1-\mu_A(x)^{\delta}\right)^n, 1-\left(1-\mu_B(x)^{\delta}\right)^n\right), \min\left(\nu_A(x)^{n\delta}, \nu_B(x)^{n\delta}\right)\right\rangle : x \in X\right\} \\
= & nA \cup nB.
\end{aligned}$$

The proof of (vi) is similar to that of (v). □

4. The operators $D_\alpha(A)$, $F_{\alpha,\beta}(A)$ and $G_{\alpha,\beta}(A)$

Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ denote a $GIFS_B$.

13. Definition. Let $\alpha \in [0,1]$ and $A \in GIFS_B$. We define the operator of $D_\alpha(A)$ as

$$D_\alpha(A) = \left\{\left\langle x, \left(\mu_A(x)^{\delta} + \alpha\pi_A(x)^{\delta}\right)^{\frac{1}{\delta}}, \left(\nu_A(x)^{\delta} + (1-\alpha)\pi_A(x)^{\delta}\right)^{\frac{1}{\delta}}\right\rangle : x \in X\right\}.$$

Clearly, $D_\alpha(A)$ is a $GIFS_B$.

4.1. Theorem. For every $GIFS_B A$ and for every $\alpha, \beta \in [0,1]$, we have

i. $\alpha \leq \beta \Rightarrow D_\alpha(A) \subset D_\beta(A)$,
ii. $D_0(A) = \Box A$,
iii. $D_1(A) = \Diamond A$.

Proof. The proof of (i) is immediate.
(ii) We have

$$\begin{aligned} D_0(A) &= \left\{\left\langle x, \left(\mu_A(x)^\delta + 0 \times \pi_A(x)^\delta\right)^{\frac{1}{\delta}}, \left(\nu_A(x)^\delta + (1-0)\pi_A(x)^\delta\right)^{\frac{1}{\delta}}\right\rangle : x \in X\right\} \\ &= \left\{\left\langle x, \mu_A(x), \left(\nu_A(x)^\delta + \pi_A(x)^\delta\right)^{\frac{1}{\delta}}\right\rangle : x \in X\right\} \\ &= \left\{\left\langle x, \mu_A(x), \left(1 - \mu_A(x)^\delta\right)^{\frac{1}{\delta}}\right\rangle : x \in X\right\} = \Box A, \end{aligned}$$

where the penultimate equality follows since $\pi_A(x)^\delta = 1 - \mu_A(x)^\delta - \nu_A(x)^\delta$. So, (ii) follows.
(iii) We have

$$\begin{aligned} D_1(A) &= \left\{\left\langle x, \left(\mu_A(x)^\delta + 1 \times \pi_A(x)^\delta\right)^{\frac{1}{\delta}}, \left(\nu_A(x)^\delta + (1-1)\pi_A(x)^\delta\right)^{\frac{1}{\delta}}\right\rangle : x \in X\right\} \\ &= \left\{\left\langle x, \left(\mu_A(x)^\delta + \pi_A(x)^\delta\right)^{\frac{1}{\delta}}, \nu_A(x)\right\rangle : x \in X\right\} \\ &= \left\{\left\langle x, \left(1 - \nu_A(x)^\delta\right)^{\frac{1}{\delta}}, \nu_A(x)\right\rangle : x \in X\right\} = \Diamond A, \end{aligned}$$

completing the proof. □

14. Definition. Let $\alpha.\beta \in [0,1]$, where $\alpha + \beta \leq 1$. Let $A \in GIFS_B$. We define the operator of $F_{\alpha,\beta}(A)$ as

$$F_{\alpha,\beta}(A) = \left\{\left\langle x, \left(\mu_A(x)^\delta + \alpha\pi_A(x)^\delta\right)^{\frac{1}{\delta}}, \left(\nu_A(x)^\delta + \beta\pi_A(x)^\delta\right)^{\frac{1}{\delta}}\right\rangle : x \in X\right\}.$$

4.2. Theorem. For every $GIFS_B A$ and for any $\alpha, \beta \in [0,1]$, where $\alpha + \beta \leq 1$, we have

i. $F_{\alpha,\beta}(A) \in GIFS_B$,
ii. $0 \leq \gamma \leq \alpha \Rightarrow F_{\gamma,\beta}(A) \subset F_{\alpha,\beta}(A)$,
iii. $0 \leq \gamma \leq \beta \Rightarrow F_{\alpha,\beta}(A) \subset F_{\alpha,\gamma}(A)$,
iv. $D_\alpha(A) = F_{\alpha,1-\alpha}(A)$,
v. $\Box A = F_{0,1}(A)$,
vi. $\Diamond A = F_{1,0}(A)$,
vii. $\overline{F_{\alpha,\beta}\overline{A}} = F_{\beta,\alpha}(A)$.

Proof. (i) follows since

$$\begin{aligned} & \mu_{F_{\alpha,\beta}(A)}(x)^\delta + \nu_{F_{\alpha,\beta}(A)}(x)^\delta \\ = & \left[\left(\mu_A(x)^\delta + \alpha\pi_A(x)^\delta\right)^{\frac{1}{\delta}}\right]^\delta + \left[\left(\nu_A(x)^\delta + \beta\pi_A(x)^\delta\right)^{\frac{1}{\delta}}\right]^\delta \\ = & \mu_A(x)^\delta + \nu_A(x)^\delta + \pi_A(x)^\delta(\alpha + \beta) \\ \leq & \mu_A(x)^\delta + \nu_A(x)^\delta + \pi_A(x)^\delta = 1. \end{aligned}$$

The proofs of (ii) and (iii) are immediate.
(iv) follows since

$$\begin{aligned} & F_{\alpha,1-\alpha}(A) \\ = & \left\{\left\langle x, \left(\mu_A(x)^\delta + \alpha\pi_A(x)^\delta\right)^{\frac{1}{\delta}}, \left(\nu_A(x)^\delta + (1-\alpha)\pi_A(x)^\delta\right)^{\frac{1}{\delta}}\right\rangle : x \in X\right\} \\ = & D_\alpha(A). \end{aligned}$$

(v) follows by Theorem 4.1 after noting that $D_0(A) = F_{0,1}(A)$ and $D_1(A) = F_{1,0}(A)$ from (iv).
(vi) follows by Theorem 4.1 after noting that $D_0(A) = F_{0,1}(A)$ and $D_1(A) = F_{1,0}(A)$ from (iv).
(vii) since

$$F_{\beta,\alpha}(A) = \left\{ \left\langle x, \left(\mu_A(x)^\delta + \beta\pi_A(x)^\delta\right)^{\frac{1}{\delta}}, \left(\nu_A(x)^\delta + \alpha\pi_A(x)^\delta\right)^{\frac{1}{\delta}} \right\rangle : x \in X \right\}$$

and

$$F_{\alpha,\beta}\left(\overline{A}\right) = \left\{ \left\langle x, \left(\nu_A(x)^\delta + \alpha\pi_A(x)^\delta\right)^{\frac{1}{\delta}}, \left(\mu_A(x)^\delta + \beta\pi_A(x)^\delta\right)^{\frac{1}{\delta}} \right\rangle : x \in X \right\},$$

we have

$$\overline{F_{\alpha,\beta}\left(\overline{A}\right)} = \left\{ \left\langle x, \left(\mu_A(x)^\delta + \beta\pi_A(x)^\delta\right)^{\frac{1}{\delta}}, \left(\nu_A(x)^\delta + \alpha\pi_A(x)^\delta\right)^{\frac{1}{\delta}} \right\rangle : x \in X \right\}$$

and $\overline{F_{\alpha,\beta}\left(\overline{A}\right)} = F_{\beta,\alpha}(A)$. □

15. Definition. Let $\alpha, \beta \in [0,1]$ and $A \in GIFS_B$. We define the operator of $G_{\alpha,\beta}(A)$ as

$$G_{\alpha,\beta}(A) = \left\{ \left\langle x, \alpha^{\frac{1}{\delta}}\mu_A(x), \beta^{\frac{1}{\delta}}\nu_A(x) \right\rangle : x \in X \right\}.$$

4.3. Theorem. For every $GIFS_B A$, and for any real numbers $\alpha, \beta, \gamma \in [0,1]$, we have

i. $G_{\alpha,\beta}(A) \in GIFS_B$,
ii. $\alpha \leq \gamma \Rightarrow G_{\alpha,\beta}(A) \subset G_{\gamma,\beta}(A)$,
iii. $\beta \leq \gamma \Rightarrow G_{\alpha,\beta}(A) \supset G_{\alpha,\gamma}(A)$,
iv. $\tau \in [0,1] \Rightarrow G_{\alpha,\beta}\left(G_{\gamma,\tau}(A)\right) = G_{\alpha\gamma,\beta\tau}(A) = G_{\gamma,\delta}\left(G_{\alpha,\beta}(A)\right)$,
v. $G_{\alpha,\beta}\left(C(A)\right) = C\left(G_{\alpha,\beta}(A)\right)$,
vi. $G_{\alpha,\beta}\left(I(A)\right) = I\left(G_{\alpha,\beta}(A)\right)$,
vii. $\overline{G_{\alpha,\beta}\left(\overline{A}\right)} = G_{\beta,\alpha}(A)$.

Proof. (i) follows since

$$G_{\alpha,\beta}(A) = \left\{ \left\langle x, \alpha^{\frac{1}{\delta}}\mu_A(x), \beta^{\frac{1}{\delta}}\nu_A(x) \right\rangle : x \in X \right\}$$

and

$$\begin{aligned} \mu_{G_{\alpha,\beta}(A)}(x)^\delta + \nu_{G_{\alpha,\beta}(A)}(x)^\delta &= \left(\alpha^{\frac{1}{\delta}}\mu_A(x)\right)^\delta + \left(\beta^{\frac{1}{\delta}}\nu_A(x)\right)^\delta \\ &= \alpha\mu_A(x)^\delta + \beta\nu_A(x)^\delta \\ &\leq \mu_A(x)^\delta + \nu_A(x)^\delta \leq 1. \end{aligned}$$

(ii) We have

$$G_{\alpha,\beta}(A) = \left\{ \left\langle x, \alpha^{\frac{1}{\delta}}\mu_A(x), \beta^{\frac{1}{\delta}}\nu_A(x) \right\rangle : x \in X \right\}$$

and

$$G_{\gamma,\beta}(A) = \left\{ \left\langle x, \gamma^{\frac{1}{\delta}}\mu_A(x), \beta^{\frac{1}{\delta}}\nu_A(x) \right\rangle : x \in X \right\}.$$

Since $\alpha \leq \gamma$, we have $\alpha^{\frac{1}{\delta}} \leq \gamma^{\frac{1}{\delta}}$ and so $\alpha^{\frac{1}{\delta}}\mu_A(x) \leq \gamma^{\frac{1}{\delta}}\mu_A(x)$, completing the proof of (ii).
The proof of (iii) is similar to that of (ii).
(iv) We have

$$G_{\gamma,\tau}(A) = \left\{ \left\langle x, \gamma^{\frac{1}{\delta}}\mu_A(x), \tau^{\frac{1}{\delta}}\nu_A(x) \right\rangle : x \in X \right\},$$

$$\begin{aligned} G_{\alpha,\beta}\left(G_{\gamma,\tau}(A)\right) &= \left\{\left\langle x, \alpha^{\frac{1}{\delta}}\gamma^{\frac{1}{\delta}}\mu_A(x), \beta^{\frac{1}{\delta}}\tau^{\frac{1}{\delta}}\nu_A(x)\right\rangle : x \in X\right\} \\ &= \left\{\left\langle x, (\alpha\gamma)^{\frac{1}{\delta}}\mu_A(x), (\beta\tau)^{\frac{1}{\delta}}\nu_A(x)\right\rangle : x \in X\right\} \\ &= G_{\alpha\gamma,\beta\tau}(A) \end{aligned}$$

and

$$\begin{aligned} G_{\gamma,\tau}\left(G_{\alpha,\beta}(A)\right) &= \left\{\left\langle x, \gamma^{\frac{1}{\delta}}\alpha^{\frac{1}{\delta}}\mu_A(x), \tau^{\frac{1}{\delta}}\beta^{\frac{1}{\delta}}\nu_A(x)\right\rangle : x \in X\right\} \\ &= \left\{\left\langle x, (\gamma\alpha)^{\frac{1}{\delta}}\mu_A(x), (\tau\beta)^{\frac{1}{\delta}}\nu_A(x)\right\rangle : x \in X\right\} \\ &= \left\{\left\langle x, (\alpha\gamma)^{\frac{1}{\delta}}\mu_A(x), (\beta\tau)^{\frac{1}{\delta}}\nu_A(x)\right\rangle : x \in X\right\} \\ &= G_{\alpha\gamma,\beta\tau}(A), \end{aligned}$$

so

$$G_{\alpha,\beta}\left(G_{\gamma,\tau}(A)\right) = G_{\alpha\gamma,\beta\tau}(A) = G_{\gamma,\tau}\left(G_{\alpha,\beta}(A)\right).$$

(v) follows since

$$C(A) = \left\{\left\langle x, \max_{y\in X}\mu_A(y), \min_{y\in X}\nu_A(y)\right\rangle : x \in X\right\}$$

and

$$\begin{aligned} G_{\alpha,\beta}\left(C(A)\right) &= \left\{\left\langle x, \alpha^{\frac{1}{\delta}}\max_{y\in X}\mu_A(y), \beta^{\frac{1}{\delta}}\min_{y\in X}\nu_A(y)\right\rangle : x \in X\right\} \\ &= \left\{\left\langle x, \max_{y\in X}\alpha^{\frac{1}{\delta}}\mu_A(y), \min_{y\in X}\beta^{\frac{1}{\delta}}\nu_A(y)\right\rangle : x \in X\right\} \\ &= C\left(G_{\alpha,\beta}(A)\right). \end{aligned}$$

(vi) follows since

$$I(A) = \left\{\left\langle x, \min_{y\in X}\mu_A(y), \max_{y\in X}\nu_A(y)\right\rangle : x \in X\right\}$$

and

$$\begin{aligned} G_{\alpha,\beta}\left(I(A)\right) &= \left\{\left\langle x, \alpha^{\frac{1}{\delta}}\min_{y\in X}\mu_A(y), \beta^{\frac{1}{\delta}}\max_{y\in X}\nu_A(y)\right\rangle : x \in X\right\} \\ &= \left\{\left\langle x, \min_{y\in X}\alpha^{\frac{1}{\delta}}\mu_A(y), \max_{y\in X}\beta^{\frac{1}{\delta}}\nu_A(y)\right\rangle : x \in X\right\} \\ &= I\left(G_{\alpha,\beta}(A)\right), \end{aligned}$$

where $\alpha, \beta \in [0,1]$.
(vii) Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ denote a $GIFS_B$. Then,

$$\begin{aligned} &\overline{A} = \{\langle x, \nu_A(x), \mu_A(x)\rangle : x \in X\}, \\ &G_{\beta,\alpha}(A) = \left\{\left\langle x, \beta^{\frac{1}{\delta}}\mu_A(x), \alpha^{\frac{1}{\delta}}\nu_A(x)\right\rangle : x \in X\right\}, \\ &G_{\alpha,\beta}\left(\overline{A}\right) = \left\{\left\langle x, \alpha^{\frac{1}{\delta}}\nu_A(x), \beta^{\frac{1}{\delta}}\mu_A(x)\right\rangle : x \in X\right\}, \\ &\overline{G_{\alpha,\beta}\left(\overline{A}\right)} = \left\{\left\langle x, \beta^{\frac{1}{\delta}}\mu_A(x), \alpha^{\frac{1}{\delta}}\nu_A(x)\right\rangle : x \in X\right\}, \end{aligned}$$

and so $\overline{G_{\alpha,\beta}\left(\overline{A}\right)} = G_{\beta,\alpha}(A)$. □

5. Conclusions

We have introduced a new generalized IFS ($GIFS_B$) as an extension to the IFS. The basic algebraic properties of $GIFS_B$ have been presented. Some operators on $GIFS_B$ are defined and their relationship have been proved. A list of open problems is as follows: i) define the generalized fuzzy intuitionistic number, norms, distances, metrics, metric spaces, etc for the generalized IFS and study of their properties; ii) develop statistical and probabilistic tools for the generalized IFS; iii) construct an axiomatic system for the generalized IFS; iv) develop efficient algorithms and computer software for the construction of degrees of membership and nonmembership of a given generalized IFS; v) define and study the properties of generalized IF boolean algebras; vi) develop information and entropy measures corresponding to generalized IFSs; vii) develop preference theory and utility theory for the generalized IFS; viii) compare with other generalizations of the IFS.

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