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A classification theorem on totally umbilical submanifolds in a cosymplectic manifold

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Abstract

In the present paper, we study totally umbilical submanifolds of cosymplectic manifolds. We obtain a result on the classification of totally umbilical contact CR-submanifolds of a cosymplectic manifold.

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1. Introduction

The notion of CR-submanifolds of a Kaehler manifold was introduced by A. Bejancu [1]. Later on, many researchers worked on these submanifolds for different structures [5]. These submanifolds are the natural generalization of both holomorphic and totally real submanifolds of a Kaehler manifold. Totally umbilical CR-submanifolds of a Kaehler manifold have been studied by A. Bejancu [2], B.Y. Chen (see [6]), S. Deshmukh and S.I. Husain [8].

The submanifolds of a cosymplectic manifold have been studied by G.D. Ludden [10]. Recently, we have obtained some results for the existence or non-existence of warped submanifolds in a cosymplectic manifold [11]. In this paper, we classify all totally umbilical contact CR-submanifolds of a cosymplectic manifold.

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2. Preliminaries

Let $\tilde{M}$ be a $(2n+1)-$dimensional almost contact manifold with almost contact structure (ϕ, ξ, η), that is ϕ is a $(1,1)$ tensor field, ξ is a vector field and η is a $1-$form, satisfying the following properties

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \quad \eta(\xi) = 1. \tag{2.1}$$

In this case we call $(\tilde{M}, \phi, \xi, \eta)$ *an almost contact manifold*. There always exists a Riemannian metric g on an almost contact manifold $\tilde{M}$ satisfying the following compatibility condition

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.2}$$

for any X, Y tangent to $\tilde{M}$; with this metric the almost contact manifold is called an *almost contact metric manifold*.

An almost contact structure (ϕ, ξ, η) is said to be *normal* if $[\phi, \phi] + 2d\eta \otimes \xi$ vanishes identically on $\tilde{M}$, where $[\phi, \phi](X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y]$ for any vector fields $X,\ Y$ tangent to $\tilde{M}$ is the Nijenhuis tensor of ϕ.

The fundamental 2-form Φ on $\tilde{M}$ is defined as $\Phi(X, Y) = g(X, \phi Y)$, for any vector fields $X,\ Y$ tangent to $\tilde{M}$. If $\Phi = d\eta$, the almost contact structure is a *contact structure*. A normal almost contact structure with Φ closed and $d\eta = 0$ is called *cosymplectic structure*. It is well known that the cosymplectic structure is characterized by

$$\tilde{\nabla}_X \phi = 0 \quad \text{and} \quad \tilde{\nabla}_X \eta = 0, \tag{2.3}$$

where $\tilde{\nabla}$ is the Levi-Civita connection of g on $\tilde{M}$. From (2.3), it follows that $\tilde{\nabla}_X \xi = 0$.

If we denote the curvature tensor of a cosymplectic manifold $\tilde{M}$ by $\tilde{R}$, then we have

$$\tilde{R}(\phi X, \phi Y) = \tilde{R}(X, Y) \quad \text{and} \quad \tilde{R}(X, Y)\phi Z = \phi\tilde{R}(X, Y)Z. \tag{2.4}$$

Blair and Goldberg [5] studied the cosymplectic structure on a Riemannian manifold from topological viewpoint. They have given a typical example of simply connected cosymplectic manifold which is the product of a simply connected Kaehler manifold with $\mathbb{R}$. They proved that a complete simply connected cosymplectic manifold is almost contact isometric to the product of a complete simply connected Kaehler manifold with $\mathbb{R}$. On the other hand the natural example of a compact cosymplectic manifold is given by the product of a compact Kaehler manifold (V, J, h) with the circle S^1, where J is almost complex structure and h is almost Hermitian metric on V. The cosymplectic structure (ϕ, ξ, η, g) on the product manifold $\tilde{M} = V \times S^1$ is defined by

$$\phi = J \circ (pr_1)_*, \quad \xi = \frac{E}{c}, \quad \eta = c(pr_2)_*(\theta), \quad g = (pr_1)_*(h) + c^2(pr_2)_*(\theta \otimes \theta),$$

where $*$ is the symbol for tangent map and $pr_1 : \tilde{M} \to V$ and $pr_2 : \tilde{M} \to S^1$ are the projections of $V \times S^1$ onto V and S^1 respectively, θ is the length element of S^1, E is its dual vector field and c is a non-zero real number [5]. In [7], De Leon and Marrero studied compact cosymplectic manifold with positive constant ϕ-sectional curvature.

Let M be a submanifold of an almost contact metric manifold $\tilde{M}$ with induced metric g and if ∇ and $\nabla^{\perp}$ are the induced connections on the tangent bundle TM and the normal bundle $T^{\perp}M$ of M, respectively. Denote by $\mathcal{F}(M)$ the algebra of smooth functions on M and by $\Gamma(TM)$ the $\mathcal{F}(M)$-module of smooth sections of tangent bundle TM over M, then Gauss and Weingarten formulae are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.5}$$

$$\tilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \tag{2.6}$$

for each $X,\ Y \in \Gamma(TM)$ and $N \in \Gamma(T^{\perp}M)$, where h and A_N are the second fundamental form and the shape operator (corresponding to the normal vector field N) respectively for the immersion of M into $\tilde{M}$. They are related as

$$g(h(X,Y),N) = g(A_N X, Y), \tag{2.7}$$

where g denotes the Riemannian metric on $\tilde{M}$ as well as induced on M. The mean curvature vector H on M is given by

$$H = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i) \tag{2.8}$$

where n is the dimension of M and $\{e_1,\ e_2, \cdots, e_n\}$ is a local orthonormal frame of vector fields on M.

A submanifold M of a Riemannian manifold $\tilde{M}$ is said to be *totally umbilical* if

$$h(X,Y) = g(X,Y)H. \tag{2.9}$$

If $h(X,Y) = 0$ for any $X, Y \in \Gamma(TM)$ then M is said to be *totally geodesic submanifold*. If $H = 0$, then it is called *minimal submanifold*.

If M is totally umbilical, then from (2.9), the equations (2.5) and (2.6) reduce to the following equations, respectively;

$$\tilde{\nabla}_X Y = \nabla_X Y + g(X,Y)H, \tag{2.10}$$

$$\tilde{\nabla}_X N = -g(H,N)X + \nabla^{\perp}_X N. \tag{2.11}$$

Now, for any $X \in \Gamma(TM)$, we write

$$\phi X = PX + FX, \tag{2.12}$$

where PX is the tangential component and FX is the normal component of ϕX.

Similarly for any $N \in \Gamma(T^{\perp}M)$, we write

$$\phi N = BN + CN, \tag{2.13}$$

where BN is the tangential component and CN is the normal component of ϕN. The covariant derivatives of the tensor fields ϕ, P and F are respectively defined as

$$(\tilde{\nabla}_X \phi)Y = \tilde{\nabla}_X \phi Y - \phi \tilde{\nabla}_X Y, \quad \forall\ X, Y \in \Gamma(T\tilde{M}), \tag{2.14}$$

$$(\tilde{\nabla}_X P)Y = \nabla_X PY - P\nabla_X Y, \quad \forall\ X, Y \in \Gamma(TM), \tag{2.15}$$

$$(\tilde{\nabla}_X F)Y = \nabla^{\perp}_X FY - F\nabla_X Y, \quad \forall\ X, Y \in \Gamma(TM). \tag{2.16}$$

3. Contact CR-submanifolds

In this section we consider the submanifold M tangent to the structure vector field ξ and defined as follows: A submanifold M tangent to ξ is called a *contact CR-submanifold* if it admits a pair of differentiable distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$ such that $\mathcal{D}$ is invariant and its orthogonal complementary distribution $\mathcal{D}^{\perp}$ is anti-invariant i.e., $TM = \mathcal{D} \oplus \mathcal{D}^{\perp} \oplus \langle\xi\rangle$ with $\phi(\mathcal{D}_x) \subseteq \mathcal{D}_x$ and $\phi(\mathcal{D}^{\perp}_x) \subset T^{\perp}_x M$, for every $x \in M$. Thus, a contact CR-submanifold M tangent to ξ is *invariant* if $\mathcal{D}^{\perp}$ is identically zero and an *anti-invariant* if $\mathcal{D}$ is identically zero, respectively. If neither $\mathcal{D} = \{0\}$ nor $\mathcal{D}^{\perp} = \{0\}$, then M is *proper contact CR-submanifold*.

Let M be a proper contact CR-submanifold of an almost contact metric manifold $\tilde{M}$, then for any $X \in \Gamma(TM)$, we have

$$X = P_1 X + P_2 X + \eta(X)\xi, \tag{3.1}$$

where P_1 and P_2 are the orthogonal projections from TM to $\mathcal{D}$ and $\mathcal{D}^\perp$, respectively. For a contact CR-submanifold, from (2.12) and (3.1), we obtain

$$PX = \phi P_1 X \quad \text{and} \quad FX = \phi P_2 X.$$

Let M be a contact CR-submanifold of an almost contact metric manifold $\tilde{M}$. Then the normal bundle $T^\perp M$ is decomposed as

$$T^\perp M = \phi \mathcal{D}^\perp \oplus \mu, \tag{3.2}$$

where μ is the orthogonal complemetary distribution of $\phi\mathcal{D}^\perp$ in $T^\perp M$ and is a $\phi-$invariant subbundle of $T^\perp M$.

Let M be a contact CR-submanifold of a cosymplectic manifold $\tilde{M}$, then for any $Z, W \in \Gamma(\mathcal{D}^\perp)$ and $U \in \Gamma(TM)$, we have

$$g(A_{\phi W}Z, U) = g(h(Z, U), \phi W).$$

Using (2.5), we obtain

$$g(A_{\phi W}Z, U) = g(\tilde{\nabla}_U Z, \phi W) = -g(\phi\tilde{\nabla}_U Z, W).$$

By the structure equation (2.3), we get

$$g(A_{\phi W}Z, U) = -g(\tilde{\nabla}_U \phi Z, W).$$

Thus, from (2.6), we derive

$$g(A_{\phi W}Z, U) = g(A_{\phi Z}U, W) = g(h(W, U), \phi Z).$$

Again Using (2.5), we obtain

$$g(A_{\phi W}Z, U) = g(\tilde{\nabla}_W U, \phi Z) = -g(U, \tilde{\nabla}_W \phi Z).$$

Then from (2.6), we get

$$g(A_{\phi W}Z, U) = g(A_{\phi Z}W, U).$$

Hence, for a contact CR-submanifold of a cosymplectic manifold we conclude that

$$A_{\phi W}Z = A_{\phi Z}W \quad \forall\ Z,\ W \in \Gamma(\mathcal{D}^\perp). \tag{3.3}$$

Now, for any $X \in \Gamma(\mathcal{D} \oplus \langle\xi\rangle)$ and $Z, W \in \Gamma(\mathcal{D}^\perp)$, we have

$$\begin{aligned} g([Z, W], \phi X) &= g(\tilde{\nabla}_Z W - \tilde{\nabla}_W Z, \phi X) \\ &= g(\phi\tilde{\nabla}_W Z - \phi\tilde{\nabla}_Z W, X). \end{aligned}$$

Thus, from (2.14) and (2.3), we obtain

$$\begin{aligned} g([Z, W], \phi X) &= g(\tilde{\nabla}_W \phi Z - \tilde{\nabla}_Z \phi W, X) \\ &= g(A_{\phi W}Z - A_{\phi Z}W, X). \end{aligned}$$

Thus, from (3.3), we obtain $g([Z, W], \phi X) = 0$. This means that $[Z, W] \in \Gamma(\mathcal{D}^\perp)$, for any $Z, W \in \Gamma(\mathcal{D}^\perp)$, that is, $\mathcal{D}^\perp$ is integrable. Now for any $X, Y \in \Gamma(\mathcal{D} \oplus \langle\xi\rangle)$, we have

$$h(X, PY) + \nabla_X PY = \tilde{\nabla}_X PY = \tilde{\nabla}_X \phi Y.$$

As $\tilde{M}$ is cosymplectic, then by (2.14) and the structure equation (2.3), we obtain

$$h(X, PY) + \nabla_X PY = \phi\tilde{\nabla}_X Y.$$

Using (2.5), (2.12) and (2.13), we derive

$$h(X, PY) + \nabla_X PY = P\nabla_X Y + F\nabla_X Y + Bh(X, Y) + Ch(X, Y).$$

Equating the normal components, we get

$$F\nabla_X Y = h(X, PY) - Ch(X, Y). \tag{3.4}$$

Similarly,

$$F\nabla_Y X = h(Y, PX) - Ch(X, Y). \tag{3.5}$$

Thus from (3.4) and (3.5), we obtain

$$F[X, Y] = h(X, PY) - h(Y, PX). \tag{3.6}$$

Hence, we conclude that $F[X, Y] = 0$ if and only if $h(X, PY) = h(Y, PX)$, that is the distribution $\mathcal{D} \oplus \langle\xi\rangle$ is integrable if and only if $h(X, PY) = h(Y, PX)$, for all $X, Y \in \Gamma(\mathcal{D} \oplus \langle\xi\rangle)$.

We give the following main result of this section.

3.1. Theorem *Let M be a totally umbilical contact CR-submanifold of a cosymplectic manifold $\tilde{M}$. Then at least one of the following statements is true*

(i) *M is totally geodesic,*
(ii) *the anti-invariant distribution $\mathcal{D}^\perp$ is one-dimensional, i.e., dim $\mathcal{D}^\perp = 1$,*
(iii) *the mean curvature vector $H \in \Gamma(\mu)$.*

Proof. For a cosympectic manifold, we have

$$\tilde{\nabla}_Z \phi W = \phi \tilde{\nabla}_Z W,$$

for any $Z, W \in \Gamma(\mathcal{D}^\perp)$. Using (2.10) and (2.11), we derive

$$-g(H, \phi W)Z + \nabla^\perp_Z \phi W = \phi \nabla_Z W + g(Z, W)\phi H. \tag{3.7}$$

Taking the product with $Z \in \Gamma(\mathcal{D}^\perp)$ in (3.7), we get

$$g(H, \phi W)\|Z\|^2 = g(Z, W)g(H, \phi Z). \tag{3.8}$$

Interchanging Z and W in (3.8), we obtain

$$g(H, \phi Z)\|W\|^2 = g(Z, W)g(H, \phi W). \tag{3.9}$$

Thus, from (3.8) and (3.9), we deduce that

$$g(H, \phi Z) = \frac{g(Z, W)^2}{\|Z\|^2\|W\|^2} g(H, \phi Z).$$

That is

$$g(H, \phi Z)\{1 - \frac{g(Z, W)^2}{\|Z\|^2\|W\|^2}\} = 0. \tag{3.10}$$

Hence, the equation (3.10) has a solution if at least one of the followings holds

$$(i)\ H = 0 \ \text{ or } \ (ii)\ Z \parallel W \ \text{ or } \ (iii)\ H \perp \phi\mathcal{D}^\perp.$$

That is either M is totally geodesic or as Z and W are parallel to each other for any $Z, W \in \Gamma(\mathcal{D}^\perp)$ that is these two vectors are linearly dependent and hence *dim* $\mathcal{D}^\perp = 1$ or $H \in \Gamma(\mu)$, this proves the theorem completely. $\square$

3.2. Example Consider a flat manifold of real dimension 6 which have a complex Kaehler structure of dimension 3, that is $(\mathbb{C}^3, J, h)$ be a Kaehler manifold with complex structure J and Euclidean Hermitian metric h. Then $\tilde{M} = \mathbb{C}^3 \times \mathbb{R}$ is a cosymplectic manifold with the structure vector field $\xi = \frac{\partial}{\partial t}$, dual 1-form $\eta = dt$ and the metric $g = h + dt^2$. Now, consider $M = \mathbb{R}^3 \times S^1$, where S^1 is a unit circle being taken as totally real submanifold of $\mathbb{C}^3$. Then M is a contact CR-submanifold of $\tilde{M}$ with the invariant distribution $\mathcal{D} = \mathbb{R}^2$, anti-invariant distribution $\mathcal{D}^\perp = \Gamma(S^1)$ and the 1-dimensional distribution $\langle\xi\rangle = \mathbb{R}$.

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