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A Novel Hybrid Method for Singularly Perturbed Delay Differential Equations

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Abstract

The aim of the present study is to solve singularly perturbed second order linear delay differential equations by combining the flexibility of differential transform method and the efficiency of Taylor series expansion method. For this purpose, we use two-term Taylor series expansion method for delayed parameter linearization and then apply the differential transform method. Two examples are presented to demonstrate the efficiency, rapidity and reliability of the proposed hybrid method.

1. INTRODUCTION

In the present study, we consider singularly perturbed delay differential equations with boundary and interval conditions as:

$$\varepsilon y''(t) + p(t)y'(t-\delta) + q(t)y(t) = f(t), \quad 0 < t < 1, \tag{1}$$

$$y(t) = \emptyset(t), \quad -\delta \le t \le 0, \tag{2}$$

$$y(1) = \gamma, \tag{3}$$

where $0 < \varepsilon \ll 1$ is a perturbation parameter; $p(t),\ q(t),\ f(t)$ and $\emptyset(t)$ are sufficiently smooth functions; γ is constant and δ is delay parameter.

Singularly perturbed differential equations are differential equations in a class where the highest derivative is multiplied by a small parameter ε. Due to the $\varepsilon -$parameter, it is not easy to solve such equations. If there is also $\delta -$delay parameter in the equation, the solution of the equation will be more difficult. Therefore, standard numerical methods for solving singularly perturbed problems fail in providing accurate results and they are unstable for the perturbation parameter ε. Therefore, some fitted numerical methods have been developed to solve the equations such as finite difference method, Adomian decomposition method, differential transform method, finite elements method, reprocuding kernel method etc. These kinds of problems are ubiquitous in engineering and certain problems that arise in astrophysics, nonlinear dynamical systems, probability theory on algebraic structures, chemical and biochemical reactions, electrodynamics, quantum mechanics and cell growth, number theory, economics, financial mathematics, mixing problems, population models, etc. Singularly perturbed differential equations with delay term were studied by many authors. Based on previous studies, differential transform method was used in [1-4]. Perturbation methods [5] and delay differential equations were also utilized in [7, 8, 13, 14, 16, 17]. Terminal boundary-value technique, finite difference method, Galerkin finite element method, homotopy perturbation method, etc.

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were used by various researchers in solving singularly perturbed delay differential equations in [6, 9-12, 15, 18, 19].

In this study, the use of Taylor series expansion and differential transform method is proposed for singularly perturbed delay differential equations with boundary and interval conditions for the first time. Taylor series expansion method transform the delay terms to linearized terms. Then, differential transform method is applied directly or by employing an iterative methods. The differential transform method is developed by Zhou (1986), who studied linear and nonlinear differential equations in electric circuit analysis [1]. In order to show the performance of this novel hybrid method, firstly, we define this method. Finally, we examine two examples with table and figures. The proposed method provides an analytical solution in the form of series.

2. DESCRIPTION OF THE METHOD

In this section, the novel hybrid method is introduced for the solution of the singularly perturbed delay differential equations. Initially, a short overview of Taylor series expansion is explained and then the differential transform method is defined.

Firstly, Taylor series expansion is used by two term. So, the delay parameter for convection term in (1) is linearized as

$$y'(t-\delta) = y'(t) - \delta y''(t). \tag{4}$$

If the linearized convection term (4) is substituted into (1), the following equation can be obtained:

$$(\varepsilon - \delta p(t))y''(t) + p(t)y'(t) + q(t)y(t) = f(t), \quad 0 < t < 1. \tag{5}$$

Now, let us define the differential transform method as follows:
Differential transform of function $y(t)$ is defined as [4],

$$Y(k) = \frac{1}{k!}\left[\frac{d^k y(t)}{dt^k}\right]_{t=0} , \tag{6}$$

where $y(t)$ is original function and $Y(k)$ is the transformed function.
Differential inverse transform of $Y(k)$ is defined as

$$y(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!}\left[\frac{d^k y(t)}{dt^k}\right]_{t=0}. \tag{7}$$

If the expansion (7) is written as follows:

$$y(t) = \sum_{k=0}^{\infty} Y(k)\, t^k = Y(0) + Y(1)t + Y(2)t^2 + Y(3)t^3 + \cdots, \tag{8}$$

then it is called series solution of the differential transfom method.
Also, the following theorems are needed in the present study, where $Y(k)$ is the differential transform of $y(t)$:

Teorem 1. If $y(t) = \frac{dy(t)}{dt}$, then $Y(k) = \frac{(k+1)!}{k!} Y(k+1) = (k+1)Y(k+1)$.

Theorem 2. If $y(t) = \frac{d^2y(t)}{dt^2}$, then $Y(k) = \frac{(k+2)!}{k!}Y(k+2) = (k+2)(k+1)Y(k+2)$.

Theorem 3. If $y(t) = \alpha y(t)$, then $Y(k) = \alpha Y(k)$,
where α is a constant.

3. NUMERICAL EXAMPLES

In this section, we give two examples to show the advantages and effectiveness of this novel process. The algorithm, figures and tables are generated in Maple environment.

Example 1. We will consider the following singularly perturbed delay differential equations with boundary, interval conditions and left layer [18]:

$$\varepsilon y''(t) + y'(t-\delta) - y(t) = 0, \quad 0 < t < 1 \tag{9}$$
$$y(0) = 1, \; y(1) = 1. \tag{10}$$

The exact solution of this problem is given by

$$y(t) = \frac{\left(1 - e^{\frac{-1+\sqrt{1+4(\epsilon-\delta)}}{2(\epsilon-\delta)}}\right) e^{\frac{-1-\sqrt{1+4(\epsilon-\delta)}}{2(\epsilon-\delta)}t} + \left(e^{\frac{-1-\sqrt{1+4(\epsilon-\delta)}}{2(\epsilon-\delta)}} - 1\right) e^{\frac{-1+\sqrt{1+4(\epsilon-\delta)}}{2(\epsilon-\delta)}t}}{e^{\frac{-1-\sqrt{1+4(\epsilon-\delta)}}{2(\epsilon-\delta)}} - e^{\frac{-1+\sqrt{1+4(\epsilon-\delta)}}{2(\epsilon-\delta)}}}. \tag{11}$$

As a first step, we use two-term Taylor expansion for linearization as follows:

$$y'(t-\delta) = y'(t) - \delta y''(t). \tag{12}$$

If we substitute (12) into (9), then the following equation is obtained:

$$(\varepsilon - \delta)y''(t) + y'(t) - y(t) = 0, \; 0 < t < 1 \tag{13}$$
$$y(0) = 1, \;\; y(1) = 1. \tag{14}$$

In the second step, the differential transform method is applied to (13)-(14), thus we have the following differential transforms:

$$(\varepsilon - \delta)y''(t) \to Y(k) = (\varepsilon - \delta)(k+1)(k+2)Y(k+2),$$
$$y'(t) \to (k+1)Y(k+1),$$
$$y(t) \to Y(k).$$

If these differential transforms are written in equation (13), the following recurrence relation is obtained

$$Y(k+2) = \frac{Y(k) - (k+1)Y(k+1)}{(\varepsilon - \delta)(k+1)(k+2)}, \;\; k = 0, 1, 2, 3, \ldots,$$

series coefficients $Y(k)$ can be obtained as follows

$$k = 0, \qquad Y(2) = \frac{Y(0) - Y(1)}{2(\varepsilon - \delta)} = \frac{1 - c}{2(\varepsilon - \delta)},$$
$$k = 1, \qquad Y(3) = \frac{Y(1) - 2Y(2)}{6(\varepsilon - \delta)} = \frac{c - 2(\frac{1-c}{2(\varepsilon-\delta)})}{6(\varepsilon - \delta)},$$

$$k=2,\qquad Y(4)=\frac{Y(2)-3Y(3)}{12(\varepsilon-\delta)}=\frac{\frac{1-c}{2(\varepsilon-\delta)}-3\left(\frac{c-2\left(\frac{1-c}{2(\varepsilon-\delta)}\right)}{6(\varepsilon-\delta)}\right)}{12(\varepsilon-\delta)}.$$

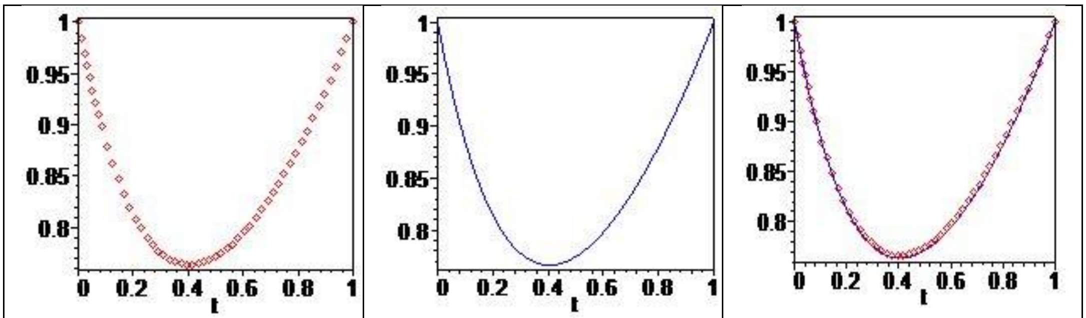
By the given initial conditions and above recurrence relation, we have that

$$Y(0)=1,\qquad Y(1)=c,$$

where c is a constant and can be calculated with y_{DTM}.
Finally, using above mentioned relations, we reach series solution of the problem (1)-(3) with 10 iterations as follows:

$$y_{DTM}=1-1.439001314t+3.048751642t^2-3.140210249t^3+2.597787998t^4-\cdots+0.02078337617t^{10}. \tag{15}$$

The computational results of the Example 1 are presented for different values of t with $\varepsilon=0.45$ and $\delta=0.05$ in Table 1. The plot of exact, DTM solution and comparison of them are shown in Figure 1.



a) EXACT solution | b) DTM solution | c) Exact (red) and DTM (blue) solution

Figure 1. *a), b) and c) for* $\varepsilon=0.45$ *and* $\delta=0.05$ *in the interval* $[0,1]$.

Table1. *Comparison of the Exact, DTM solution and Error values distrubition for* $\varepsilon=0.45,\ \delta=0.05$.

t	Exact solution	DTM solution (N=10)	Error
0.0	1.0000000000	0.9999999998	0.0000000002
0.1	0.8845103423	0.8836909193	0.0008194230
0.2	0.8141729590	0.8126972050	0.0014757540
0.3	0.7774411037	0.7754214971	0.0020196066
0.4	0.7660670621	0.7635793510	0.0024877111
0.5	0.7741895881	0.7712844118	0.0029051763
0.6	0.7976761006	0.7943960285	0.0032800721
0.7	0.8336486996	0.8300736831	0.0035750165
0.8	0.8801428410	0.8765237788	0.0036190622
0.9	0.9358617572	0.9329763895	0.0028853677
1.0	1.0000000000	0.9999999996	0.0000000004

Example 2. Let us consider the following singularly perturbed delay differential equation with boundary and interval conditions having a right boundary layer as [19]:

$$\varepsilon y''(t)-y'(t-\delta)-y(t)=0,\ \ 0<t<1, \tag{16}$$
$$y(0)=1, \tag{17}$$
$$y(1)=-1. \tag{18}$$

The exact solution is given by

$$y(t)=\frac{\left(1+e^{\frac{1+\sqrt{1+4(\epsilon+\delta)}}{2(\epsilon+\delta)}}\right)e^{\left(\frac{1-\sqrt{1+4(\epsilon+\delta)}}{2(\epsilon+\delta)}\right)t}-\left(e^{\frac{1-\sqrt{1+4(\epsilon+\delta)}}{2(\epsilon+\delta)}}+1\right)e^{\left(\frac{1+\sqrt{1+4(\epsilon+\delta)}}{2(\epsilon+\delta)}\right)t}}{e^{\frac{1+\sqrt{1+4(\epsilon+\delta)}}{2(\epsilon+\delta)}}-e^{\frac{1-\sqrt{1+4(\epsilon+\delta)}}{2(\epsilon+\delta)}}}.$$

As the first step, using two-term Taylor expansion as follows:

$$y'(t-\delta)=y'(t)-\delta y''(t). \tag{19}$$

If we substitute (19) into (16), then we obtain as follows:

$$(\varepsilon+\delta)y''(t)-y'(t)-y(t)=0,\ \ 0<t<1, \tag{20}$$
$$y(0)=1, \tag{21}$$
$$y(1)=-1. \tag{22}$$

Now, by using step by step the differential transform method for the problem (20)-(22), we can obtain

$$(\varepsilon+\delta)y''(t)\to Y(k)=(\varepsilon+\delta)(k+1)(k+2)Y(k+2),$$
$$y'(t)\to(k+1)Y(k+1),$$
$$y(t)\to Y(k).$$

When we write these differential transforms in (20), we have the following recurrence relation:

$$Y(k+2)=\frac{(k+1)Y(k+1)+Y(k)}{(\varepsilon+\delta)(k+1)(k+2)}.$$

When this relation is used for $k=0,1,2,3,...$, we can find differential transforms $Y(k)$.
It is also necessary to obtain $Y(0)$ and $Y(1)$ using initial conditions and recurrence relation, we deduce that

$$Y(0)=1,\ Y(1)=c,$$

where c is a constant. It can be obtained with y_{DTM} solution.
If these differential transforms are written in (8), the solution of the problem (20)-(22) can be illustrated with the following numerical series solution with 10 iterations:

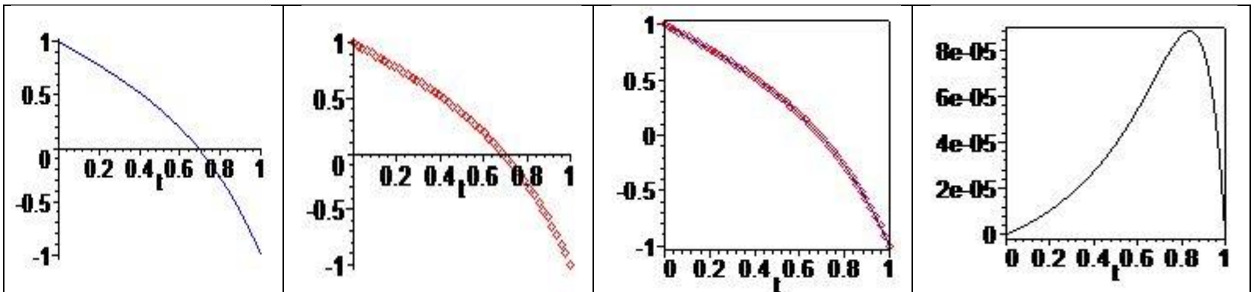
$$y_{DTM}=1-1.084773057t-0.0831108400t^2-0.4088218093t^3-0.2139830505t^4-0.1239954913t^5-0.0545072226t^6-\cdots 0.000568460424t^{10}. \tag{23}$$

Figure 2 shows DTM, exact solution and comparison of them. In Table 2, we explain DTM, exact solution and error distrubition for different values of t.

Table 2. *Comparison of the Exact, DTM solution and Error values for* $\varepsilon=0.01,\ \delta=0.5$.

t	**Exact solution**	**DTM solution (N=10)**	**Error**
0.0	1.0000000000	0.9999999996	0.0000000004
0.1	0.8902648064	0.8902600691	0.0000047373
0.2	0.7760751556	0.7760645517	0.0000106039
0.3	0.6539884687	0.6539704970	0.0000179717
0.4	0.5196450786	0.5196177591	0.0000273195
0.5	0.3674749768	0.3674357545	0.0000392223
0.6	0.1903147702	0.1902606336	0.0000541366
0.7	-0.0210929127	-0.2116432602	0.0000714132
0.8	-0.2787538865	-0.2788399390	0.0000860539

0.9	-0.5982782742	-0.5983581081	0.0000798339
1.0	-1.0000000000	-0.9999999997	0.0000000003



a) EXACT solution *b) DTM solution* *c) Exact (red) and DTM (blue) solution* *d) Error distrubition*

Figure 2. *a), b), c) and d) for* $\varepsilon = 0.01$ *and* $\delta = 0.5$ *in the interval* $[0,1]$.

4. CONCLUSION

In this study, the exact and numerical solutions of problems were compared with the new hybrid method. Theoretical consideration was described and two examples (with left and right boundary layers) were solved to demonstrate the validity of the method. This proposed method provided the desired accurate results only in a few terms and in a series form of the solution and demonstrated more fast and simple performance when compared to the known methods. The results of this performance can be observed in from Tables 1 and 2, and Figures 1 and 2.

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CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

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