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Basic Properties of Sumudu Transformation and Its Application to Some Partial Differential Equations

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Abstract

One of the solution methods of ordinary and partial differential equations is integral transform. Newly introduced Sumudu transform provides an alternative integral transform which gives us an efficient tool to solve initial-boundary value problems. In this work, it is obtained solutions to some dynamic problems which arise in physics and engineering.

Keywords: Integral transform, Laplace transform, Sumudu transform, Convolution.

1. INTRODUCTION

In the study of problems to ordinary and partial differential equations, integral transforms methods such as Laplace, Fourier, Hilbert, or Stieltjes transforms, have a significant role. Transforms such as Sumudu and Elzaki resemble Laplace transforms, and by use of these transforms, differential equation systems have been solved by transforming to algebraic equations. Sumudu transform, introduced by Watugala in 1993, has been used frequently to solve ordinary differential equations.

Sumudu transform yields results in some cases where other methods fail, especially arise in control theory [5]. Furthermore, for some problems Sumudu transform gives better results compared to Laplace transform.

In this study, Sumudu transforms are rapplied to both ordinary and partial differential equations, after giving its properties, and relations between Laplace transform.

2. DEFINITION AND PROPERTIES OF SUMUDU TRANSFORMATION

Definition 2.1. Let A be a function set defined by

$$A = \{f(t) \,|\, \exists M, \tau_1, \text{and/or } \tau_2 > 0, \text{ such that } |f(t)| < Me^{|t|/\tau_j}, \text{if } t \in (-1)^j \times [0,\infty)\}, \tag{2.1}$$

where M is a constant and τ_1, τ_2 are finite constants or infinite, [1]-[3]. For a function of exponential order, Sumudu integral transform is defined in various forms as follows;

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$$G(u)=S[f(t)]=\begin{cases}\int_0^\infty f(ut)e^{-t}dt \quad , \quad 0\le u<\tau_2 \\ \int_0^\infty f(ut)e^{-t}dt \quad , \quad -\tau_1<u\le 0 ,\end{cases} \tag{2.2}$$

$$G(u)=S[f(t)]=\int_0^\infty f(ut)e^{-t}dt \quad , \quad u\in(-\tau_1,\tau_2), \tag{2.3}$$

$$G(u)=S[f(t)]=\frac{1}{u}\int_0^\infty f(t)e^{-\frac{t}{u}}dt \ , \ u\in(-\tau_1,\tau_2). \tag{2.4}$$

Let $f(x,t)$ be continuous and of exponential order. Then Sumudu transform of $f(x,t)$ is written as

$$G(x,u)=S[f(x,t)]=\frac{1}{u}\int_0^\infty f(x,t)e^{-\frac{t}{u}}dt \ . \tag{2.5}$$

An Example . Apply Sumudu transform to function $f(t)=\cos(at)\ \ t\ge 0$.

By integration by parts we find

$$\begin{aligned}S[\cos(at)]&=\int_0^\infty \cos(uat)e^{-t}dt\\ &=1-(ua)^2\int_0^\infty \cos(uat)e^{-t}dt\\ &=\frac{1}{1+(au)^2}.\end{aligned}$$

Theorem 2.2. [1] Let $f(t)\in A$ with Sumudu transform $G(u)$. Then,

$$S\left[\frac{1}{t}\int_0^t f(\tau)d\tau\right]=\frac{1}{u}\int_0^u G(v)dv. \tag{2.6}$$

Proof. By considering right-hand side of (2.6), it is written as

$$\begin{aligned}\frac{1}{u}\int_0^u G(v)dv&=\frac{1}{u}\int_0^u\int_0^\infty f(vt)e^{-t}dtdv\\ &=\frac{1}{u}\int_0^\infty e^{-t}\int_0^u f(vt)dvdt. \end{aligned} \tag{2.7}$$

If we make change of variables in the last integral, we obtain the result:

$$\begin{aligned}&=\int_0^\infty \frac{e^{-t}}{u}\int_0^{ut} f(w)\frac{dw}{t}dt=\int_0^\infty \frac{1}{ut}e^{-t}\int_0^{ut} f(w)dwdt\\ &=\int_0^\infty \frac{1}{ut}\left[\int f(w)dw\right]_0^{ut} e^{-t}dt=S\left[\frac{1}{t}\int_0^t f(\tau)d\tau\right].\end{aligned} \tag{2.8}$$

Theorem 2.3. Let Sumudu transform of $f(t)\in A$ is $G(u)$. Then

$$S\left[e^{at}f(t)\right]=\frac{1}{1-au}G\left(\frac{u}{1-au}\right). \tag{2.9}$$

[1].

Proof. From (2.3) Sumudu transform of $f(t)\in A$ is

$$\begin{aligned}S\left[e^{at}f(t)\right]&=\int_0^\infty e^{aut}f(ut)e^{-t}dt\\ &=\int_0^\infty f(ut)e^{-t(1-au)}dt.\end{aligned} \tag{2.10}$$

Making change of variable, gives

$w=t(1-au)$, $\left(t=\dfrac{w}{(1-au)}\right)$ then we obtain

$$\begin{aligned}S\left[e^{at}f(t)\right]&=\int_0^\infty f\left(\frac{uw}{1-au}\right)e^{-w}\frac{dw}{1-au}\\ &=\frac{1}{1-au}\int_0^\infty f\left(\frac{uw}{1-au}\right)e^{-w}dw\\ &=\frac{1}{1-au}G\left(\frac{u}{1-au}\right).\end{aligned} \tag{2.11}$$

Teorem 2.4. [1] Let $f(t)\in A$ with Sumudu transform $G(u)$. Then,

$$S\left[t\frac{df(t)}{dt}\right]=u\frac{dG(u)}{du}. \tag{2.12}$$

For the inverse Sumudu transform, we use the equation between Laplace and Sumudu transform. Inverse Laplace transform is

$$\begin{aligned}L^{-1}[F(s)]&=\frac{1}{2\Pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}F(s)ds\\ &=\Sigma\text{Residues}\left[e^{st}F(s)\right],\end{aligned} \tag{2.13}$$

see [4].

Theorem 2.5. Let $G(u)$ be the Sumudu transform of $f(t)\in A$ such that

(i) $\dfrac{G(1/u)}{u}$, is a meromorphic function, with singularities having $\text{Re}(u)<\gamma$, and

(ii) there exists a circular region Γ, with radius R and positive constants, M and K, with

$$\left|\frac{G(1/u)}{u}\right| < MR^{-k}, \quad (2.14)$$

then the function $f(t)$ is given by [3].

$$S^{-1}[G(u)] = \frac{1}{2\Pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{ut}\frac{G(1/u)}{u}du$$

$$= \sum \text{Residues}\left[e^{ut}\frac{G(1/u)}{u}\right]. \quad (2.15)$$

3. SUMUDU TRANSFORMATION DERIVATIVE, INTEGRAL AND LAPLACE TRANSFORMATION DUALITY

P In this section, we give the connections between Laplace and Sumudu transforms. Furthmore, for a given $f(t)\in A$, Sumudu transform of derivative and integral of that function are introduced.

Theorem 3.1. [1] Let $f(t)\in A$ be a function whose Laplace transform is $F(s)$. Then the following relation is valid;

$$G(u) = \frac{F(1/u)}{u}. \quad (3.1)$$

Corollary 3.2. For $x>0$, Sumudu transform of t^{x-1} is

$$G(u) = S(t^{x-1}) = \Gamma(x)u^{x-1}. \quad (3.2)$$

Corollary 3.3. [1] Let $f(t)\in A$ be function whose Laplace and Sumudu transforms are $F(s)$ and $G(u)$, respectively. Then

$$F(s) = \frac{G(1/s)}{s}. \quad (3.3)$$

Theorem 3.4. [1] Let $F_1(u)$ and $G_1(u)$ be Laplace and Sumudu transforms of the derivative of function $f(t)\in A$. Then

$$G_1(u) = \frac{G(u)-f(0)}{u}. \quad (3.4)$$

Theorem 3.5. [1] Let $G_n(u)$ and $F_n(u)$ be Sumudu and Laplace transforms of the nth derivative of the function, $f(t)\in A$, respectively $n\geq 1$. Then the following

$$G_n(u) = \frac{G(u)}{u^n} - \sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{u^{n-k}}. \quad (3.5)$$

Lemma 3.6. Let $f(x,t)$ be a piecewise continuous function having exponential order. If $Y(x,u)$ is Sumudu transform of $f(x,t)$, then Sumudu transforms of partial derivatives of that function are given as follows,

$$\text{(i)}\, S\left[\frac{\partial f(x,t)}{\partial t}\right] = \frac{1}{u}[Y(x,u)-f(x,0)], \quad (3.6)$$

$$\text{(ii)}\, S\left[\frac{\partial f(x,t)}{\partial x}\right] = \frac{d[Y(x,u)]}{dx}, \quad (3.7)$$

$$\text{(iii)}\, S\left[\frac{\partial^2 f(x,t)}{\partial t^2}\right] = \frac{1}{u^2}Y(x,u) - \frac{1}{u^2}f(x,0) - \frac{1}{u}\frac{\partial f(x,0)}{\partial t}, \quad (3.8)$$

$$\text{(iv)}\, S\left[\frac{\partial^2 f(x,t)}{\partial x^2}\right] = \frac{d^2[Y(x,u)]}{dx^2}, \quad (3.9)$$

Theorem 3.7. [1] Let $F^1(s)$ and $G^1(u)$ denote the Laplace and the Sumudu transforms of the integral of $f(t)$, respectively, $W(t) = \int_0^t f(\tau)d\tau$ then the following relation holds

$$G^1(u) = S[W(t)] = uG(u). \quad (3.10)$$

4. APPLICATIONS

In this section, some examples of initial value problem for both ordinary and partial differential equations are given.

Example 4.1. Find the solution to the initial value problem

$y''(t) + y'(t) - 2y(t) = 3t$, $(t\geq 0)$;

$y(0)=3$, $y'(0)=0$.

Solution. Applying Sumudu transform to both sides of the equation gives

$$\frac{G(u)}{u^2} - \frac{y(0)}{u^2} - \frac{y'(0)}{u} + \frac{G(u)-y(0)}{u} - 2G(u) = 3u. \quad (4.1)$$

By using initial conditions, we obtain

$$\frac{G(u)}{u^2}-\frac{3}{u^2}+\frac{G(u)-3}{u}-2G(u)=3u. \qquad (4.2)$$

Last equation yields $G(u)$ as

$$\begin{aligned} G(u)&=\frac{3u^3+3u+3}{-2u^2+u+1}\\ &=-\frac{3}{2}u-\frac{3}{4}+\frac{3}{4(2u+1)}+\frac{3}{1-u}. \end{aligned} \qquad (4.3)$$

Consequently, it is obtained the solution by applying inverse transform

$$S^{-1}[G(u)]=S^{-1}\left[-\frac{3}{2}u-\frac{3}{4}+\frac{3}{4(2u+1)}+\frac{3}{1-u}\right]$$

$$y(t)=-\frac{3}{2}t-\frac{3}{4}+\frac{3}{4}e^{-2t}+3e^{t}. \qquad (4.4)$$

Example 4.2. Solve the initial-boundary value problem of the wave equation:

$$y_{tt}-c^2y_{xx}=0 \qquad 0<x<3,\ t>0,$$

(4.5) $y(x,0)=\sin\left(\frac{\pi}{3}x\right)\ 0\le x\le 3,$

(4.6)

$$y_t(x,0)=0,\ 0\le x\le 3, \qquad (4.7)$$

$$y(0,t)=0,\ y(3,t)=0\ t\ge 0. \qquad (4.8)$$

Solution. Sumudu transform of function $y(x,t)$ *is*

$$S[y(x,t)]=\frac{1}{u}\int_0^\infty y(x,t)e^{\frac{-t}{u}}dt=Y(x,u)=Y. \qquad (4.9)$$

If Sumudu transform is applied on both sides of the equation, it is deduced as

$$S[y_{tt}]-c^2S[y_{xx}]=0. \qquad (4.10)$$

Then substituting initial and boundary conditions into the last equality, one gets an equation with variable x. Solution to homogeneous part of this equation is

$$Y_h(x,u)=c_1e^{\frac{1}{cu}x}+c_2e^{\frac{-1}{cu}x}. \qquad (4.11)$$

It is easily obtained particular solution of the non-homogeneous differential equations as

$$Y_p(x,t)=A\sin\left(\frac{\pi x}{3}\right)+B\cos\left(\frac{\pi x}{3}\right), \qquad (4.12)$$

$$Y_p(x,u)=\frac{1}{u^2c^2\frac{\pi^2}{9}+1}\sin\left(\frac{\pi x}{3}\right). \qquad (4.13)$$

Finally taking inverse Sumudu transform to the last function, we found the solution,

$$y(x,t)=\sin\left(\frac{\pi x}{3}\right)\cos\left(\frac{\pi c}{3}t\right). \qquad (4.14)$$

Example 4.3. Determine the solution to following IBVP of the non-homogeneous wave equation

$$y_{tt}=c^2y_{xx}+\sin(\pi x), \qquad (4.15)$$

$$y(x,0)=0, \qquad (4.16)$$

$$y_t(x,0)=0, \qquad (4.17)$$

$$y(0,t)=0=y(1,t), \qquad (4.18)$$

$0<x<1,\ t>0.$

Solution. Applying Sumudu transform to the equation gives

$$\begin{aligned} &\frac{1}{u^2}Y(x,u)-\frac{1}{u^2}y(x,0)-\frac{1}{u}\frac{\partial y}{\partial t}(x,0)\\ &=c^2\frac{d^2Y}{dx^2}(x,u)+\sin(\pi x). \end{aligned} \qquad (4.19)$$

By means of the initial conditions, it reduces to

$$u^2c^2Y_{xx}-Y=-u^2\sin(\pi x). \qquad (4.20)$$

General solution to homogeneous part of the equation is

$$Y_h(x,u)=c_1e^{\frac{1}{cu}x}+c_2e^{\frac{-1}{cu}x}. \qquad (4.21)$$

Then particular solution of (4.20) is obtained easily as

$$Y_p=A\sin(\pi x)+B\cos(\pi x), \qquad (4.22)$$

where $A=\frac{u^2}{u^2c^2\pi^2+1}$ and $B=0$.

Therefore, the general solution of the transformed equation (4.20) is

$$Y(x,u)=c_1e^{\frac{1}{cu}x}+c_2e^{\frac{-1}{cu}x}+\frac{u^2}{u^2c^2\pi^2+1}\sin(\pi x). \tag{4.23}$$

Plugging the boundary conditions into the last function gives

$$Y(x,u)=\frac{u^2}{u^2c^2\pi^2+1}\sin(\pi x). \tag{4.24}$$

Finally, if we take inverse Sumudu transform (4.24), we arrive the solution to initial and boundary value problem as

$$y(x,t)=\sin(\pi x)\frac{1}{(c\pi)^2}\left(1-\cos(c\pi t)\right). \tag{4.25}$$

Example 4.4.

Let $c,k,a\in\mathbb{R}$ .Solve

$$\frac{1}{c^2}y_{tt}-y_{xx}=k\sin\left(\frac{\pi x}{a}\right),\quad 0<x<a,\quad t>0, \tag{4.26}$$

$$y(x,0)=0=y_t(x,0),\quad 0<x<a, \tag{4.27}$$

$$y(0,t)=0=y(a,t),\quad t>0, \tag{4.28}$$

$$y(0,t)=0=y(a,t),\quad t>0. \tag{4.29}$$

Solution. Applying Sumudu transform to the equation gives

$$\frac{1}{c^2u^2}Y(x,u)-\frac{1}{c^2u^2}y(x,0)-\frac{1}{c^2u}\frac{\partial y}{\partial t}(x,0)-c^2\frac{d^2Y}{dx^2}(x,u)=k\sin(\frac{\pi x}{a}). \tag{4.30}$$

By means of the initial conditions, it reduces to

$$c^2u^2Y_{xx}-Y=-c^2u^2k\sin\left(\frac{\pi x}{a}\right). \tag{4.31}$$

General solution to homogeneous part of the equation is

$$Y_h(x,u)=c_1e^{\frac{1}{cu}x}+c_2e^{\frac{-1}{cu}x}. \tag{4.32}$$

Then particular solution of solution of (4.31) is obtained easily as

$$Y_p=A\sin(\frac{\pi x}{a})+B\cos(\frac{\pi x}{a}), \tag{4.33}$$

where

$$A=\frac{kc^2u^2}{u^2\left(\frac{\pi c}{a}\right)^2+1}\quad\text{and } B=0\quad .$$

Therefore, the general solution of the transformed equation (4.31) is

$$Y(x,u)=c_1e^{\frac{1}{cu}x}+c_2e^{\frac{-1}{cu}x}+\frac{kc^2u^2}{u^2\left(\frac{\pi c}{a}\right)^2+1}\sin(\frac{\pi x}{a}). \tag{4.34}$$

Plugging the boundary conditions into the last function gives

$$Y(x,u)=\frac{kc^2u^2}{u^2\left(\frac{\pi c}{a}\right)^2+1}\sin(\frac{\pi x}{a}). \tag{4.35}$$

Finally, if we take inverse Sumudu transform (4.35), we arrive the solution to initial and boundary value problem as

$$Y(x,u)=\frac{ka^2}{\pi^2}\sin(\frac{\pi x}{a})\left(1-\cos(\frac{\pi ct}{a})\right).$$

5. CONCLUSION AND REMARKS

In this work, it is shown the properties of newly introduced integral transform as Sumudu transform and its applications to the some ordinary and partial differential equations.

Furthermore, it can be stated that this transform may be used as a tool to study solutions to system of the differential equations.

6. REFERENCES

[1] Belgacem, F.B.M., Karaballi, A.A., and Kalla, S.L.,Analytical investigations of the Sumudu transform and applications to integral production equations. Mathemetical Problems in Engineering, no.3,103-118, 2003.

[2] Weerakoon, S.,Application of Sumudu transform to partial differential equations. Int. J. Math. Educ. Sci. Technol., vol.25, no.2, 277-283, 1994.

[3] Belgacem, F.B.M., Karaballi, A.A., Sumudu transform fundamental properties investigations and applications.Hindawi Publishing Corporation Journal of Applied Mathematic and Stochstic Analysis Volume, Article ID 91083, 1-23,DOI 10.1155/JAMSA/2006/91083, 2006.

[4] Belgacem, F. B. M., Introducing and analysing deeper Sumudu properties. Nonlinear Studies, vol. 13, no. 1, 23-41, 2006.

[5] Watugala, G. K., Sumudu transform: a new integral transform to solve differential equations and control engineering problems. International Journal of Mathematical Education in Science and Technology, no.1, 35-43, 1993.

[6] Eltayeb, H., Kılıçman, A. 2010. A note on the Sumudu transforms and differential equations., Applied Mathematical Sciences, 4(22), 1089-1098.

[7] Bulut, Hasan; Başkonuş, H. Mehmet; Tuluce, Seyma. 2012. The solutions of partial differential equations with variable coefficient by Sumudu transform method. In: AIP Conference Proceedings. AIP, p. 91-95.

[8] Belgacem, Fethi Bin Muhammad. 2009. Sumudu applications to Maxwell's equations. PIERS Online, 5.4: 355-360.

[9] Eltayeb, H., Kılıçman, A., Fisher, B. 2010. A new integral transform and associated distributions. Integral Transforms and Special Functions, 21(5), 367-379.