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On 3-Dimensional Trans-Sasakian Manifold Admitting a Semi-Symmetric Metric Connection

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Abstract

The purpose of the present paper is to study 3-dimensional trans-Sasakian manifold admitting a semi-symmetric metric connection. Here we mainly study locally ϕ-symmetric and locally ϕ-concircularly symmetric 3-dimensional trans-Sasakian manifold admitting a semi-symmetric metric connection. Moreover, we examine our results and the results of [1] and [2] by constructing some examples.

1. INTRODUCTION

In the Gray-Hervella [3] classification of almost Hermitian manifold, there appears 16 different types of structures on the almost Hermitian manifold. Using the structure in the class W_4, trans-Sasakian structure $(\phi,\xi,\eta,g,\alpha,\beta)$ [4] was introduced. In general, a trans-Sasakian manifold $(M,\phi,\xi,\eta,g,\alpha,\beta)$ is called trans-Sasakian manifold of type (α,β), where α and β are smooth functions defined on M. The trans-Sasakian manifold of type $(0,0)$, $(0,\beta)$ and $(\alpha,0)$ are cosymplectic, β-Kenmotsu and α-Sasakian manifolds respectively [5, 6].

From then, many important contributions have appeared on the geometry of trans-Sasakian manifold. In particular, extensive work is done on the geometry and existence of $3-$dimensional trans-Sasakian manifold [7-15] with different restrictions on curvature and smooth functions α and β. Note that, a $3-$dimensional trans-Sasakian manifold with α,β constant, is either α-Sasakian or β-Kenmotsu or cosymplectic manifold. In addition, some authors studied some curvature properties and D-homothetic deformation of trans-Sasakian structure [16, 17]. Moreover, the present author and Sari also studied on invariant submanifolds of a trans-Sasakian manifold and a nearly trans-Sasakian manifold [18, 19].

In [20], Hayden introduced semi-symmetric metric connection on a Riemannian manifold. In [21], Yano obtained a relation between the semi-symmetric metric connection $\overline{\nabla}$ and the Levi-Civita connection ∇. Since then many geometers studied semi-symmetric metric connection on Riemannian manifold and trans-

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Sasakian manifold [1, 2, 22-25]. Recently, authors in [2] given an example of a 3-dimensional trans-Sasakian manifold admitting a semi-symmetric metric connection with all the values of semi-symmetric connections equals to zero (i.e., every $\overline{\nabla}_{E_i}E_j = 0$ for $1 \leq i, j \leq 3$), so that all other conditions are trivially true. Now the next questions arises that, is it possible to construct examples of $3-$dimensional trans-Sasakian manifold admitting a semi-symmetric metric connection with some values of semi-symmetric metric connections are non-zero (i.e., some $\overline{\nabla}_{E_i}E_j \neq 0$ for $1 \leq i, j \leq 3$)? Furthermore, if this construction of example is possible then, does it justifies your results? In this article we have given answers to these questions by constructing examples and verifying some results.

The paper is structured as follows: After introduction, in Section 2 we gave a brief account of trans-Sasakian manifold and semi-symmetric metric connection on it. In the next two sections we study locally ϕ-symmetric and locally concircularly ϕ-symmetric $3-$dimensional trans-Sasakian manifold admitting semi-symmetric metric connection. We obtained the expression for scalar curvature under ϕ-concircularly semi-symmetric $3-$dimensional trans-Sasakian manifold with respect to semi-symmetric metric connection in section 5. In section 6, we have constructed examples of $3-$dimensional trans-Sasakian manifold with respect to semi-symmetric metric connection which substantiate our results as well as some results of [1, 2].

2. PRELIMINARIES

Let M be an almost contact metric manifold with an almost contact metric structure (ϕ, ξ, η, g), that is, a tensor field ϕ of type (1,1), a vector field ξ, a 1-form η, and a Riemannian metric g such that

$$\phi^2 = -I + \eta o \xi, \ \ \eta(\xi) = 1, \ \eta \circ \phi = 0, \ \phi\xi = 0 \tag{1}$$

$$g(\phi U, \phi V) = g(U,V) - \eta(U)\eta(V) \tag{2}$$

for all differentiable vector fields U and V. If there are smooth functions α, β on an almost contact metric manifold (M, ϕ, ξ, η, g) satisfying

$$(\nabla_U \phi)V = \alpha\{g(U,V)\xi - \eta(V)U\} + \beta\{g(\phi U, V)\xi - \eta(V)\phi U\} \tag{3}$$

then it is said to be a trans-Sasakian manifold, and ∇ is the Levi-Civita connection with respect to the metric g. In view of (1) and (3), we have

$$\nabla_U \xi = -\alpha\phi U + \beta(U - \eta(U)\xi), (\nabla_U \eta)(V) = -\alpha g(\phi U, V) + \beta g(\phi U, \phi V). \tag{4}$$

From now on we will denote a $3-$dimensional trans-Sasakian manifold by *M*. In *M*, the following relations hold [8, 9]:

$$2\alpha\beta + \xi\alpha = 0,$$

$$S(U,V) = \{\frac{r}{2} + \xi\beta - (\alpha^2 - \beta^2)\}g(U,V)$$
$$-\{\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2)\}\eta(U)\eta(V) - \{V\beta + (\phi U)\alpha\}\eta(V),$$

$$R(U,V)Z = \{\frac{r}{2} + 2\xi\beta - 2(\alpha^2 - \beta^2)\}\{g(V,Z)U - g(U,Z)V\}$$

$$-g(V,Z)[\{\frac{r}{2}+\xi\beta-3(\alpha^2-\beta^2)\}\eta(U)\xi-\eta(U)(\phi grad\alpha-grad\beta)$$
$$+(U\beta+(\phi U)\alpha)\xi]+g(U,Z)[\{\frac{r}{2}+\xi\beta-3(\alpha^2-\beta^2)\}\eta(V)\xi$$
$$-\eta(V)(\phi grad\alpha-grad\beta)+(V\beta+(\phi V)\alpha)\xi]-[(Z\beta+(\phi Z)\alpha)\eta(V)$$
$$+(V\beta+(\phi V)\alpha)\eta(Z)+\{\frac{r}{2}+\xi\beta-3(\alpha^2-\beta^2)\}\eta(V)\eta(Z)]U+[(Z\beta+(\phi Z)\alpha)\eta(U)$$
$$+(U\beta+(\phi U)\alpha)\eta(Z)+\{\frac{r}{2}+\xi\beta-3(\alpha^2-\beta^2)\}\eta(U)\eta(Z)]V.$$

From here after we consider α and β to be constants, then the above relations become

$$S(U,V)=\{\frac{r}{2}-(\alpha^2-\beta^2)\}g(U,V)-\{\frac{r}{2}-3(\alpha^2-\beta^2)\}\eta(U)\eta(V),$$

$$QU=\{\frac{r}{2}-(\alpha^2-\beta^2)\}U-\{\frac{r}{2}-3(\alpha^2-\beta^2)\}\eta(U)\xi,$$

$$R(U,V)Z=\{\frac{r}{2}-2(\alpha^2-\beta^2)\}\{g(V,Z)U-g(U,Z)V\}$$
$$+\{\frac{r}{2}-3(\alpha^2-\beta^2)\}\{g(U,Z)\eta(V)-g(V,Z)\eta(U)\}\xi$$
$$+\{\frac{r}{2}-3(\alpha^2-\beta^2)\}\{\eta(U)\eta(Z)V-\eta(V)\eta(Z)U\}, \tag{5}$$

$$R(U,V)\xi=(\alpha^2-\beta^2)\{\eta(V)U-\eta(U)V\}$$

for all $U,V,Z\in\chi(M)$.

The relation between semi-symmetric metric connection $\overline{\nabla}$ and the Levi-Civita connection ∇ on M is given by

$$\overline{\nabla}_U V=\nabla_U V+\eta(V)U-g(U,V)\xi. \tag{6}$$

From now on we will denote the semi-symmetric metric connection by $\overline{\nabla}$, and the Levi-Civita connection by ∇ on M. The relation between Riemannian curvature tensor $\overline{R}$, Ricci curvature tensor $\overline{S}$ and scalar curvature $\overline{r}$ of $\overline{\nabla}$ and those with respect to ∇ for M [2] were given by

$$\overline{R}(U,V)Z=R(U,V)Z-\alpha\{g(\phi U,Z)V-g(\phi V,Z)U\}-\alpha\{g(U,Z)\phi V-g(V,Z)\phi U\}$$
$$-(\beta+1)\{\eta(U)\eta(Z)V-\eta(V)\eta(Z)U\}-(\beta+1)\{g(U,Z)\eta(V)$$
$$-g(V,Z)\eta(U)\}\xi+(2\beta+1)\{g(U,Z)V-g(V,Z)U\}, \tag{7}$$

$$\overline{S}(V,Z)=S(V,Z)+\alpha g(\phi V,Z)+(\beta+1)\eta(V)\eta(Z)-(3\beta+1)g(V,Z),$$

$$\overline{r}=r-8\beta-2. \tag{8}$$

Proposition 1: *In M admitting the semi-symmetric metric connection $\overline{\nabla}$, following relations are true*

$$(\overline{\nabla}_U\phi)V = \alpha\{g(U,V)\xi - \eta(V)U\} - (\beta+1)\{g(U,\phi V)\xi + \eta(V)\phi U, \tag{9}$$

$$\overline{\nabla}_W\xi = -\alpha\phi W + (\beta+1)(W - \eta(W)\xi), \tag{10}$$

$$(\overline{\nabla}_U\eta)V = -\alpha g(U,\phi V) + (\beta+1)\{g(U,V) - \eta(U)\eta(V)\}. \tag{11}$$

Proof. Above proposition is obtained by (1), (3), (4) and (8) by a straight forward computation.

3. LOCALLY ϕ-SYMMETRIC 3-DIMENSIONAL TRANS-SASAKIAN MANIFOLD

Definition 1: *M admitting the $\overline{\nabla}$ is said to be locally ϕ-symmetric if the condition $\phi^2((\overline{\nabla}_W\overline{R})(U,V)Z) = 0$ holds, for all vector fields U,V,Z,W orthogonal to ξ.*

We know that

$$(\overline{\nabla}_W\overline{R})(U,V)Z = \overline{\nabla}_W(\overline{R}(U,V)Z) - \overline{R}(\overline{\nabla}_W U,V)Z - \overline{R}(U,\overline{\nabla}_W V)Z - \overline{R}(U,V)\overline{\nabla}_W Z. \tag{12}$$

By virtue of (6), above equation turns into

$$\begin{aligned}(\overline{\nabla}_W\overline{R})(U,V)Z = {} & (\nabla_W\overline{R})(U,V)Z + \eta(\overline{R}(U,V)Z)W - g(W,\overline{R}(U,V)Z)\xi \\ & -\eta(U)\overline{R}(W,V)Z + g(W,U)\overline{R}(\xi,V)Z - \eta(V)\overline{R}(U,W)Z \\ & + g(W,V)\overline{R}(U,\xi)Z - \eta(Z)\overline{R}(U,V)W + g(W,Z)\overline{R}(U,V)\xi.\end{aligned} \tag{13}$$

Using (3), (4) and (7) in (13), we get

$$\begin{aligned}(\overline{\nabla}_W\overline{R})(U,V)Z = {} & (\nabla_W R)(U,V)Z - \alpha[\alpha\{(g(W,U)V - g(W,V)U)\eta(Z) \\ & + (\eta(V)U - \eta(U)V)g(Z,W) + (g(W,V)\xi - \eta(V)W)g(U,Z) \\ & + (\eta(U)W - g(W,U)\xi)g(V,Z)\}] + \beta\{(g(\phi W,U)V - g(\phi W,V)U)\eta(Z) \\ & - (\eta(V)U + \eta(U)V)g(Z,\phi W) + (g(\phi W,V)\xi - \eta(V)\phi W)g(U,Z) \\ & - (\eta(U)\phi W - g(\phi W,U)\xi)g(V,Z)\}] - (\beta+1)[\alpha\{(g(\phi W,V)U \\ & - g(\phi W,U)V)\eta(Z) + (\eta(V)U - \eta(U)V)g(Z,\phi W) - (g(\phi W,V)\xi \\ & + \eta(V)\phi W)g(U,Z) + (\eta(U)\phi W + g(\phi W,U)\xi)g(V,Z)\} + \beta\{(g(\phi W,\phi U)V \\ & - g(\phi W,\phi V)U)\eta(Z) + (\eta(U)V - \eta(V)U)g(\phi Z,\phi W) + (g(\phi W,\phi V)g(U,Z) \\ & - g(\phi W,\phi U)g(V,Z))\xi + (g(U,Z)\eta(V) - g(V,Z)\eta(U))W + (g(V,Z)\eta(U)\eta(W) \\ & - g(U,Z)\eta(V)\eta(W))\xi] + \eta(\overline{R}(U,V)Z)W - g(W,\overline{R}(U,V)Z)\xi \\ & -\eta(U)\overline{R}(W,V)Z + g(W,U)\overline{R}(\xi,V)Z - \eta(V)\overline{R}(U,W)Z \\ & + g(W,V)\overline{R}(U,\xi)Z - \eta(Z)\overline{R}(U,V)W + g(W,Z)\overline{R}(U,V)\xi.\end{aligned} \tag{14}$$

Applying ϕ^2 on both sides of the above equation and assuming that all vector fields U,V,Z,W are orthogonal to ξ, one can obtain

$$\phi^2((\overline{\nabla}_W\overline{R})(U,V)Z) = \phi^2((\nabla_W R)(U,V)Z).$$

$$\phi^2((\overline{\nabla}_W\overline{R})(U,V)Z) = \phi^2((\nabla_W R)(U,V)Z). \tag{15}$$

Thus, we state the following:

Theorem 1: *M of type (α,β) is locally ϕ-symmetric with respect to $\overline{\nabla}$ if and only if M is locally ϕ-symmetric with respect to ∇ provided α and β are constants.*

U. C. De and Avijit Sarkar [8] have proved the following theorem:

Theorem 2: *M of type (α,β) is locally ϕ-symmetric if and only if the scalar curvature is constant provided α and β are constants.*

In view of above theorem, one can restate the Theorem 1 as.

Theorem 3: *M of type (α,β) is locally ϕ-symmetric with respect to $\overline{\nabla}$ if and only if the scalar curvature is constant, provided α and β are constants.*

4. LOCALLY ϕ-CONCIRCULARLY SYMMETRIC $3-$DIMENSIONAL TRANS-SASAKIAN MANIFOLD

Concircular transformations have been introduced by Yano [26] as conformal transformations preserving geodesic circles. A geodesic circle is a curve in M such that the first curvature is constant and the second curvature is identically zero. The concircular curvature tensor C is an interesting invariant of a concircular transformation [26]. Concircular curvature tensor on M is given by

$$C(U,V)Z = R(U,V)Z - \frac{r}{6}[g(V,Z)U - g(U,Z)V]. \tag{16}$$

Definition 2: *M admitting $\overline{\nabla}$ is said to be locally ϕ-concircularly symmetric if the condition* $\phi^2((\overline{\nabla}_W \overline{C})(U,V)Z) = 0$ holds, for all vector fields U,V,Z,W orthogonal to ξ, where $\overline{C}$ is the concircular curvature tensor with respect to $\overline{\nabla}$ given by

$$\overline{C}(U,V)Z = \overline{R}(U,V)Z - \frac{\overline{r}}{6}[g(V,Z)U - g(U,Z)V]. \tag{17}$$

Using (10) and (16) in (17) gives

$$\begin{aligned}\overline{C}(U,V)Z = C(U,V)Z &- \alpha\{g(\phi U,Z)V - g(\phi V,Z)U\} - \alpha\{g(U,Z)\phi V - g(V,Z)\phi U\} \\ &- (\beta+1)\{\eta(U)\eta(Z)V - \eta(V)\eta(Z)U\} - (\beta+1)\{g(U,Z)\eta(V) - g(V,Z)\eta(U)\}\xi \\ &+ (2\beta+1)\{g(U,Z)V - g(V,Z)U\} + \frac{4\beta+1}{3}[g(V,Z)U - g(U,Z)V].\end{aligned} \tag{18}$$

In view of (1), (5) and (16), it follows that

$$C(U,V)\xi = \{(\alpha^2-\beta^2) - \frac{r}{6}\}\{\eta(V)U - \eta(U)V\}, \tag{19}$$

$$C(\xi,V)Z = \{(\alpha^2-\beta^2) - \frac{r}{6}\}\{g(V,Z)\xi - \eta(Z)V\}, \tag{20}$$

$$\eta(C(U,V)Z) = \{(\alpha^2-\beta^2) - \frac{r}{6}\}\{g(V,Z)\eta(U) - g(U,Z)\eta(V)\}. \tag{21}$$

Taking covariant differentiation of (18) with respect to W and using (6), (9)-(11) and (19)-(21), we get

$$
\begin{aligned}
(\overline{\nabla}_W \overline{C})(U,V)Z = {} & (\nabla_W C)(U,V)Z - g(W,C(U,V)Z)\xi - \eta(U)C(W,V)Z \\
& -\eta(V)C(U,W)Z - \eta(Z)C(U,V)W + \{(\alpha^2-\beta^2)-\frac{r}{6}\}\{g(V,Z)\eta(U)W \\
& -g(U,Z)\eta(V)W + g(U,W)g(V,Z)\xi - g(U,W)\eta(Z)V \\
& -g(V,W)g(U,Z)\xi + g(V,W)\eta(Z)U + g(Z,W)\eta(V)U \\
& -g(Z,W)\eta(U)V\} - \alpha^2\{(g(W,U)V - g(W,V)U)\eta(Z) + (\eta(V)U \\
& -\eta(U)V)g(Z,W) + (g(W,V)\xi - \eta(V)W)g(U,Z) + (\eta(U)W \\
& -g(W,U)\xi)g(V,Z)\} + 2\alpha(\beta+1)\{g(\phi W,Z)\eta(U)V \\
& -g(\phi W,Z)\eta(V)U + g(U,Z)\eta(V)\phi W - g(V,Z)\eta(U)\phi W\} \\
& -(\beta+1)^2[(g(U,W)-\eta(U)\eta(W))\eta(Z)V + (g(Z,W)-\eta(Z)\eta(W))\eta(U)V \\
& -(g(V,W)-\eta(V)\eta(W))\eta(Z)U - (g(Z,W)-\eta(Z)\eta(W))\eta(V)U + (g(V,W) \\
& -\eta(V)\eta(W))g(U,Z)\xi - (g(U,W)-\eta(U)\eta(W))g(V,Z)\xi \\
& +(g(U,Z)\eta(V)-g(V,Z)\eta(U))W - (g(U,Z)\eta(V)-g(V,Z)\eta(U))\eta(W)\xi]. \qquad (22)
\end{aligned}
$$

Applying ϕ^2 on both sides of (22) and considering that $U,V,Z,W \in \xi^{\perp}$, (22) turns into

$$\phi^2((\overline{\nabla}_W \overline{C})(U,V)Z) = \phi^2((\nabla_W C)(U,V)Z). \qquad (23)$$

Hence, we have the following;

Theorem 4: *M of type (α,β) is locally ϕ-concircularly symmetric with respect to $\overline{\nabla}$ if and only if M is locally ϕ-concircularly symmetric with respect to ∇ provided α and β are constants.*

Taking into an account of (16) and considering that $U,V,Z,W \in \xi^{\perp}$, (23) gives

$$\phi^2((\overline{\nabla}_W \overline{C})(U,V)Z) = \phi^2((\nabla_W R)(U,V)Z) - \frac{dr(W)}{6}\{g(V,Z)U - g(U,Z)V\}. \qquad (24)$$

From the above equation we can state the following;

Theorem 5: *Let M be admitting $\overline{\nabla}$ with (α and β) constant. Then in M, if any two of the following statements hold then the remaining statement holds.*
(a) M is locally ϕ-concircularly symmetric with respect to $\overline{\nabla}$
(b) M is locally ϕ-concircularly symmetric with respect to ∇.
(c) r is constant.

M with respect to $\overline{\nabla}$ is said to be horizontal ξ-concircularly flat if it satisfies $\overline{C}(U,V)\xi = 0$, for all $U,V \in \xi^{\perp}$. Putting $Z=\xi$ and considering $U,V \in \xi^{\perp}$, (18) gives

$$\overline{C}(U,V)\xi = C(U,V)\xi.$$

Thus we have the following assertion;

Theorem 6: *M of type $(\alpha\ ,\ \beta)$ is horizontally ξ-concircularly flat with respect to $\overline{\nabla}$ if and only if M is horizontally ξ-concircularly flat with respect to ∇ provided α and β are constants.*

5. ϕ-CONCIRCULARLY SEMI-SYMMETRIC $3-$DIMENSIONAL TRANS-SASAKIAN MANIFOLD

Definition 3: *A Riemannian manifold N is said to be ϕ-concircularly semi-symmetric if*

$$C(U,V)\cdot\phi=0, \tag{25}$$

holds on N.

Definition 4: *M admitting $\overline{\nabla}$ is said to be ϕ-concircularly semi-symmetric if*

$$\overline{C}(U,V)\cdot\phi=0, \tag{26}$$

for every vector fields U,V,Z on M.

Lemma 1: *Let M be admitting $\overline{\nabla}$ with α and β constants. Then for every vector fields U,V and Z, the following relation holds:*

$$\begin{aligned}\overline{R}(U,V)\phi Z-\phi\overline{R}(U,V)Z=&\{\alpha^2-(\beta+1)^2-(\beta+1)\}\{g(\phi U,Z)V-g(V,Z)\phi X\\&+g(U,Z)\phi V-g(Z,\phi V)U\}+2\alpha(\beta+2)\{g(V,Z)U\\&-g(U,Z)V\}+\alpha\{g(U,Z)\eta(V)\xi-g(V,Z)\eta(U)\xi+\eta(Z)\eta(U)V\\&-\eta(Z)\eta(V)U\}-\alpha(\beta+1)\{g(U,\phi Z)\phi V+g(Z,\phi V)\phi U\\&-g(V,\phi Z)\phi U-g(Z,\phi U)\phi V\}-(\beta+1)\{g(U,\phi Z)\eta(V)\xi\\&-\eta(U)g(V,\phi Z)\xi+\eta(V)\eta(Z)\phi U-\eta(Z)\eta(U)\phi V\}.\end{aligned} \tag{27}$$

***Proof*.** Above lemma can be easily proved with the help of (1), (6) and (9)-(11).

Theorem 7: If M *is a ϕ-concircularly semi-symmetric admitting $\overline{\nabla}$ with α and β constant, then the scalar curvature is given by* $\bar{r}=6(\alpha^2-(\beta+1)^2)$.

Proof. Suppose that M is ϕ-concircularly semi-symmetric. Then, from (26) we have

$$(\overline{C}(U,V)\cdot\phi)Z=\overline{C}(U,V)\phi Z-\phi\overline{C}(U,V)Z=0. \tag{28}$$

In view of (17) and (27), (28) turns into

$$\begin{aligned}&\{\alpha^2-(\beta+1)^2-(\beta+1)\}\{g(\phi U,Z)V-g(V,Z)\phi U+g(U,Z)\phi V\\&-g(Z,\phi V)U\}+2\alpha(\beta+2)\{g(V,Z)U-g(U,Z)V\}+\alpha\{g(U,Z)\eta(V)\xi\\&-g(V,Z)\eta(U)\xi+\eta(Z)\eta(U)V-\eta(Z)\eta(V)U\}-\alpha(\beta+1)\{g(U,\phi Z)\phi V\\&+g(Z,\phi V)\phi U-g(V,\phi Z)\phi U-g(Z,\phi U)\phi V\}-(\beta+1)\{g(U,\phi Z)\eta(V)\xi\\&-\eta(U)g(V,\phi Z)\xi+\eta(V)\eta(Z)\phi U-\eta(Z)\eta(U)\phi V\}\\&-\frac{\bar{r}}{6}\{g(V,\phi Z)U-g(U,\phi Z)V-g(V,Z)\phi U+g(U,Z)\phi V\}=0.\end{aligned} \tag{29}$$

Replacing U by ϕU and then taking inner product of the resultant equation with W, we get

$$\{\alpha^2-(\beta+1)^2-(\beta+1)\}\{g(\phi^2U,Z)g(V,W)-g(V,Z)g(\phi^2U,W)$$
$$+g(\phi U,Z)g(\phi V,W)-g(Z,\phi V)g(\phi U,W)\}+2\alpha(\beta+2)\{g(Y,Z)g(\phi U,W)$$
$$-g(\phi U,Z)g(V,W)\}+\alpha\{g(\phi U,Z)\eta(V)\eta(W)-g(V,Z)\eta(\phi U)\eta(W)$$
$$+\eta(Z)\eta(\phi U)g(V,W)-\eta(Z)\eta(V)g(\phi U,W)\}-\alpha(\beta+1)\{g(\phi U,\phi Z)g(\phi Y,W)$$
$$+g(Z,\phi V)g(\phi^2U,W)-g(V,\phi Z)g(\phi^2U,W)-g(Z,\phi^2U)g(\phi V,W)\}$$
$$-(\beta+1)\{g(\phi U,\phi Z)\eta(V)\eta(W)-\eta(\phi U)g(V,\phi Z)\eta(W)$$
$$+\eta(V)\eta(Z)g(\phi^2U,W)-\eta(Z)\eta(\phi U)g(\phi V,W)\}-\frac{\bar{r}}{6}\{g(V,\phi Z)g(\phi U,W)$$
$$-g(\phi U,\phi Z)g(V,W)-g(V,Z)g(\phi^2U,W)+g(\phi U,Z)g(\phi V,W)\}=0. \tag{30}$$

Using (1) and (2) in (30) and then contracting over X and W, we obtain

$$2\alpha g(V,\phi Z)+(2\alpha^2-2(\beta+1)^2-\frac{\bar{r}}{3})\eta(V)\eta(Z)=0.$$

Further contraction of the above equation gives

$$\bar{r}=6(\alpha^2-(\beta+1)^2). \tag{31}$$

Hence the proof.

6. EXAMPLES

Example 1 :

We consider a $3-$ dimensional manifold $M=\{(x_1,x_2,x_3)\in\mathbb{R}^3: x_3\neq 0\}$, where (x_1,x_2,x_3) are the standard coordinates in $\mathbb{R}^3$. Let $\{E_1,E_2,E_3\}$ be a linearly independent global frame field on M given by

$$E_1=e^{-x_3}(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}),\quad E_2=e^{-x_3}(-\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}),\quad E_3=\frac{\partial}{\partial x_3}.$$

Let g be the Riemannian metric defined by

$$g(E_1,E_2)=g(E_1,E_3)=g(E_2,E_3)=0,$$
$$g(E_1,E_1)=g(E_2,E_2)=g(E_3,E_3)=1.$$

If η is the 1-form defined by $\eta(W)=g(W,E_3)$ for any vector field $W\in\chi(M)$ and if ϕ is the (1, 1)-tensor field defined by

$$\phi(E_1)=E_2,\ \phi(E_2)=-E_1,\ \phi(E_3)=0,$$

then using the linearity of ϕ and g, we have

$$\eta(E_3)=1,$$

$$\phi^2 U = -U + \eta(U)E_3,$$
$$g(\phi U, \phi V) = g(U,V) - \eta(U)\eta(V)$$

for any $U, V \in \chi(M)$. Now we have

$$[E_1, E_2] = 0,\ [E_1, E_3] = E_1,\ [E_2, E_3] = E_2.$$

Using the Koszul's formula we get the following;

$$\nabla_{E_1} E_1 = -E_3,\ \ \nabla_{E_2} E_1 = 0,\ \ \nabla_{E_3} E_1 = 0,\ \nabla_{E_1} E_2 = 0,\ \ \nabla_{E_2} E_2 = -E_3,\ \ \nabla_{E_3} E_2 = 0,$$
$$\nabla_{E_1} E_3 = E_1,\ \ \nabla_{E_2} E_3 = E_2,\ \ \nabla_{E_3} E_3 = 0.$$

Clearly (ϕ, ξ, η, g) satisfy (3) and (4) with $\alpha = 0$ and $\beta = 1$. Thus M is a trans-Sasakian manifold. We also have

$$\overline{\nabla}_{E_1} E_1 = -2E_3,\ \ \overline{\nabla}_{E_2} E_1 = 0,\ \ \overline{\nabla}_{E_3} E_1 = 0,\ \overline{\nabla}_{E_1} E_2 = 0,\ \ \overline{\nabla}_{E_2} E_2 = -2E_3,\ \ \overline{\nabla}_{E_3} E_2 = 0,$$
$$\overline{\nabla}_{E_1} E_3 = 2E_1,\ \ \overline{\nabla}_{E_2} E_3 = 2E_2,\ \ \overline{\nabla}_{E_3} E_3 = 0.$$

With the help of the above it is easy to verify that

$$R(E_1,E_2)E_3 = 0,\quad R(E_2,E_3)E_3 = -E_2,\quad R(E_1,E_3)E_3 = -E_1,$$
$$R(E_1,E_2)E_2 = -E_1,\quad R(E_2,E_3)E_2 = E_3,\quad R(E_1,E_3)E_2 = 0,$$
$$R(E_1,E_2)E_1 = E_2,\quad R(E_2,E_3)E_1 = 0,\quad R(E_1,E_3)E_1 = E_3$$

and

$$\overline{R}(E_1,E_2)E_3 = 0,\quad \overline{R}(E_2,E_3)E_3 = -2E_2,\quad \overline{R}(E_1,E_3)E_3 = -2E_1,$$
$$\overline{R}(E_1,E_2)E_2 = -4E_1,\quad \overline{R}(E_2,E_3)E_2 = 2E_3,\quad \overline{R}(E_1,E_3)E_2 = 0,$$
$$\overline{R}(E_1,E_2)E_1 = 4E_2,\quad \overline{R}(E_2,E_3)E_1 = 0,\quad \overline{R}(E_1,E_3)E_1 = 2E_3.$$

Since E_1, E_2, E_3 forms a basis, any vector field $U, V, Z \in \chi(M)$ can be written as $U = a_1E_1 + b_1E_2 + c_1E_3$, $V = a_2E_1 + b_2E_2 + c_2E_3$, $Z = a_3E_1 + b_3E_2 + c_3E_3$, where $a_i, b_i, c_i \in \mathbb{R}^+$ (the set of all positive real numbers), $i = 1,2,3$. Using the expressions of the curvature tensors, we find values of Riemannian curvature tensor, Ricci tensor and scalar curvature with respect to ∇ and $\overline{\nabla}$ as;

$$\begin{aligned} R(U,V)Z = &[-\{a_1b_2 - b_1a_2\}b_3 + \{c_1a_2 - a_1c_2\}c_3]E_1 \\ &+[-\{b_1a_2 - a_1b_2\}a_3 + \{c_1b_2 - b_1c_2\}c_3]E_2 \\ &+[-\{c_1a_2 - a_1c_2\}a_3 - \{c_1b_2 - b_1c_2\}b_3]E_3, \end{aligned}$$

$$\begin{aligned} \overline{R}(U,V)Z = &[-4\{a_1b_2 - b_1a_2\}b_3 + 2\{c_1a_2 - a_1c_2\}c_3]E_1 \\ &+[-4\{b_1a_2 - a_1b_2\}a_3 + 2\{c_1b_2 - b_1c_2\}c_3]E_2 \\ &+[-2\{c_1a_2 - a_1c_2\}a_3 - 2\{c_1b_2 - b_1c_2\}b_3]E_3, \end{aligned}$$

$$S(E_1,E_1)=S(E_2,E_2)=S(E_3,E_3)=-2,$$

$$\bar{S}(E_1,E_1)=\bar{S}(E_2,E_2)=-6, \bar{S}(E_3,E_3)=-4,$$
$$r=-6, \quad \bar{r}=-16.$$

Clearly equation (8) is satisfied for values of β, r and $\bar{r}$ obtained. By virtue of values of Ricci tensors, it follows that the scalar curvature of the manifold is constant with respect to both ∇ and $\bar{\nabla}$. Also from the above expressions of curvature tensor and scalar curvatures, one can easily see that

$$\phi^2((\bar{\nabla}_{E_i}\bar{R})(U,V)Z)=\phi^2((\nabla_{E_i}R)(U,V)Z)=0,$$

$$\phi^2((\bar{\nabla}_{E_i}\bar{C})(U,V)Z)=\phi^2((\nabla_{E_i}C)(U,V)Z)=0.$$

Therefore, this upholds the Theorem-1 and Theorem-3 of section 3 and Theorem-4 of section 4.
Moreover, the manifold under consideration satisfies

$$\bar{R}(U,V)Z=-\bar{R}(V,U)Z, \quad \bar{R}(U,V)Z+\bar{R}(V,Z)U+\bar{R}(Z,U)V=0.$$

Hence, from the above equations one can see that this example verifies the equation (15) and the Theorem 3.2 of [1]. Further, it follows from the expressions of curvature tensor and Ricci tensor with respect to ∇ and $\bar{\nabla}$ that

$$P(E_1,E_2)E_3=P(E_1,E_3)E_3=P(E_2,E_3)E_3=0,$$

$$\bar{P}(E_1,E_2)E_3=\bar{P}(E_1,E_3)E_3=\bar{P}(E_2,E_3)E_3=0.$$

Hence, M is ξ-projectively flat with respect to both ∇ and $\bar{\nabla}$. Thus, Theorem 1 of [2] is verified.

Example 2 :

We consider $3-$dimensional manifold $M=\{(x_1,x_2,x_3)\in\mathbb{R}^3: x_3\neq 0\}$. The vector fields are chosen $E_1=-e^{-x_3}\frac{\partial}{\partial x_1}$, $E_2=e^{-x_3}\frac{\partial}{\partial x_2}$, $E_3=\frac{\partial}{\partial x_3}$, which are linearly independent at each point of M. Let g be the Riemannian metric defined by

$$g=e^{2x_3}(dx_1\otimes dx_1+dx_2\otimes dx_2)+\eta\otimes\eta,$$

where η is the $1-$form defined by $\eta(U)=g(U,E_3)$, for any vector field U on M. Hence, $\{E_1,E_2,E_3\}$ is an orthonormal basis of M. We defined the $(1,1)$ tensor field ϕ as

$$\phi\left(X\frac{\partial}{\partial x_1}+Y\frac{\partial}{\partial x_2}\right)+Z\frac{\partial}{\partial x_3}=\left(Y\frac{\partial}{\partial x_1}-X\frac{\partial}{\partial x_2}\right).$$

Thus, we have $\phi(E_1)=E_2$, $\phi(E_2)=-E_1$ and $\phi(E_3)=0$. The linearity property of ϕ and g yields that

$$\eta(E_3)=1, \quad \phi^2X=-X+\eta(X)E_3, \quad g(\phi U,\phi V)=g(U,V)-\eta(U)\eta(V),$$

for any vector fields U, V on M. The $1-$forms η is closed. In addition, we have $\Phi = -e^{2x_3} dx_1 \wedge dx_2$. Hence, $d\Phi = -2e^{2x_3} dx_3 \wedge dx_1 \wedge dx_2 = 2\eta \wedge \Phi$. It can be seen that $M(\phi,\xi,\eta,g)$ is normal. Thus for $E_3 = \xi$, $M(\phi,\xi,\eta,g)$ is a trans-Sasakian manifold of type $(0,-1)$. Moreover, we get

$$[E_i,\xi] = E_i\,,\ \ [E_i,E_j] = 0\,,\ \ i,j = 1,2$$

and others are zero. Therefore, non-zero terms of $\overline{\nabla}$ on M becomes

$$\overline{\nabla}_{E_i}E_i = -2\xi,\ \overline{\nabla}_{E_i}\xi = 2E_i,\ \ i = 1,2.$$

With a similar approach in the Example 6.1, we can see that this example also verifies locally ϕ-symmetric, locally concircularly ϕ-symmetric, first Bianchi identity, skew symmetric property of Riemannian curvature tensor and ξ-projectively flat $3-$dimensional trans-Sasakian manifold M admitting a $\overline{\nabla}$. This connection coincide the connection of example in our paper. Hence this example provides all since other calculations are the same.

7. CONLUSION

From the above examples, we can see the existence of $3-$dimensional trans-Sasakian manifold M admitting a semi-symmetric metric connection satisfying locally ϕ-symmetric, locally concircularly ϕ-symmetric, first Bianchi identity, skew symmetric property of Riemannian curvature tensor and ξ-projectively flatness conditions with constants α and β. For the future study one can think of above conditions or different curvature conditions on M with functions α and β.

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CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

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